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SOME INEQUALITIES REGARDING A GENERALIZATION OF IOACHIMESCU'S CONSTANT

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Abstract. The purpose of this paper is to evaluate the limit $\mathscr{I}(a)$ of the sequence

$$\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{a+1}}+\cdots+\frac{1}{\sqrt{a+n-1}}-2(\sqrt{a+n-1}-\sqrt{a})\right)_{n\in\mathbb{N}},$$

where $a\in(0,+\infty)$. We give some lower and upper estimates for

$$\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{a+1}}+\cdots+\frac{1}{\sqrt{a+n-1}}-2(\sqrt{a+n-1}-\sqrt{a})-\mathscr{I}(a),\ \ n\in\mathbb{N}.$$

1. Introduction

In the problem proposed by A. G. Ioachimescu [8] in 1895, it is asked to be shown that the sequence $(S_n)_{n\in\mathbb{N}}$, defined by $S_n=1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}-2\sqrt{n}$, for each $n\in\mathbb{N}$, is convergent and its limit lies between -2 and -1.

There have been given many generalizations and other results regarding Ioachimescu's problem in the literature (see, for example, [1], [2], [3], [4, Theorem 1, parts a) and b)], [5, problem 3, p. 534], [6, problem 3.1, p. 431], [7, problem P2, parts (i) and (ii)], [9, pp. 27–33], [10], [11], [12]).
As it is mentioned in [4, p. 199], M. Bătineţu-Giurgiu proposed in 1992 the following problem. Let $a,r\in(0,+\infty)$ and $(a_n)_{n\in\mathbb{N}}$ be the sequence defined by $a_n=a+(n-1)r$, for each $n\in\mathbb{N}$. Show that the sequence $\left(r\sum\limits_{k=1}^{n}\frac{1}{\sqrt{a_k}}-2\sqrt{a_n}\right)_{n\in\mathbb{N}}$ is convergent.

We consider the sequence $(I_n)_{n\in\mathbb{N}}$ defined by $I_n=1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}-2(\sqrt{n}-1)$, for each $n\in\mathbb{N}$. Also, we denote the limit of $(I_n)_{n\in\mathbb{N}}$ by $\mathscr{I}$ and we call it Ioachimescu's constant.

In Section 2 we present a generalization of Ioachimescu's constant as the limit of the sequence $\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{a+1}}+\cdots+\frac{1}{\sqrt{a+n-1}}-2(\sqrt{a+n-1}-\sqrt{a})\right)_{n\in\mathbb{N}}$, where $a\in(0,+\infty)$, and we denote this limit by $\mathscr{I}(a)$. In Section 3 we give estimates for $I_n-\mathscr{I}$, $n\in\mathbb{N}$, and in Section 4 we give some estimates for $\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{a+1}}+\cdots+\frac{1}{\sqrt{a+n-1}}-2(\sqrt{a+n-1}-\sqrt{a})-\mathscr{I}(a)$, $n\in\mathbb{N}\setminus\{1\}$.

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2. The number $\mathscr{I}(a)$

THEOREM 2.1. ([10], Theorem 2) *Let* $a \in (0,+\infty)$. *We consider the sequences* $(x_n(a))_{n\in\mathbb{N}}$ *and* $(y_n(a))_{n\in\mathbb{N}}$ *defined by*

$$x_n(a) = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{a+1}} + \cdots + \frac{1}{\sqrt{a+n-1}} - 2(\sqrt{a+n} - \sqrt{a})$$

and

$$y_n(a) = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{a+1}} + \cdots + \frac{1}{\sqrt{a+n-1}} - 2(\sqrt{a+n-1} - \sqrt{a}),$$

for each $n \in \mathbb{N}$.

Then:

(i) *the sequences* $(x_n(a))_{n\in\mathbb{N}}$ *and* $(y_n(a))_{n\in\mathbb{N}}$ *are convergent to the same number, which we denote by* $\mathscr{I}(a)$, *and satisfy the inequalities* $x_n(a) < x_{n+1}(a) < \mathscr{I}(a) < y_{n+1}(a) < y_n(a)$, *for each* $n \in \mathbb{N}$;

(ii) $0 < \dfrac{1}{\sqrt{a}} - 2(\sqrt{a+1} - \sqrt{a}) < \mathscr{I}(a) < \dfrac{1}{\sqrt{a}}$;

(iii) $\lim\limits_{n\to\infty} \sqrt{n}(\mathscr{I}(a) - x_n(a)) = \dfrac{1}{2}$ *and* $\lim\limits_{n\to\infty} \sqrt{n}(y_n(a) - \mathscr{I}(a)) = \dfrac{1}{2}$.

