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AN AREA INEQUALITY FOR ELLIPSES INSCRIBED IN QUADRILATERALS

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Abstract. If E is any ellipse inscribed in a convex quadrilateral, $Ð$, then we prove that $\frac{\text{Area}(E)}{\text{Area}(Ð)} \leqslant \frac{\pi}{4}$, and equality holds if and only if $Ð$ is a parallelogram and E is tangent to the sides of $Ð$ at the midpoints. We also prove that the foci of the unique ellipse of maximal area inscribed in a parallelogram, $Ð$, lie on the orthogonal least squares line for the vertices of $Ð$. This does not hold in general for convex quadrilaterals.

1. Introduction

It is well known (see [1], [4], or [5]) that there is a unique ellipse inscribed in a given triangle, T, tangent to the sides of T at their respective midpoints. This is often called the midpoint or Steiner inellipse, and it can be characterized as the inscribed ellipse having maximum area. In addition, one has the following inequality. If E denotes any ellipse inscribed in T, then

$$\frac{\text{Area}(E)}{\text{Area}(T)} \leqslant \frac{\pi}{3\sqrt{3}}, \tag{1}$$

with equality if and only if E is the midpoint ellipse. In [5] the authors also discuss a connection between the Steiner ellipse and the orthogonal least squares line for the vertices of T.

DEFINITION 1. The line, $£$, is called a line of best fit for n given points $z_1, \ldots, z_n$ in $\mathbb{C}$, if $£$ minimizes $\sum_{j=1}^{n} d^2(z_j, l)$ among all lines l in the plane. Here $d(z_j, l)$ denotes the distance(Euclidean) from z_j to l.

In [5] the authors prove the following results.

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THEOREM 2. *Suppose* z_j *are points in* $\mathbb{C}, g = \frac{1}{n}\sum_{j=1}^{n} z_j$ *is the centroid, and* $Z = \sum_{j=1}^{n}(z_j - g)^2 = \sum_{j=1}^{n} z_j^2 - ng^2$.

(a) If $Z = 0$*, then every line through* g *is a line of best fit for the points* $z_1, \ldots, z_n$.

(b) If $Z \neq 0$*, then the line, £, thru* g *that is parallel to the vector from* $(0,0)$ *to* $\sqrt{Z}$ *is the unique line of best fit for* $z_1, \ldots, z_n$.

THEOREM 3. *If* z_1, z_2*, and* z_3 *are the vertices of a nonequilateral triangle,* T*, and £ minimizes* $\sum_{k=1}^{3} d^2(z_k, l)$ *among all lines* l *in the plane, then the major axis of the Steiner ellipse lies on £.*

A proof of Theorem 3 goes all the way back to Coolidge in 1913 (see [5]).

The purpose of this paper is to attempt to generalize (1) and Theorem 3 to ellipses inscribed in convex quadrilaterals. Many of the results in this paper use results from two earlier papers of the author about ellipses inscribed in quadrilaterals. In particular, in [2] we proved the following results.

THEOREM 4. *Let Đ be a convex quadrilateral in the* xy *plane and let* M_1 *and* M_2 *be the midpoints of the diagonals of Đ. Let* Z *be the open line segment connecting* M_1 *and* M_2*. If* $(h,k) \in Z$ *then there is a unique ellipse with center* (h,k) *inscribed in Đ.*

THEOREM 5. *Let Đ be a convex quadrilateral in the* xy *plane. Then there is a unique ellipse of maximal area inscribed in Đ.*

The following general result about ellipses is essentially what appears in [6], except that the cases with $A = B$ were added by the author.

LEMMA 1. *Let* E *be an ellipse with equation* $Ax^2 + By^2 + 2Cxy + Dx + Ey + F = 0$*, and let* ϕ *denote the counterclockwise angle of rotation from the line thru the center parallel to the* x *axis to the major axis of* E*. Then*

$$\phi = \begin{cases} 0 & \text{if } C = 0 \text{ and } A < B \\ \frac{\pi}{2} & \text{if } C = 0 \text{ and } A > B \\ \frac{1}{2}\cot^{-1}\left(\frac{A-B}{2C}\right) & \text{if } C \neq 0 \text{ and } A < B \\ \frac{\pi}{2} + \frac{1}{2}\cot^{-1}\left(\frac{A-B}{2C}\right) & \text{if } C \neq 0 \text{ and } A > B \\ \frac{\pi}{4} & \text{if } C < 0 \text{ and } A = B \\ \frac{3\pi}{4} & \text{if } C > 0 \text{ and } A = B \end{cases}$$

In [3] we also derived the following results about ellipses inscribed in parallelograms.

