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SOME NEW REFINED HARDY–TYPE INEQUALITIES WITH KERNELS

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Abstract. By using the notion of the subdifferential of a convex function, we state and prove a new general refined weighted Hardy-type inequality for convex functions and the integral operator with a non-negative kernel. We point out that the obtained result generalizes and refines the classical one-dimensional Hardy's, Pólya–Knopp's, and Hardy–Hilbert's inequalities, as well as related dual inequalities. We show that our results may be seen as generalizations of some recent results related to Riemann-Liouville's and Weyl's operator, as well as a generalization and a refinement of the so-called Godunova's inequality.

1. Introduction

First, we recall some well-known integral inequalities. If $p>1$, then Hardy's integral inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx \tag{1.1}$$

holds for all non-negative functions $f\in L^p(\mathbb{R}_+)$, where $\mathbb{R}_+=(0,\infty)$. Another important inequality, closely related to (1.1), is Hardy–Hilbert's inequality,

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\,dx\right)^p dy \leqslant \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p \int_0^\infty f^p(t)\,dt, \tag{1.2}$$

which holds for $p>1$ and non-negative functions $f\in L^p(\mathbb{R}_+)$. Moreover, we mention Pólya–Knopp's inequality,

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)\,dt\right) dx < e\int_0^\infty f(x)\,dx, \tag{1.3}$$

for positive functions $f\in L^1(\mathbb{R}_+)$. Since (1.3) can be obtained from (1.1) by rewriting it with the function f replaced with $f^{\frac{1}{p}}$ and then by letting $p\to\infty$, Pólya–Knopp's

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inequality may be considered as a limiting case of Hardy's inequality. Observe that the constants $\left(\frac{p}{p-1}\right)^p$, $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p$, and e, respectively appearing on the right-hand sides of (1.1), (1.2), and (1.3), are the best possible, that is, none of them can be replaced with any smaller constant.

Since being discovered, the above inequalities have been discussed by several authors, who either gave their alternative proofs using different techniques, or applied, refined and generalized them in various ways. Further information and remarks concerning the rich history, development, generalizations, and applications of inequalities (1.1) – (1.3) can be found e.g. in the monographs [12, 20, 21, 24, 26], expository papers [6, 14, 19], and the references cited therein. Besides, we also emphasize the papers [3–5, 7–9, 11, 15, 16, 18, 25], all of which to some extent have guided us in the research presented in this paper.

In particular, recently, it was pointed out by S. Kaijser et al. in [15] that both (1.1) and (1.3) are just special cases of a much more general Hardy-Knopp's type inequality,

$$\int_0^\infty \Phi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{dx}{x} \leqslant \int_0^\infty \Phi(f(x))\,\frac{dx}{x}, \tag{1.4}$$

where Φ is a convex function on $\mathbb{R}_+$ and $f:\mathbb{R}_+\to\mathbb{R}$ is a locally integrable positive function. Note that (1.4) follows by a standard application of Jensen's inequality and Fubini's theorem.

On the other hand, E. K. Godunova, [11] (see also [26, Chapter VIII, p. 233]), proved that the inequality

$$\int_{\mathbb{R}_+^n} \Phi\left(\frac{1}{x_1\cdots x_n}\int_{\mathbb{R}_+^n} l\left(\frac{y_1}{x_1},\dots,\frac{y_n}{x_n}\right) f(y_1,\dots,y_n)\,d\mathbf{y}\right)\frac{d\mathbf{x}}{x_1\cdots x_n} \leqslant \int_{\mathbb{R}_+^n} \frac{\Phi(f(\mathbf{x}))}{x_1\cdots x_n}\,d\mathbf{x} \tag{1.5}$$

holds for a non-negative function $l:\mathbb{R}_+^n\to\mathbb{R}_+$, such that $\int_{\mathbb{R}_+^n} l(\mathbf{x})\,d\mathbf{x}=1$, a convex function $\Phi:[0,\infty)\to[0,\infty)$, and a non-negative function f on $\mathbb{R}_+^n$, such that the function $x\mapsto \frac{\Phi(f(\mathbf{x}))}{x_1\cdots x_n}$ is integrable on $\mathbb{R}_+^n$.