REMARK 2.1. The sequence $(y_n(a))_{n\in\mathbb{N}}$ from Theorem 2.1, for $a=1$, becomes the sequence $(I_n)_{n\in\mathbb{N}}$, so $\mathscr{I}(1) = \mathscr{I}$.

REMARK 2.2. Taking into account the inequalities from part (i) of Theorem 2.1, using the computer program MAPLE, we obtain, for example:

$1.2241\ldots = x_{1000}\left(\frac{1}{4}\right) < \mathscr{I}\left(\frac{1}{4}\right) < y_{1000}\left(\frac{1}{4}\right) = 1.2557\ldots$;
$1.0208\ldots = x_{1000}\left(\frac{1}{3}\right) < \mathscr{I}\left(\frac{1}{3}\right) < y_{1000}\left(\frac{1}{3}\right) = 1.0524\ldots$;
$0.7935\ldots = x_{1000}\left(\frac{1}{2}\right) < \mathscr{I}\left(\frac{1}{2}\right) < y_{1000}\left(\frac{1}{2}\right) = 0.8251\ldots$;
$0.5238\ldots = x_{1000}(1) < \mathscr{I}(1) < y_{1000}(1) = 0.5554\ldots$;
$0.3522\ldots = x_{1000}(2) < \mathscr{I}(2) < y_{1000}(2) = 0.3838\ldots$;
$0.2808\ldots = x_{1000}(3) < \mathscr{I}(3) < y_{1000}(3) = 0.3124\ldots$;
$0.2394\ldots = x_{1000}(4) < \mathscr{I}(4) < y_{1000}(4) = 0.2709\ldots$.

THEOREM 2.2. ([10], Theorem 3) *Let* $a \in (0,+\infty)$. *We consider the sequences* $(x_n(a))_{n\in\mathbb{N}}$ *and* $(y_n(a))_{n\in\mathbb{N}}$ *from the enunciation of Theorem* 2.1, *of which limit we denoted by* $\mathscr{I}(a)$.

Then:

(i) $\dfrac{1}{2\sqrt{a+n}} < \mathscr{I}(a) - x_n(a) < \dfrac{1}{\sqrt{a+n} + \sqrt{a+n-1}}$, *for each* $n \in \mathbb{N}$;

(ii) $\dfrac{1}{\sqrt{a+n} + \sqrt{a+n-1}} < y_n(a) - \mathscr{I}(a) < \dfrac{1}{2\sqrt{a+n-1}}$, *for each* $n \in \mathbb{N}$.

3. Lower and upper estimates for $I_n - \mathscr{I}$

THEOREM 3.1. *Let $p \in \mathbb{N}$. We consider the sequence $(I_n)_{n\in\mathbb{N}}$, presented in Section 1, of which limit we denoted by $\mathscr{I}$.*

Then

$$\frac{1}{2\sqrt{n}+\alpha_p} < I_n - \mathscr{I} < \frac{1}{2\sqrt{n}+\beta},$$

for each $n \in \mathbb{N}$, $n \geqslant p$, with $\alpha_p = \sqrt{4p+3} - \sqrt{p+1} - \sqrt{p}$ and $\beta = 0$.

Proof. We consider the sequence $(u_n(r))_{n\in\mathbb{N}}$ defined by

$$u_n(r) = I_n - \frac{1}{2\sqrt{n}+r},$$

for each $n \in \mathbb{N}$, where $r > -2$. We have

$$\begin{aligned}
u_{n+1}(r) - u_n(r) &= \frac{1}{\sqrt{n+1}} - 2(\sqrt{n+1}-\sqrt{n}) - \frac{1}{2\sqrt{n+1}+r} + \frac{1}{2\sqrt{n}+r} \\
&= \frac{1}{\sqrt{n+1}} - \frac{2}{\sqrt{n+1}+\sqrt{n}} + \frac{2(\sqrt{n+1}-\sqrt{n})}{(2\sqrt{n+1}+r)(2\sqrt{n}+r)} \\
&= \frac{1}{\sqrt{n+1}+\sqrt{n}}\left[\frac{-1}{\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})} + \frac{2}{(2\sqrt{n+1}+r)(2\sqrt{n}+r)}\right] \\
&= \frac{2(n+1) - 2\sqrt{n(n+1)} - 2r(\sqrt{n+1}+\sqrt{n}) - r^2}{\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})^2(2\sqrt{n+1}+r)(2\sqrt{n}+r)},
\end{aligned}$$