LEMMA 2. *Let Z be the rectangle with vertices* $(0,0)$, $(l,0)$, $(0,k)$, *and* (l,k), *where* $l,k>0$.

(A) The general equation of an ellipse, Ψ, inscribed in Z is given by

$$k^2x^2+l^2y^2-2l(k-2v)xy-2lkvx-2l^2vy+l^2v^2=0, 0<v<k.$$

(B) The corresponding points of tangency of Ψ are

$$\left(\frac{lv}{k},0\right), (0,v), \left(\frac{l}{k}(k-v),k\right), \text{and } (l,k-v).$$

(C) If a and b denote the lengths of the semi–major and semi–minor axes, respectively, of Ψ, then

$$a^2=\frac{2l^2(k-v)v}{k^2+l^2-\sqrt{(k^2+l^2)^2-16l^2(k-v)v}} \qquad \text{and}$$
$$b^2=\frac{2l^2(k-v)v}{k^2+l^2+\sqrt{(k^2+l^2)^2-16l^2(k-v)v}}.$$

PROPOSITION 6. *Let $Đ$ be the parallelogram with vertices* $O=(0,0)$, $P=(l,0)$, $Q=(d,k)$, *and* $R=(d+l,k)$, *where* $l,k>0, d\geqslant 0$. *The general equation of an ellipse, Ψ, inscribed in $Đ$ is given by*

$$k^3x^2+\left(k(d+l)^2-4dlv\right)y^2-2k(k(d+l)-2lv)xy$$
$$-2k^2lvx+2klv(d-l)y+kl^2v^2=0, 0<v<k.$$

Of course, any two triangles are affine equivalent, while the same is not true of quadrilaterals–thus it is not surprising that not all of the results about ellipses inscribed in triangles extend nicely to quadrilaterals. For example, there is not necessarily an ellipse inscribed in a given quadrilateral, $Đ$, which is tangent to the sides of $Đ$ at their respective midpoints. There is such an ellipse, which we call the midpoint ellipse, when $Đ$ is a parallelogram (see Proposition 8). We are able to prove an inequality (see Theorem 7), similar to (1), which holds for all convex quadrilaterals. If E is any ellipse inscribed in a quadrilateral, $Đ$, then $\dfrac{\text{Area}(E)}{\text{Area}(Đ)}\leqslant\dfrac{\pi}{4}$, and equality holds if and only if $Đ$ is a parallelogram and E is the midpoint ellipse.

Not suprisingly, Theorem 3 also does not extend in general to ellipses inscribed in convex quadrilaterals. However, such a characterization does hold again when $Đ$ is a parallelogram. We prove in Theorem 10 that the foci of the unique ellipse of maximal area inscribed in a parallelogram, $Đ$, lie on the orthogonal least squares line for the vertices of $Đ$. It is also well known that if $p(z)$ is a cubic polynomial with roots at the vertices of a triangle, then the roots of $p'(z)$ are the foci of the Steiner inellipse. A proof of this fact goes all the way back to Siebeck in 1864 (see [5]) and is known in the literature as Marden's Theorem. Now it is easy to show that the orthogonal least squares line, $\pounds$, from Theorem 2 is identical to the line through the roots of $p'(z)$. Thus Marden's Theorem implies Theorem 3 and hence is a stronger statement than Theorem 3. There is an obvious way to try to generalize such a result to convex quadrilaterals, $Đ$. If $p(z)$ is a quartic polynomial with roots at the vertices of $Đ$, must the foci of the unique ellipse of maximal area inscribed in $Đ$ equal the roots of $p''(z)$? We give an example in section 3 that shows that such a stronger statement does not hold for parallelograms, or even for rectangles.

2. An Area Inequality

THEOREM 7. *Let E be any ellipse inscribed in a convex quadrilateral, $Đ$. Then $\frac{Area(E)}{Area(Đ)} \leqslant \frac{\pi}{4}$, and equality holds if and only if $Đ$ is a parallelogram and E is tangent to the sides of $Đ$ at the midpoints.*

Before proving Theorem 7, we need the following lemma.