Finally, K. Krulić et al. [18] unified the above results by studying the measure spaces $(\Omega_1,\Sigma_1,\mu_1)$, $(\Omega_2,\Sigma_2,\mu_2)$, and the general integral operator A_k defined by

$$A_k f(x) = \frac{1}{K(x)}\int_{\Omega_2} k(x,y) f(y)\,d\mu_2(y),\ x\in\Omega_1, \tag{1.6}$$

where $f:\Omega_2\longrightarrow\mathbb{R}$ is a measurable function, $k:\Omega_1\times\Omega_2\to\mathbb{R}$ is measurable and non-negative, and

$$K(x) = \int_{\Omega_2} k(x,y)\,d\mu_2(y) > 0,\ x\in\Omega_1. \tag{1.7}$$

Just by using Jensen's inequality and Fubini's theorem, they elegantly proved the weighted inequality

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))\, d\mu_1(x) \leqslant \int_{\Omega_2} v(y)\Phi(f(y))\, d\mu_2(y), \tag{1.8}$$

where $u : \Omega_1 \to \mathbb{R}$ is a non-negative measurable function, $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, v is defined on Ω_2 by

$$v(y) = \int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}\, d\mu_1(x), \tag{1.9}$$

Φ is a convex function on an interval $I \subseteq \mathbb{R}$, and $f : \Omega_2 \to \mathbb{R}$ is such that $f(y) \in I$, for all $y \in \Omega_2$. Some important and useful modular inequalities related to (1.8) can be found e.g. in [13, 17, 22].

Although the papers [15] and [18] are fairly new, the role of Jensen's inequality in such type of results was known much earlier, e.g. already in [23]. However, the idea to use the notion of the subdifferential of the convex function Φ, instead of Jensen's inequality, is new and it enables us to get a refinement of (1.8). Therefore, in this paper we consider the operator (1.6) and prove a refined general weighted Hardy–type inequality with a kernel, related to an arbitrary convex function. Further, we point out that our result unifies, generalizes and refines inequalities (1.1), (1.2), (1.3), (1.4), (1.5), and (1.8), as well as some further results from the literature. Finally, applying it to some important particular measure spaces, kernels, weights, and convex functions, we derive a series of new refined Hardy–type inequalities.

The paper is organized in the following way. After this Introduction, in Section 2 we introduce some necessary notation and state and prove our main result in this paper: a general refined weighted Hardy–type inequality with a non-negative kernel and an arbitrary convex (or concave) function. Especially, we show that our relation can be regarded as a refinement of (1.8). In the same section, we discuss some particular cases of the obtained general inequality, related to power and exponential functions, and to the simplest possible kernel – the one with separate variables. Further, in the next two sections, our general results are applied to various one-dimensional settings and the Lebesgue measure. Namely, in Section 3 we derive refined Hardy and Pólya–Knopp–type inequalities, as well as their dual inequalities, while Section 4 is dedicated to new refined Hardy–Hilbert–type inequalities. Among other interesting new results, in Section 3 we obtain new refined inequalities related to Riemann–Liouville's and Weyl's operator. On the other hand, in Section 4 we get new refined Hardy–Hilbert's and Hardy–Littlewood–Pólya's inequality. The paper is concluded with Section 5, where a particular multidimensional setting is analyzed. As a special case, a new refined Godunova–type inequality is obtained.

Conventions. Throughout this paper, all measures are assumed to be positive and σ-finite, and all functions to be measurable, while expressions of the form $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, and $\frac{a}{\infty}$, where $a \in \mathbb{R}$, are taken to be equal to zero. By $|\Omega|_\mu$ we denote the measure of a measurable set Ω with respect to the measure μ. In particular, we use the symbol $|\ |_1$ as an abbreviation for $\|\ \|_{L^1(\Omega,\mu)}$. In addition, by a weight function (shortly: a weight) we mean a non-negative measurable function on the actual set, while an interval in $\mathbb{R}$ stands for any convex subset of $\mathbb{R}$. Moreover, $\operatorname{Int} I$ denotes the interior of an interval $I \subseteq \mathbb{R}$. Finally, inequalities like (2.1) are interpreted to mean that if the left-hand side is finite, so is the right-hand side and the inequality holds.