for each $n \in \mathbb{N}$. Set $\lambda_n := \sqrt{n(n+1)}$ and $\mu_n := \sqrt{n+1}+\sqrt{n}$, for any $n \in \mathbb{N}$. Clearly, $\lambda_n^2 = n(n+1)$ and $\mu_n^2 = 2n+1+2\lambda_n$, for each $n \in \mathbb{N}$. It is not difficult to see that $n = \frac{\mu_n^2 - 2\lambda_n - 1}{2}$ and $\mu_n^2 - 4\lambda_n = \frac{1}{\mu_n^2}$, for each $n \in \mathbb{N}$. So,

$$\begin{aligned}
u_{n+1}(r) - u_n(r) &= \frac{\mu_n^2 - 4\lambda_n - 2r\mu_n - r^2 + 1}{\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})^2(2\sqrt{n+1}+r)(2\sqrt{n}+r)} \\
&= \frac{-2r\mu_n^3 + (1-r^2)\mu_n^2 + 1}{\mu_n^2\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})^2(2\sqrt{n+1}+r)(2\sqrt{n}+r)},
\end{aligned}$$

for any $n \in \mathbb{N}$.

We consider the function $f : (-2,+\infty) \times [\sqrt{2}+1,+\infty) \to \mathbb{R}$, defined by

$$f(r,\mu) = -2r\mu^3 + (1-r^2)\mu^2 + 1,$$

for each $(r,\mu) \in (-2,+\infty) \times [\sqrt{2}+1,+\infty)$.

One of the two solutions of the equation $-\mu_p^2 t^2 - 2\mu_p^3 t + \mu_p^2 + 1 = 0$ is $\alpha_p = \sqrt{4p+3} - \sqrt{p+1} - \sqrt{p}$. We have $\dfrac{\partial f}{\partial \mu}(r,\mu) = -2\mu(3r\mu + r^2 - 1)$, for each $(r,\mu) \in (-2,+\infty) \times [\sqrt{2}+1,+\infty)$. It is not difficult to verify that $\dfrac{\partial f}{\partial \mu}(\alpha_p,\mu) < 0$, for any

$\mu \in [\mu_p, +\infty)$, which means that $0 = f(\alpha_p, \mu_p) > f(\alpha_p, \mu)$, for each $\mu \in (\mu_p, +\infty)$. So, $u_{n+1}(\alpha_p) - u_n(\alpha_p) < 0$, for each $n \in \mathbb{N}$, $n \geqslant p+1$, i.e. the sequence $(u_n(\alpha_p))_{n \geqslant p+1}$ is strictly decreasing. Also, because $f(\alpha_p, \mu_p) = 0$, it is clear that $u_{p+1}(\alpha_p) = u_p(\alpha_p)$. We have $\lim\limits_{n\to\infty} u_n(\alpha_p) = \mathscr{I}$. Now we can write that

$$\mathscr{I} < u_n(\alpha_p) = I_n - \frac{1}{2\sqrt{n} + \alpha_p},$$

for each $n \in \mathbb{N}$, $n \geqslant p$.

We have that $f(0, \mu) = \mu^2 + 1 > 0$, for each $\mu \in [\sqrt{2} + 1, +\infty)$. So, $u_{n+1}(0) - u_n(0) > 0$, for each $n \in \mathbb{N}$, i.e. the sequence $(u_n(0))_{n \in \mathbb{N}}$ is strictly increasing. We have $\lim\limits_{n\to\infty} u_n(0) = \mathscr{I}$. Now we can write that

$$I_n - \frac{1}{2\sqrt{n}} = u_n(0) < \mathscr{I},$$

for each $n \in \mathbb{N}$. □

REMARK 3.1. The sequence $(\alpha_p)_{p \in \mathbb{N}}$ defined by $\alpha_p = \sqrt{4p+3} - \sqrt{p+1} - \sqrt{p}$, for each $p \in \mathbb{N}$, is strictly decreasing.