LEMMA 3. *Suppose that s and t are positive real numbers with $s+t>1$ and $s \neq 1 \neq t$. Let $h_a = \frac{st+t-2s-1+\sqrt{(t-1)^2+s^2(t^2-t+1)-s(t^2-3t+2)}}{6(t-1)}$. Then $h_a \in I =$ the open interval with endpoints $\frac{1}{2}$ and $\frac{1}{2}s$.*

Proof.

$$h_a - \frac{1}{2} = \frac{st-2(s+t-1)+\sqrt{(t-1)^2+s^2(t^2-t+1)-s(t^2-3t+2)}}{6(t-1)}, \qquad (2)$$

and

$$h_a - \frac{1}{2}s = \frac{\sqrt{(t-1)^2+s^2(t^2-t+1)-s(t^2-3t+2)}-(2st-(s+t-1))}{6(t-1)}. \qquad (3)$$

There are four cases to consider: $s,t>1$, $s<1<t$, $t<1<s$, and $s,t<1$. We prove the first two cases, the proof of the other two being similar.

Case 1: $s,t>1$, which implies that $I = \left(\frac{1}{2}, \frac{1}{2}s\right)$

By (2), $h_a - \frac{1}{2} > 0 \iff$

$$\sqrt{(t-1)^2+s^2(t^2-t+1)-s(t^2-3t+2)} > 2(s+t-1)-st,$$

which always holds since

$$(t-1)^2+s^2(t^2-t+1)-s(t^2-3t+2)-(2(s+t-1)-st)^2$$
$$= 3(t-1)(s-1)(s+t-1) > 0.$$

By (3), $h_a - \frac{1}{2}s < 0 \iff$

$$\sqrt{(t-1)^2 + s^2(t^2-t+1) - s(t^2-3t+2)} < 2st - (s+t-1),$$

which always holds since

$$(2st - (s+t-1))^2 - \left((t-1)^2 + s^2(t^2-t+1) - s(t^2-3t+2)\right) \\ = 3st\,(t-1)\,(s-1) > 0.$$

Case 2: $s < 1 < t$, which implies that $I = \left(\frac{1}{2}s, \frac{1}{2}\right)$

By (2), $h_a - \frac{1}{2} < 0 \iff$

$$\sqrt{(t-1)^2 + s^2(t^2-t+1) - s(t^2-3t+2)} < 2\,(s+t-1) - st,$$

which always holds since

$$(2\,(s+t-1) - st)^2 - \left((t-1)^2 + s^2(t^2-t+1) - s(t^2-3t+2)\right) \\ = 3\,(t-1)\,(1-s)\,(s+t-1) > 0.$$

By (3), $h_a - \frac{1}{2}s > 0 \iff$

$$\sqrt{(t-1)^2 + s^2(t^2-t+1) - s(t^2-3t+2)} > 2st - (s+t-1),$$

which always holds since

$$\left((t-1)^2 + s^2(t^2-t+1) - s(t^2-3t+2)\right) - (2st - (s+t-1))^2 \\ = 3st\,(t-1)\,(1-s) > 0.$$

Proof (of Theorem 7). We shall prove Theorem 7 when $\mathcal{D}$ is not a trapezoid, though it certainly holds in that case as well. Since ratios of areas of ellipses and four–sided convex polygons are preserved under one–one affine transformations, we may assume, without loss of generality, that the vertices of $\mathcal{D}$ are $(0,0), (1,0), (0,1)$, and (s,t) for some positive real numbers s and t. Furthermore, since $\mathcal{D}$ is convex and is not a trapezoid, it follows easily that

$$s + t > 1 \text{ and } s \neq 1 \neq t. \tag{4}$$

It also follows easily that Area($\mathcal{D}$) $= \frac{1}{2}(s+t)$. Let E denote any ellipse inscribed in $\mathcal{D}$ and $A_E = \text{Area}(E)$. The midpoints of the diagonals of $\mathcal{D}$ are $M_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $M_2 = \left(\frac{1}{2}s, \frac{1}{2}t\right)$. Let I denote the open interval with endpoints $\frac{1}{2}$ and $\frac{1}{2}s$. By Theorem

4 the locus of centers of ellipses inscribed in $Ð$ is precisely the set $\{(h, L(h), h \in I\}$, where

$$L(x) = \frac{1}{2}\frac{s-t+2x(t-1)}{s-1} \tag{5}$$

is the line thru M_1 and M_2. In proving Theorem 5 ([2], Theorem 3.3), we also showed that

$$A_E^2 = \frac{\pi^2}{4(s-1)^2}A(h), A(h) = (2h-1)(s-2h)(s+2h(t-1)),\ h \in I. \tag{6}$$