2. New general refined Hardy-type inequalities with kernels

For readers' convenience, we introduce some necessary notation and recall some basic facts about convex functions. Let I be an interval in $\mathbb{R}$ and $\Phi : I \longrightarrow \mathbb{R}$ be a convex function. For $x \in I$, by $\partial\Phi(x)$ we denote the subdifferential of Φ at x, that is, the set $\partial\Phi(x) = \{\alpha \in \mathbb{R} : \Phi(y) - \Phi(x) - \alpha(y-x) \geqslant 0, y \in I\}$. It is well-known that $\partial\Phi(x) \neq \emptyset$ for all $x \in \operatorname{Int} I$. More precisely, at each point $x \in \operatorname{Int} I$ we have $-\infty < \Phi'_-(x) \leqslant \Phi'_+(x) < \infty$ and $\partial\Phi(x) = [\Phi'_-(x), \Phi'_+(x)]$, while the set on which Φ is not differentiable is at most countable. Moreover, each function $\varphi : I \longrightarrow \mathbb{R}$ such that $\varphi(x) \in \partial\Phi(x)$, whenever $x \in \operatorname{Int} I$, is increasing on $\operatorname{Int} I$.

Now, we are ready to state and prove the central result of this paper, that is, a new refined general weighted Hardy-type inequality with a non-negative kernel and related to an arbitrary convex function. It is given in the following theorem.

THEOREM 2.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, u be a weight function on Ω_1, k a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (1.7). Suppose that $K(x) > 0$ for all $x \in \Omega_1$, that the function $x \mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, and that v is defined on Ω_1 by (1.9). If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality*

$$\begin{aligned}\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y) &- \int_{\Omega_1} u(x)\Phi(A_k f(x))\,d\mu_1(x)\\ &\geqslant \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y)\,\big|\,|\Phi(f(y)) - \Phi(A_k f(x))|\\ &\qquad - |\varphi(A_k f(x))| \cdot |f(y) - A_k f(x)|\,\big|\,d\mu_2(y)\,d\mu_1(x) \qquad (2.1)\end{aligned}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $f(y) \in I$ for all $y \in \Omega_2$, where $A_k f$ is defined on Ω_1 by (1.6).

Proof. First, note that for an arbitrary $x \in \Omega_1$ and the function $h_x : \Omega_2 \to \mathbb{R}$ defined by $h_x(y) = f(y) - A_k f(x)$, we have

$$\begin{aligned}\int_{\Omega_2} k(x,y)h_x(y)\,d\mu_2(y) &= \int_{\Omega_2} k(x,y)f(y)\,d\mu_2(y) - \int_{\Omega_2} k(x,y)A_k f(x)\,d\mu_2(y)\\ &= K(x)A_k f(x) - A_k f(x)\int_{\Omega_2} k(x,y)\,d\mu_2(y) = 0. \qquad (2.2)\end{aligned}$$

Our next step is to show that $A_k f(x) \in I$, for all $x \in \Omega_1$. Otherwise, let $x_0 \in \Omega_1$ be such that $A_k f(x_0) \notin I$. In this setting, since I is an interval in $\mathbb{R}$ and $f(\Omega_2) \subseteq I$, the function h_{x_0} is either strictly positive or strictly negative on Ω_2 and the product $k(x_0,y)h_{x_0}(y)$ has a constant sign on Ω_2. Moreover, by assumptions of Theorem 2.1 we have $K(x_0) > 0$, so there exists a set $\tilde{\Omega}_2 \in \Sigma_2$ such that $|\tilde{\Omega}_2|_{\mu_2} > 0$ and $k(x_0,y) > 0$ for all $y \in \tilde{\Omega}_2$. Therefore, $\int_{\Omega_2} k(x_0,y)h_{x_0}(y)\,d\mu_2(y) \neq 0$, which contradicts (2.2). Thus $A_k f(x) \in I$, $x \in \Omega_1$. In particular, if $A_k f(x)$ is an endpoint of I for some $x \in \Omega_1$ (in cases when I is not an open interval), then h_x (or $-h_x$) will be a non-negative function whose integral over Ω_2, with respect to the measure μ_2, is equal to 0. Hence, $h_x \equiv 0$, that is, $f(y) = A_k f(x)$ holds for μ_2–a.e. $y \in \Omega_2$.