THEOREM 3.2. *We consider the sequence $(I_n)_{n \in \mathbb{N}}$, presented in Section 1, of which limit we denoted by $\mathscr{I}$.*

Then

$$\frac{1}{2\sqrt{n} + \alpha} \leqslant I_n - \mathscr{I} < \frac{1}{2\sqrt{n} + \beta},$$

for each $n \in \mathbb{N}$, with $\alpha = \dfrac{2\mathscr{I} - 1}{1 - \mathscr{I}}$ and $\beta = 0$.

Moreover, the constants α and β are the best possible with this property.

Proof. We follow the proof of Theorem 3.1, with $p = 2$, regarding the lower estimate. One of the two solutions of the equation $-\mu_2^2 t^2 - 2\mu_2^3 t + \mu_2^2 + 1 = 0$ is $\alpha_2 = \sqrt{11} - \sqrt{3} - \sqrt{2}$. It is not difficult to verify that $\dfrac{\partial f}{\partial \mu}(\alpha_2, \mu) < 0$, for any $\mu \in [\sqrt{2} + 1, +\infty)$, which means that

$$f(\alpha_2, \mu_1) > 0 = f(\alpha_2, \mu_2) > f(\alpha_2, \mu),$$

for each $\mu \in (\mu_2, +\infty)$. Therefore

$$\mathscr{I} < u_n(\alpha_2) = I_n - \frac{1}{2\sqrt{n} + \alpha_2},$$

for each $n \in \mathbb{N} \setminus \{1\}$. Clearly, because $f(\alpha_2, \mu_1) > 0$, we have that $u_2(\alpha_2) - u_1(\alpha_2) > 0$. It is easy to see that $u_1(\alpha_2) = I_1 - \frac{1}{2+\alpha_2} < \mathscr{I}$, hence the lower estimate does not hold for $n = 1$, with α_2.

We have $\alpha_1 > \alpha > \alpha_2$. With $\alpha = \dfrac{2\mathscr{I}-1}{1-\mathscr{I}}$ we have the equality

$$\frac{1}{2+\alpha} = I_1 - \mathscr{I}.$$

Taking into account that $\dfrac{\partial f}{\partial r}(r,\mu) = -2\mu^2(\mu+r) < 0$, for any $(r,\mu) \in (-2,+\infty) \times [\sqrt{2}+1,+\infty)$, we are able to write that $0 \geqslant f(\alpha_2,\mu) > f(\alpha,\mu)$, for each $\mu \in [\mu_2,+\infty)$. So, $u_{n+1}(\alpha) - u_n(\alpha) < 0$, for each $n \in \mathbb{N}\setminus\{1\}$, i.e. the sequence $(u_n(\alpha))_{n\geqslant 2}$ is strictly decreasing. We have $\lim\limits_{n\to\infty} u_n(\alpha) = \mathscr{I}$. Now we are able to write that

$$\mathscr{I} < u_n(\alpha) = I_n - \frac{1}{2\sqrt{n}+\alpha},$$

for each $n \in \mathbb{N}\setminus\{1\}$. □

4. Some estimates for $y_n(a) - \mathscr{I}(a)$

THEOREM 4.1. *Let* $a \in (0,+\infty)$. *We consider the sequence* $(y_n(a))_{n\in\mathbb{N}}$ *from the enunciation of Theorem* 2.1*, of which limit we denoted by* $\mathscr{I}(a)$.

Then

$$\frac{1}{2\sqrt{a+n-\frac{4}{5}}} < y_n(a) - \mathscr{I}(a) < \frac{1}{2\sqrt{a+n-\frac{5}{6}}},$$

for each $n \in \mathbb{N}\setminus\{1\}$.

Proof. We consider the sequences $(u_n(a))_{n\in\mathbb{N}}$ and $(v_n(a))_{n\in\mathbb{N}}$ defined by

$$u_n(a) = y_n(a) - \frac{1}{2\sqrt{a+n-\frac{4}{5}}}$$

and

$$v_n(a) = y_n(a) - \frac{1}{2\sqrt{a+n-\frac{5}{6}}},$$

for each $n \in \mathbb{N}$.