Setting $A'(h) = 0$ yields

$$h = h_a = \frac{st+t-2s-1 \pm \sqrt{(t-1)^2+s^2(t^2-t+1)-s(t^2-3t+2)}}{6(t-1)}.$$

By Lemma 3, the positive square root always yields $h_a \in I$. Thus

$$h_a = \frac{st+t-2s-1+\sqrt{(t-1)^2+s^2(t^2-t+1)-s(t^2-3t+2)}}{6(t-1)}$$

is the unique point in I such that $A(h) \leqslant A(h_a), h \in I$. Let

$$b(s,t) = (st-(s+t-1))^2 + st(s+t-1).$$

A simple computation using (6) shows that

$$A(h_a) = \frac{1}{27(t-1)^2}\left(2ts-s-t+1-\sqrt{b(s,t)}\right) \times \left(ts-2t-2s+2+\sqrt{b(s,t)}\right)\left(s+ts+t-1+\sqrt{b(s,t)}\right).$$

Thus by (6) and the fact that Area($Ð$) $= \frac{1}{2}(s+t)$, we have

$$\frac{A_E^2}{(\text{Area}(Ð))^2} = \frac{\pi^2}{27}\left(2ts-s-t+1-\sqrt{b(s,t)}\right) \times \frac{\left(ts-2t-2s+2+\sqrt{b(s,t)}\right)\left(s+ts+t-1+\sqrt{b(s,t)}\right)}{(s-1)^2(t-1)^2(s+t)^2}. \tag{7}$$

Clearly $\frac{A_E^2}{(\text{Area}(Ð))^2}$ is **symmetric** in s and t. Thus we may assume, without loss of generality, that

$$s \leqslant t. \tag{8}$$

It is convenient to make the change of variable

$$u = s+t-1, v = st. \tag{9}$$

Solving (9) for s and t yields

$$s = u+1-\frac{1}{2}\left(u+1\pm\sqrt{(u+1)^2-4v}\right), \quad t=\frac{1}{2}\left(u+1\pm\sqrt{(u+1)^2-4v}\right).$$

By (8),

$$s=\frac{1}{2}\left(u+1-\sqrt{(u+1)^2-4v}\right), \quad t=\frac{1}{2}\left(u+1+\sqrt{(u+1)^2-4v}\right). \tag{10}$$

Substituting for s and t using (10) gives $b(s,t)=c(u,v)$, where $c(u,v)=u^2-uv+v^2$. Then

$$\frac{A_E^2}{(\text{Area}(\mathcal{D}))^2}=\frac{\pi^2}{27}\frac{(2v-u-\sqrt{c(u,v)})(v-2u+\sqrt{c(u,v)})(v+u+\sqrt{c(u,v)})}{(v-u)^2(u+1)^2},$$

which simplifies to

$$\frac{A_E^2}{(\text{Area}(\mathcal{D}))^2}=\frac{\pi^2}{27}d(u,v), \tag{11}$$

where

$$d(u,v)=\frac{(v-2u)(2v-u)(u+v)+2(c(u,v))^{3/2}}{(u+1)^2(v-u)^2}. \tag{12}$$

We consider two cases: $s>1$ and $0<s<1$

Case 1: $s>1$

Let $w=\frac{u}{v}$. Then by (12) and some simplification, we have

$$d(u,v)=\frac{v}{(u+1)^2}z(w), \tag{13}$$

where

$$z(w)=\frac{(1-2w)(2-w)(1+w)+2\left(w^2-w+1\right)^{3/2}}{(1-w)^2}.$$

Note that by (4), $u>0$, and $(u+1)^2-4v=(s+t)^2-4st=(s-t)^2\geqslant 0$, which implies

$$\frac{v}{(u+1)^2}\leqslant\frac{1}{4}. \tag{14}$$

Also, $1<s \iff 2<u+1-\sqrt{(u+1)^2-4v} \iff \sqrt{(u+1)^2-4v}<u-1 \iff (u+1)^2-4v<(u-1)^2 \iff u<v$, which implies that $0<w<1$.