To prove inequality (2.1), observe that for all $r \in \operatorname{Int} I$, $s \in I$, and any function $\varphi : I \to \mathbb{R}$ such that $\varphi(x) \in \partial\Phi(x)$ for $x \in \operatorname{Int} I$, we have

$$\Phi(s) - \Phi(r) - \varphi(r)(s-r) \geqslant 0.$$

Therefrom,

$$\begin{aligned} \Phi(s) - \Phi(r) - \varphi(r)(s-r) &= |\Phi(s) - \Phi(r) - \varphi(r)(s-r)| \\ &\geqslant \big|\, |\Phi(s) - \Phi(r)| - |\varphi(r)| \cdot |s-r| \,\big|. \end{aligned} \tag{2.3}$$

Substituting $r = A_k f(x)$ and $s = f(y)$ in (2.3), for $A_k f(x) \in \operatorname{Int} I$ we get

$$\begin{aligned} &\Phi(f(y)) - \Phi(A_k f(x)) - \varphi(A_k f(x))\,(f(y) - A_k f(x)) \\ &\qquad \geqslant \big|\, |\Phi(f(y)) - \Phi(A_k f(x))| - |\varphi(A_k f(x))| \cdot |f(y) - A_k f(x)| \,\big|. \end{aligned} \tag{2.4}$$

The above analysis provides (2.4) to hold even if $A_k f(x)$ is an endpoint of I. In that case, both sides of inequality (2.4) are equal to 0 for μ_2–a.e. $y \in \Omega_2$.

Multiplying (2.4) by $u(x)\frac{k(x,y)}{K(x)} \geqslant 0$ for a fixed $x \in \Omega_1$, and then integrating it over Ω_2 and Ω_1 respectively, we obtain

$$\begin{aligned} &\int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\Phi(f(y))\,d\mu_2(y)\,d\mu_1(x) \\ &\quad - \int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\Phi(A_k f(x))\,d\mu_2(y)\,d\mu_1(x) \\ &\quad - \int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\varphi(A_k f(x))\,(f(y) - A_k f(x))\,d\mu_2(y)\,d\mu_1(x) \\ &\geqslant \int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\,\big|\, |\Phi(f(y)) - \Phi(A_k f(x))| \\ &\quad - |\varphi(A_k f(x))| \cdot |f(y) - A_k f(x)| \,\big|\, d\mu_2(y)\,d\mu_1(x). \end{aligned} \tag{2.5}$$

By using Fubini's theorem and the definition (1.9) of the weight v, the first integral on the left-hand side od (2.5) becomes

$$\int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\Phi(f(y))\,d\mu_2(y)\,d\mu_1(x) = \int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y), \tag{2.6}$$

while for the second integral on the left-hand side of (2.5) we have

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\Phi(A_kf(x))\,d\mu_2(y)\,d\mu_1(x)\\
&=\int_{\Omega_1} u(x)\Phi(A_kf(x))\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\,d\mu_2(y)\right)d\mu_1(x)\\
&=\int_{\Omega_1} u(x)\Phi(A_kf(x))\,d\mu_1(x). \qquad (2.7)
\end{aligned}$$