We have

$$u_{n+1}(a) - u_n(a) = \frac{1}{\sqrt{a+n}} - 2(\sqrt{a+n} - \sqrt{a+n-1}) - \frac{1}{2\sqrt{a+n+\frac{1}{5}}} + \frac{1}{2\sqrt{a+n-\frac{4}{5}}},$$

for each $n \in \mathbb{N}$. Set $x := a+n$. We have

$$\begin{aligned} E(x) &:= \frac{1}{\sqrt{x}} - 2(\sqrt{x} - \sqrt{x-1}) - \frac{1}{2\sqrt{x+\frac{1}{5}}} + \frac{1}{2\sqrt{x-\frac{4}{5}}} \\ &= \frac{1}{\sqrt{x}} - \frac{2}{\sqrt{x}+\sqrt{x-1}} + \frac{\sqrt{x+\frac{1}{5}} - \sqrt{x-\frac{4}{5}}}{2\sqrt{\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)}} \end{aligned}$$

$$= \frac{-1}{\sqrt{x}(\sqrt{x}+\sqrt{x-1})^2} + \frac{1}{2\sqrt{\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)}\left(\sqrt{x+\frac{1}{5}}+\sqrt{x-\frac{4}{5}}\right)}$$

$$= \frac{\sqrt{x}(2x-1+2\sqrt{x(x-1)}) - 2\sqrt{\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)}\left(\sqrt{x+\frac{1}{5}}+\sqrt{x-\frac{4}{5}}\right)}{2\sqrt{x}(\sqrt{x}+\sqrt{x-1})^2\sqrt{\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)}\left(\sqrt{x+\frac{1}{5}}+\sqrt{x-\frac{4}{5}}\right)}$$

$$= \frac{x(2x-1+2\sqrt{x(x-1)})^2 - 4\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)\left(\sqrt{x+\frac{1}{5}}+\sqrt{x-\frac{4}{5}}\right)^2}{2\sqrt{x}(\sqrt{x}+\sqrt{x-1})^2\sqrt{\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)}\left(\sqrt{x+\frac{1}{5}}+\sqrt{x-\frac{4}{5}}\right)}$$

$$\times \frac{1}{\sqrt{x}(2x-1+2\sqrt{x(x-1)}) + 2\sqrt{\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)}\left(\sqrt{x+\frac{1}{5}}+\sqrt{x-\frac{4}{5}}\right)}.$$

Also, we have that

$$\begin{aligned}
F(x) &:= x(2x-1+2\sqrt{x(x-1)})^2 - 4\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)\left(\sqrt{x+\frac{1}{5}}+\sqrt{x-\frac{4}{5}}\right)^2 \\
&= -\frac{4}{5}x^2 + \frac{1}{5}x - \frac{48}{125} \\
&\quad -4\left[2\left(x^2-\frac{3}{5}x-\frac{4}{25}\right)\sqrt{\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)} - x(2x-1)\sqrt{x(x-1)}\right] \\
&= -\frac{4}{5}x^2 + \frac{1}{5}x - \frac{48}{125} \\
&\quad -4\cdot\frac{4\left(x^2-\frac{3}{5}x-\frac{4}{25}\right)^2\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right) - x^3(2x-1)^2(x-1)}{2\left(x^2-\frac{3}{5}x-\frac{4}{25}\right)\sqrt{\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)} + x(2x-1)\sqrt{x(x-1)}} \\
&= -\frac{4}{5}x^2 + \frac{1}{5}x - \frac{48}{125} \\
&\quad -4\cdot\frac{\frac{4}{5}x^5 - \frac{13}{5}x^4 + \frac{61}{25}x^3 - \frac{48}{125}x^2 - \frac{576}{3125}x - \frac{256}{15625}}{2\left(x^2-\frac{3}{5}x-\frac{4}{25}\right)\sqrt{\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)} + x(2x-1)\sqrt{x(x-1)}} \\
&= -\left[(x-1)\left(\frac{4}{5}x+\frac{3}{5}\right) + \frac{123}{125}\right] \\
&\quad -4\cdot\frac{(x-2)^2\left(\frac{4}{5}x^3+\frac{3}{5}x^2+\frac{41}{25}x+\frac{472}{125}\right) + \frac{26124}{3125}x - \frac{236256}{15625}}{2\left(x^2-\frac{3}{5}x-\frac{4}{25}\right)\sqrt{\left(x+\frac{1}{5}\right)\left(x-\frac{4}{5}\right)} + x(2x-1)\sqrt{x(x-1)}}.
\end{aligned}$$

It follows that $F(x) < 0$ and therefore $E(x) < 0$, for any $x \in (2,+\infty)$. This means that $u_{n+1}(a) - u_n(a) < 0$, for each $n \in \mathbb{N}\setminus\{1\}$, i.e. the sequence $(u_n(a))_{n\geqslant 2}$ is strictly decreasing. Using this, as well as the fact that $\lim_{n\to\infty} u_n(a) = \mathscr{I}(a)$, we are able to write that

$$\mathscr{I}(a) < u_n(a) = y_n(a) - \frac{1}{2\sqrt{a+n-\frac{4}{5}}},$$

for each $n \in \mathbb{N} \setminus \{1\}$.