$$z'(w)=\frac{2w^3-6w^2+9w-1+\left(2w^2-5w-1\right)\sqrt{w^2-w+1}}{(w-1)^3}=0$$

$$\Rightarrow \left(2w^3-6w^2+9w-1\right)^2-\left(2w^2-5w-1\right)^2\left(w^2-w+1\right)=0 \Rightarrow 27w(w-1)^3=0,$$

which has no solution in $(0,1)$. $z(0)=4$ and $\lim_{w\to 1} z(w)=\frac{27}{4}$. Thus

$$z(w)<\frac{27}{4},\ 0<w<1, \tag{15}$$

which implies, by (14), that $\frac{v}{(u+1)^2}z(w)\leqslant \frac{1}{4}\frac{27}{4}=\frac{27}{16}$. $\frac{\text{Area}\,(E)}{\text{Area}(Ð)}<\frac{\pi}{4}$ then follows immediately by (11) and (13).

Case 2: $0<s<1$

Now $s<1$ implies that $v<u$ and $v>0$ since $s,t>0$. Now we let $w=\frac{v}{u}$, which again implies that $0<w<1$. Then by (11) and some simplification, we have

$$d(u,v)=\frac{u}{(u+1)^2}z(w). \tag{16}$$

Since $(u-1)^2\geqslant 0 \Rightarrow (u+1)^2-4u\geqslant 0$, we have

$$\frac{u}{(u+1)^2}\leqslant\frac{1}{4}. \tag{17}$$

By (15) and (17), $\frac{u}{(u+1)^2}z(w)\leqslant\frac{27}{16}$, and $\frac{\text{Area}\,(E)}{\text{Area}(Ð)}<\frac{\pi}{4}$ then follows immediately by (11) and (16). We have proven that $\frac{\text{Area}\,(E)}{\text{Area}(Ð)}<\frac{\pi}{4}$ when $Ð$ is not a trapezoid. Using a limiting argument, one can then show immediately that $\frac{\text{Area}\,(E)}{\text{Area}(Ð)}\leqslant\frac{\pi}{4}$ when $Ð$ is a trapezoid. However, that still does not give the strict inequality when $Ð$ is not a parallelogram. We shall omit the details here, but the author will provide them upon request. To finish the proof of Theorem 7, we need to consider the case when $Ð$ is a parallelogram. That case will follow from Proposition 8 and Theorem 9 below.

First we prove that, just like any triangle in the plane, there is an ellipse tangent to a parallelogram at the midpoints of its sides.

PROPOSITION 8. *The ellipse of maximal area inscribed in a parallelogram, Ğ, is tangent to Ğ at the midpoints of the four sides.*

Proof. Proposition 8 was proven in ([3], Theorem 3) when Ğ is a rectangle. The reader can prove it directly themselves using Lemma 2. Ratios of areas of ellipses, points of tangency, and midpoints of line segments are preserved under one–one affine transformations. Since any given parallelogram is affinely equivalent to a rectangle, that proves Proposition 8 for parallelograms in general.

THEOREM 9. *Let E be any ellipse inscribed in a parallelogram, Ğ. Then* $\frac{Area\,(E)}{Area(\breve{G})}\leqslant\frac{\pi}{4}$, *with equality if and only if E is the midpoint ellipse.*

Proof. As noted earlier, ratios of areas of ellipses and four–sided convex polygons, are preserved under one–one affine transformations. In addition, points of tangency and midpoints of line segments are also preserved under one–one affine transformations. Thus we may assume that Ğ is a rectangle(or even a square, though we don't need that much simplification here), Z, with vertices $(0,0),(l,0),(0,k),$and (l,k), where $l,k>0$. Let E be any ellipse inscribed in Z, and let a and b denote the lengths of the semi–major and semi–minor axes, respectively, of E. By Lemma 2(C),

$$\begin{aligned} a^2b^2 &= \frac{2l^2(k-v)v}{k^2+l^2-\sqrt{(k^2+l^2)^2-16l^2(k-v)v}} \\ &\quad\times\frac{2l^2(k-v)v}{k^2+l^2+\sqrt{(k^2+l^2)^2-16l^2(k-v)v}} \\ &= \frac{4l^4(k-v)^2v^2}{(k^2+l^2)^2-((k^2+l^2)^2-16l^2(k-v)v)} \\ &= \frac{4l^4(k-v)^2v^2}{16l^2(k-v)v}=\frac{1}{4}l^2(k-v)v. \end{aligned}$$