Finally, applying (2.2) we similarly get

$$\begin{aligned}
&\int_{\Omega_1}\int_{\Omega_2} u(x)\frac{k(x,y)}{K(x)}\varphi(A_kf(x))\,(f(y)-A_kf(x))\,d\mu_2(y)\,d\mu_1(x)\\
&=\int_{\Omega_1}\frac{u(x)}{K(x)}\varphi(A_kf(x))\left(\int_{\Omega_2} k(x,y)h_x(y)\,d\mu_2(y)\right)d\mu_1(x)=0, \qquad (2.8)
\end{aligned}$$

so (2.1) holds by combining (2.5), (2.6), (2.7), and (2.8). □

REMARK 2.1. Let Φ be a concave function (that is, $-\Phi$ is convex). Then for all $r\in \operatorname{Int} I$ and $s\in I$ we have

$$\Phi(r)-\Phi(s)-\varphi(r)(r-s)\geqslant 0,$$

and (2.3) reads

$$\begin{aligned}
\Phi(r)-\Phi(s)-\varphi(r)(r-s) &= |\Phi(r)-\Phi(s)-\varphi(r)(r-s)|\\
&\geqslant \big|\,|\Phi(s)-\Phi(r)|-|\varphi(r)|\cdot|s-r|\,\big|,
\end{aligned}$$

where φ is an arbitrary real function on I such that $\varphi(x)\in\partial\Phi(x)=[\Phi'_+(x),\,\Phi'_-(x)]$, for all $x\in \operatorname{Int} I$. Hence, in this setting, (2.1) holds with its left-hand side replaced with

$$\int_{\Omega_1} u(x)\Phi(A_kf(x))\,d\mu_1(x)-\int_{\Omega_2} v(y)\Phi(f(y))\,d\mu_2(y).$$

REMARK 2.2. Since the right-hand side of (2.1) is non-negative, we get (1.8) as an immediate consequence of Theorem 2.1 and Remark 2.1. Consequently, our new result can be regarded as a refinement of the general weighted Hardy-type inequality (1.8). The same holds also for a concave function Φ.

Although (2.1) holds for all convex (or concave) functions, some choices of Φ are of particular interest. Namely, we shall consider power and exponential functions. To start with, let $p\in\mathbb{R}\setminus\{0\}$ and the function $\Phi:\mathbb{R}_+\to\mathbb{R}$ be defined by $\Phi(x)=x^p$. Obviously, $\varphi(x)=\Phi'(x)=px^{p-1}$, $x\in\mathbb{R}_+$, so Φ is convex for $p\in\mathbb{R}\setminus[0,1)$, concave for $p\in(0,1]$, and affine, that is, both convex and concave for $p=1$. In this setting, we get the following consequence of Theorem 2.1.

COROLLARY 2.1. *Let* $\Omega_1,\Omega_2,\mu_1,\mu_2,u,k,K$, *and* v *be as in Theorem* 2.1. *Let* $p\in\mathbb{R}$ *be such that* $p\neq 0$, $f:\Omega_2\to\mathbb{R}$ *be a non-negative measurable function* (*positive for* $p<0$), A_kf *be defined by* (1.6), *and*

$$R_{p,k}f(x,y)=\left|\left|f^p(y)-A_k^pf(x)\right|-|p|\cdot|A_kf(x)|^{p-1}\,|f(y)-A_kf(x)|\right|, \tag{2.9}$$

for $x\in\Omega_1$, $y\in\Omega_2$. *If* $p\geqslant 1$ *or* $p<0$, *then the inequality*

$$\begin{aligned}\int_{\Omega_2} v(y)f^p(y)\,d\mu_2(y)-\int_{\Omega_1} u(x)A_k^pf(x)\,d\mu_1(x)\\ \geqslant \int_{\Omega_1}\frac{u(x)}{K(x)}\int_{\Omega_2} k(x,y)R_{p,k}f(x,y)\,d\mu_2(y)\,d\mu_1(x)\end{aligned} \tag{2.10}$$

holds, while for $p\in(0,1)$ *relation* (2.10) *holds with*

$$\int_{\Omega_1} u(x)A_k^pf(x)\,d\mu_1(x)-\int_{\Omega_2} v(y)f^p(y)\,d\mu_2(y)$$

on its left-hand side.