We have

$$v_{n+1}(a) - v_n(a) = \frac{1}{\sqrt{a+n}} - 2(\sqrt{a+n} - \sqrt{a+n-1}) - \frac{1}{2\sqrt{a+n+\frac{1}{6}}} + \frac{1}{2\sqrt{a+n-\frac{5}{6}}},$$

for each $n \in \mathbb{N}$. Set $x := a+n$. We have

$$\begin{aligned}
G(x) &:= \frac{1}{\sqrt{x}} - 2(\sqrt{x} - \sqrt{x-1}) - \frac{1}{2\sqrt{x+\frac{1}{6}}} + \frac{1}{2\sqrt{x-\frac{5}{6}}} \\
&= \frac{1}{\sqrt{x}} - \frac{2}{\sqrt{x}+\sqrt{x-1}} + \frac{\sqrt{x+\frac{1}{6}} - \sqrt{x-\frac{5}{6}}}{2\sqrt{\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)}} \\
&= \frac{-1}{\sqrt{x}(\sqrt{x}+\sqrt{x-1})^2} + \frac{1}{2\sqrt{\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)}\left(\sqrt{x+\frac{1}{6}} + \sqrt{x-\frac{5}{6}}\right)} \\
&= \frac{\sqrt{x}(2x-1+2\sqrt{x(x-1)}) - 2\sqrt{\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)}\left(\sqrt{x+\frac{1}{6}} + \sqrt{x-\frac{5}{6}}\right)}{2\sqrt{x}(\sqrt{x}+\sqrt{x-1})^2\sqrt{\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)}\left(\sqrt{x+\frac{1}{6}} + \sqrt{x-\frac{5}{6}}\right)} \\
&= \frac{x(2x-1+2\sqrt{x(x-1)})^2 - 4\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)\left(\sqrt{x+\frac{1}{6}} + \sqrt{x-\frac{5}{6}}\right)^2}{2\sqrt{x}(\sqrt{x}+\sqrt{x-1})^2\sqrt{\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)}\left(\sqrt{x+\frac{1}{6}} + \sqrt{x-\frac{5}{6}}\right)} \\
&\quad \times \frac{1}{\sqrt{x}(2x-1+2\sqrt{x(x-1)}) + 2\sqrt{\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)}\left(\sqrt{x+\frac{1}{6}} + \sqrt{x-\frac{5}{6}}\right)}.
\end{aligned}$$

Also, we have that

$$\begin{aligned}
H(x) &:= x(2x-1+2\sqrt{x(x-1)})^2 - 4\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)\left(\sqrt{x+\frac{1}{6}} + \sqrt{x-\frac{5}{6}}\right)^2 \\
&= \frac{1}{3}x - \frac{10}{27} + 4\left[x(2x-1)\sqrt{x(x-1)} - 2\left(x^2 - \frac{2}{3}x - \frac{5}{36}\right)\sqrt{\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)}\right] \\
&= \frac{1}{3}x - \frac{10}{27} + 4 \cdot \frac{x^3(2x-1)^2(x-1) - 4\left(x^2-\frac{2}{3}x-\frac{5}{36}\right)^2\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)}{x(2x-1)\sqrt{x(x-1)} + 2\left(x^2-\frac{2}{3}x-\frac{5}{36}\right)\sqrt{\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)}}
\end{aligned}$$

$$= \frac{1}{3}x - \frac{10}{27} + 4 \cdot \frac{\frac{4}{3}x^4 - \frac{55}{27}x^3 + \frac{55}{108}x^2 + \frac{25}{162}x + \frac{125}{11664}}{x(2x-1)\sqrt{x(x-1)} + 2\left(x^2 - \frac{2}{3}x - \frac{5}{36}\right)\sqrt{\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)}}$$

$$= \frac{1}{3}x - \frac{10}{27} + 4 \cdot \frac{(x-2)\left(\frac{4}{3}x^3 + \frac{17}{27}x^2 + \frac{191}{108}x + \frac{299}{81}\right) + \frac{86237}{11664}}{x(2x-1)\sqrt{x(x-1)} + 2\left(x^2 - \frac{2}{3}x - \frac{5}{36}\right)\sqrt{\left(x+\frac{1}{6}\right)\left(x-\frac{5}{6}\right)}}.$$