Hence $\text{Area}(E)=\pi a^2b^2=\frac{\pi}{2}l\sqrt{v(k-v)}$. Since the function $v(k-v)$ attains its unique maximum on $(0,v)$ when $v=\frac{k}{2}$, the unique ellipse of maximal area, E_A, satisfies $\text{Area}(E_A)=\frac{\pi}{2}l\sqrt{\frac{k}{2}\left(k-\frac{k}{2}\right)}=\frac{\pi}{4}lk$, and thus $\frac{\text{Area}(E_A)}{\text{Area}(Z)}=\frac{\pi}{4}$. Furthermore, $\frac{\text{Area}(E)}{\text{Area}(Z)}$ $<\frac{\pi}{4}$ when $E\neq E_A$. By Lemma 2(B), letting $v=\frac{k}{2}$ gives the points of tangency $\left(\frac{l}{2},0\right)$, $\left(0,\frac{k}{2}\right)$, $\left(\frac{l}{2},k\right)$, and $\left(l,\frac{k}{2}\right)$, which are the midpoints of Z. That completes the proof of Theorem 9 as well as the equality part of Theorem 7.

3. Orthogonal least squares

If l is a line in the plane, we let $d(z_k,l)$ denote the Euclidean distance from z_k to l.

THEOREM 10. *Let Ğ be the parallelogram with vertices* z_1,z_2,z_3,z_4

(A) If Ğ is not a square, then there is a unique line, £, which minimizes $\sum_{k=1}^{4} d^2(z_k,l)$ *among all lines,* l. *Furthermore, the foci of the midpoint ellipse for Ğ lie on £.*

(B) If Ğ is a square, then the midpoint ellipse is a circle, and every line through the center of Ğ is a line of best fit for the vertices of Ğ.

REMARK 1. (1) For part (A), an equivalent statement is that the foci of the midpoint ellipse for Ğ lie on the line through the second derivative of the polynomial with roots at the vertices of Đ.

(2) Note that £ is **not** the standard regression line in general.

Proof. The line through the foci of an ellipse is **not** preserved in general under nonsingular affine transformations of the plane, but it is preserved under translations and rotations. Thus we can assume that the vertices of Ğ are $O=(0,0), P=(l,0), Q=(d,k)$, and $R=(l+d,k)$, where $l,k>0, d\geqslant 0$. Using complex notation for the vertices, $z_1=0$, $z_2=l$, $z_3=d+ki$, and $z_4=l+d+ki$, the centroid, g, of Ğ is given by $g=\frac{1}{4}\sum_{k=1}^{4} z_k=\frac{1}{2}(l+d+ki)$, and $Z=\sum_{j=1}^{4}(z_j-g)^2=\sum_{j=1}^{4}\left(z_j-\frac{1}{2}(l+d+ki)\right)^2$, which simplifies to

$$Z=d^2+l^2-k^2+2idk. \tag{18}$$

It is well known, and easy to show, that the center of any ellipse inscribed in Ğ equals the point of intersection of the diagonals of Ğ, which is $\left(\frac{1}{2}(d+l),\frac{1}{2}k\right)$. Also, as noted in the proof of Theorem 9, $v=\frac{k}{2}$ yields the unique ellipse of maximal area, E_A, inscribed in Ğ. Let the major axis line of E_A refer to the line through the foci of E_A. Thus the major axis line of E_A passes through g. We take care of some special cases first.

• $d=0$: Then Ğ is a rectangle. If $l=k$, then Ğ is a square and letting $v=\frac{k}{2}$ shows that E_A is a circle. Also, $Z=0$. Thus every line through the center of Ğ is a line of best fit for the vertices of Ğ. We may assume now that $l\neq k$, which implies that $Z\neq 0$. Since $g=\frac{1}{2}(l+ki)$, the major axis line of E_A passes through g, and $Z=l^2-k^2$. If $l>k$, then $\sqrt{Z}=\sqrt{l^2-k^2}$ is real, and thus the line, £, thru g parallel to the vector from $(0,0)$ to $\sqrt{Z}$ has slope 0. By Lemma 2(A), the major axis line of any ellipse inscribed in Ğ is parallel to the x axis. Since the major axis line passes through g, it must be identical to £. If $l<k$, then $\sqrt{Z}=\sqrt{l^2-k^2}$ is imaginary, and thus £ is vertical. By Lemma 2(A) again, the major axis line of any ellipse inscribed in Ğ is parallel to the y axis. Again, the major axis line must be identical to £.

• Assume now that $d\neq 0$, which implies that $Z\neq 0$ since $dk\neq 0$.

Letting $v=\frac{k}{2}$ in Proposition 6 gives the coefficients for the equation of E_A. In particular,

$$A=k^3,\quad B=k\left(d^2+l^2\right),\quad C=-k^2d. \tag{19}$$

Note that $C<0$ for all d,k, and l, and $A<B \iff k^3<k\left(d^2+l^2\right) \iff d^2+l^2-k^2>0$ since $k>0$. Let ϕ denote the counterclockwise angle of rotation from the line thru the center parallel to the x axis to the major axis of E_A. We use the formula

$$\frac{\operatorname{Im}\sqrt{Z}}{\operatorname{Re}\sqrt{Z}}=\frac{|Z|-\operatorname{Re}Z}{\operatorname{Im}Z},\quad \operatorname{Im}Z\neq 0. \tag{20}$$

If $k^2-d^2-l^2=0$, then by (18) and (20), $Z=2idk \Rightarrow \dfrac{\operatorname{Im}\sqrt{Z}}{\operatorname{Re}\sqrt{Z}}=\dfrac{2dk}{2dk}=1$. By (19), $A=B$ and $C<0$, which implies, by Lemma 1, that $\phi=\dfrac{\pi}{4}$. Thus the major axis line has slope equal to $\tan\phi=\tan\dfrac{\pi}{4}=1$, which is the same slope as that of $\pounds$. Since both the major axis line and $\pounds$ pass through g, they must be identical. Assume now that $k^2-d^2-l^2\neq 0$, which implies that $A\neq B$.

$|Z|^2=\left(d^2+l^2-k^2\right)^2+(2dk)^2=\left((k+l)^2+d^2\right)\left((k-l)^2+d^2\right)$, which implies that

$$|Z|=\sqrt{((k+l)^2+d^2)\,((k-l)^2+d^2)}.$$

Hence

$$\frac{|Z|-\operatorname{Re}Z}{\operatorname{Im}Z}=\frac{\sqrt{((k+l)^2+d^2)\,((k-l)^2+d^2)}-\left(d^2+l^2-k^2\right)}{2dk}. \tag{21}$$

By Lemma 1, $\cot 2\phi=\dfrac{A-B}{2C}$ or $\cot(2\phi-\pi)=\dfrac{A-B}{2C}$, which implies that $\tan 2\phi=\dfrac{2C}{A-B}$ or $\tan(2\phi-\pi)=\dfrac{2C}{A-B}$. Since $\tan(x-\pi)=\tan x$, in either case,

$$\frac{2\tan\phi}{1-\tan^2\phi}=\frac{2C}{A-B},$$

which implies that

$$\frac{\tan\phi}{1-\tan^2\phi}=\frac{C}{A-B}=\frac{-k^2d}{k^3-k\left(d^2+l^2\right)}=\frac{dk}{d^2+l^2-k^2}$$

by (19). The equation $\dfrac{\tan\phi}{1-\tan^2\phi}=r$ has solution

$$\tan\phi=\frac{1}{2r}\left(-1\pm\sqrt{1+4r^2}\right). \tag{22}$$

Letting $r=\dfrac{dk}{d^2+l^2-k^2}$ yields $1+4r^2=\dfrac{\left(d^2+(k-l)^2\right)\left(d^2+(k+l)^2\right)}{\left(k^2-d^2-l^2\right)^2}$. By (22),

$$\tan\phi=\frac{k^2-d^2-l^2}{2dk}\left(1\pm\frac{\sqrt{((k-l)^2+d^2)\,((k+l)^2+d^2)}}{k^2-d^2-l^2}\right).$$

There are two cases to consider.

Case 1: $k^2-d^2-l^2>0$

Let $S=\{(d,k,l:d>0,k>0,l>0\}$, which is a connected set. $k^2-d^2-l^2=0\Rightarrow\phi=\dfrac{\pi}{4}\Rightarrow\tan\phi>0$. Also, $\tan\phi=0\iff r=0$. But $r\neq 0$ since $dk\neq 0$. Thus $\tan\phi>0$ on S. Also,

$$4k^2d^2>0$$

implies that

$$\left((k-l)^2+d^2\right)\left((k+l)^2+d^2\right)-\left(k^2-d^2-l^2\right)^2>0,$$

which implies that

$$\sqrt{((k-l)^2+d^2)\,((k+l)^2+d^2)}>k^2-d^2-l^2.$$

Hence

$$\tan\phi=\frac{k^2-d^2-l^2}{2dk}\left(1+\frac{\sqrt{((k-l)^2+d^2)\,((k+l)^2+d^2)}}{k^2-d^2-l^2}\right) \tag{23}$$

since $\dfrac{k^2-d^2-l^2}{2dk}\left(1-\dfrac{\sqrt{((k-l)^2+d^2)\,((k+l)^2+d^2)}}{k^2-d^2-l^2}\right)<0.$