It follows that $H(x) > 0$ and therefore $G(x) > 0$, for any $x \in (2, +\infty)$. This means that $v_{n+1}(a) - v_n(a) > 0$, for each $n \in \mathbb{N} \setminus \{1\}$, i.e. the sequence $(v_n(a))_{n \geqslant 2}$ is strictly increasing. Using this, as well as the fact that $\lim_{n\to\infty} v_n(a) = \mathscr{I}(a)$, we are able to write that

$$y_n(a) - \frac{1}{2\sqrt{a+n-\frac{5}{6}}} = v_n(a) < \mathscr{I}(a),$$

for each $n \in \mathbb{N} \setminus \{1\}$. □

REMARK 4.1. We have

$$\frac{1}{\sqrt{a+n} + \sqrt{a+n-1}} < \frac{1}{2\sqrt{a+n-\frac{4}{5}}}$$

and

$$\frac{1}{2\sqrt{a+n-\frac{5}{6}}} < \frac{1}{2\sqrt{a+n-1}},$$

for each $n \in \mathbb{N} \setminus \{1\}$. So, the lower estimate and the upper estimate from Theorem 4.1 are finer than those from part (ii) of Theorem 2.2, for $n \in \mathbb{N} \setminus \{1\}$.

REMARK 4.2. From Theorem 4.1 it follows that

$$v_n(a) = y_n(a) - \frac{1}{2\sqrt{a+n-\frac{5}{6}}} < \mathscr{I}(a) < y_n(a) - \frac{1}{2\sqrt{a+n-\frac{4}{5}}} = u_n(a),$$

for each $n \in \mathbb{N} \setminus \{1\}$. Taking into account these inequalities, using the computer program MAPLE, we obtain, for example:

$1.2399636\ldots = v_{1000}\left(\frac{1}{4}\right) < \mathscr{I}\left(\frac{1}{4}\right) < u_{1000}\left(\frac{1}{4}\right) = 1.2399638\ldots$;
$1.0366172\ldots = v_{1000}\left(\frac{1}{3}\right) < \mathscr{I}\left(\frac{1}{3}\right) < u_{1000}\left(\frac{1}{3}\right) = 1.0366175\ldots$;
$0.8093148\ldots = v_{1000}\left(\frac{1}{2}\right) < \mathscr{I}\left(\frac{1}{2}\right) < u_{1000}\left(\frac{1}{2}\right) = 0.8093151\ldots$;
$0.5396455\ldots = v_{1000}(1) < \mathscr{I}(1) < u_{1000}(1) = 0.5396459\ldots$;
$0.3680726\ldots = v_{1000}(2) < \mathscr{I}(2) < u_{1000}(2) = 0.3680730\ldots$;
$0.2966403\ldots = v_{1000}(3) < \mathscr{I}(3) < u_{1000}(3) = 0.2966407\ldots$;
$0.2551885\ldots = v_{1000}(4) < \mathscr{I}(4) < u_{1000}(4) = 0.2551888\ldots$.

COROLLARY 4.1. *We consider the sequence* $(I_n)_{n\in\mathbb{N}}$, *presented in Section* 1, *of which limit we denoted by* $\mathscr{I}$.

Then

$$\frac{1}{2\sqrt{n+\frac{1}{5}}} < I_n - \mathscr{I} < \frac{1}{2\sqrt{n+\frac{1}{6}}},$$

for each $n \in \mathbb{N}$.

Proof. We take $a = 1$ in Theorem 4.1 and it is not difficult to see from the proof of this theorem that the estimates hold for $n = 1$ too in this case. □

Having in view the results obtained so far, it would be an interesting problem to be found the best possible constants α and β, with the property that

$$\frac{1}{2\sqrt{n+\alpha}} \leqslant I_n - \mathscr{I} \leqslant \frac{1}{2\sqrt{n+\beta}},$$

for each $n \in \mathbb{N}$.

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