Case 2: $k^2-d^2-l^2<0$

Then $\dfrac{k^2-d^2-l^2}{2dk}\left(1-\dfrac{\sqrt{((k-l)^2+d^2)\,((k+l)^2+d^2)}}{k^2-d^2-l^2}\right)<0$, so again (23) holds.

By (21),

$$\frac{|Z|-\operatorname{Re}Z}{\operatorname{Im}Z}=\frac{k^2-d^2-l^2}{2dk}\times\left(1+\frac{\sqrt{((k-l)^2+d^2)\,((k+l)^2+d^2)}}{k^2-d^2-l^2}\right).$$

Thus the major axis line and $\pounds$ each have slope given by the right hand side of (23). Since both the major axis line and $\pounds$ pass through g, they must be identical. That completes the proof of (A). (B) follows immediately from the fact that the ellipse of maximal area inscribed in a square is a circle.

We show in the following example that a version of Marden's Theorem does not hold for ellipses inscribed in parallelograms.

EXAMPLE 11. Let $Đ$ be the rectangle with vertices $z_1=0$, $z_2=1$, $z_3=1+2i$, $z_4=2i$. Letting $l=1$, $k=2$, and $v=\dfrac{k}{2}=1$ in Lemma 2 yields the equation $4x^2+y^2-4x-2y+1=0$ for the maximal area ellipse, E_A, inscribed in $Đ$. Rewriting the equation as $4\left(x-\dfrac{1}{2}\right)^2+(y-1)^2=1$ shows that the foci of E_A are $\left(\dfrac{1}{2},1+\dfrac{\sqrt{3}}{2}\right)$ and $\left(\dfrac{1}{2},1-\dfrac{\sqrt{3}}{2}\right)$. If

$$Q(z)=z(z-1)(z-(1+2i))(z-2i)),$$

then the roots of $Q''(z)$ are $\dfrac{1}{2}+\dfrac{1}{2}i$ and $\dfrac{1}{2}+\dfrac{3}{2}i$, which would give the points $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$ and $\left(\dfrac{1}{2},\dfrac{3}{2}\right)$. Hence the roots of $Q''(z)$ do **not** give the foci of the maximal area ellipse.

REMARK 2. Some heuristic reasoning shows that a version of Marden's Theorem does not hold for ellipses inscribed in convex quadrilaterals. For if it did, then when the quadrilateral is tangential(contains an inscribed circle), Q'' would have to have a double root. But a square is tangential, while Q'' has distinct roots.

REMARK 3. For an ellipse, E, circumscribed about a convex quadrilateral, Ɖ (that is, passing through the vertices of Ɖ), one can prove a result similar to Theorem 7: $\frac{A_E}{\text{Area}(Ɖ)} \geqslant \frac{\pi}{2}$. We shall have more to say about this in a future paper, but we sketch the idea of the proof here. We thank Grant Keady of the University of Western Australia for the general approach. There is an affine map, A, which maps E onto a circle, C. Let $\tilde{Ɖ}$ denote $A(Ɖ)$, so that $\tilde{Ɖ}$ is a cyclic quadrilateral. Since $\frac{A_E}{\text{Area}(Ɖ)} = \frac{A_C}{\text{Area}(\tilde{Ɖ})}$, it suffices to prove that $\frac{A_C}{\text{Area}(\tilde{Ɖ})} \geqslant \frac{\pi}{2}$. That in turn follows from the following results.

(1) A square is the quadrilateral of maximal area which can be inscribed in a given circle.

and

(2) Let R be the rectangle of maximal area which can be inscribed in a given semi–circle of radius r. Then the area of R equals r^2.

We also note that this same approach might lead to a shorter proof of Theorem 7. Use again an affine map, A, which maps E onto a circle, C. This time $\tilde{Ɖ} = A(Ɖ)$ is a tangential quadrilateral, so that one can make use of known results about the areas of circles inscribed in tangential quadrilaterals. We leave it to the reader to fill in the details.

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