REMARK 2.3. Note that relation (2.10) is trivial for $p=1$, since both of its sides are equal to 0.

On the other hand, for the convex function $\Phi:\mathbb{R}\to\mathbb{R}$, $\Phi(x)=e^x$, we have $\varphi(x)=\Phi'(x)=e^x$ and the following new general refined weighted Pólya–Knopp–type inequality with a kernel, which is a generalization and refinement of the classical Pólya–Knopp's inequality (1.3).

COROLLARY 2.2. *Let* $\Omega_1,\Omega_2,\mu_1,\mu_2,u,k,K$, *and* v *be as in Theorem* 2.1 *and let* $p\in\mathbb{R}$, $p\neq 0$. *Then the inequality*

$$\begin{aligned}\int_{\Omega_2} v(y)f^p(y)\,d\mu_2(y)-\int_{\Omega_1} u(x)G_k^pf(x)\,d\mu_1(x)\\ \geqslant \int_{\Omega_1}\frac{u(x)}{K(x)}\int_{\Omega_2} k(x,y)S_{p,k}f(x,y)\,d\mu_2(y)\,d\mu_1(x)\end{aligned} \tag{2.11}$$

holds for all positive measurable functions f *on* Ω_2, *where* $G_kf(x)$ *and* $S_{p,k}f(x,y)$ *are for* $x\in\Omega_1$ *and* $y\in\Omega_2$ *respectively defined by*

$$G_kf(x)=\exp\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)\ln f(y)\,d\mu_2(y)\right)$$

and

$$S_{p,k}f(x,y)=\left|\left|f^p(y)-G_k^pf(x)\right|-|p|\,G_k^p(x)\left|\ln\frac{f(y)}{G_kf(x)}\right|\right|. \tag{2.12}$$

In particular, for $p = 1$ *we get*

$$\int_{\Omega_2} v(y)f(y)\,d\mu_2(y) - \int_{\Omega_1} u(x)G_kf(x)\,d\mu_1(x) \geqslant \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,y) \times \\ \times \left| |f(y) - G_kf(x)| - G_k(x) \left| \ln \frac{f(y)}{G_kf(x)} \right| \right| d\mu_2(y)\,d\mu_1(x). \tag{2.13}$$

Moreover, relations (2.11) *and* (2.13) *are equivalent.*

Proof. Apply (2.1) with $\Phi : \mathbb{R} \to \mathbb{R}$, $\Phi(x) = e^x$, and replace the function f with $p \ln f$. Note that $G_kf = \exp(A_k(\ln f))$ and $G_kf^p = G_k^p f$, so equivalence of (2.11) and (2.13) is evident. □

To conclude this section, we consider the simplest kernels k, that is, those with separate variables. As a corollary of Theorem 2.1 in this setting, we get a refined general Jensen's inequality.

COROLLARY 2.3. *Suppose* Ω *is a measure space with a positive* σ*-finite measure* μ, $m \in L^1(\Omega,\mu)$ *is a non-negative function such that* $|m|_1 > 0$, *a real function* Φ *is convex on an interval* $I \subseteq \mathbb{R}$, *and* $\varphi : I \to \mathbb{R}$ *is any function such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \operatorname{Int} I$. *Then the inequality*

$$\int_{\Omega} m(y)\Phi(f(y))\,d\mu(y) - |m|_1\Phi(A_mf) \\ \geqslant \int_{\Omega} m(y)\,\big|\,|\Phi(f(y)) - \Phi(A_mf)| - |\varphi(A_mf)| \cdot |f(y) - A_mf|\,\big|\,d\mu(y) \tag{2.14}$$

holds for all measurable functions $f : \Omega \to \mathbb{R}$ *with values in* I, *where*

$$A_mf = \frac{1}{|m|_1} \int_{\Omega} m(y)f(y)\,d\mu(y).$$

If the function Φ *is concave, the order of integrals on the left-hand side of* (2.14) *is reversed.*

Proof. Suppose that in Theorem 2.1 we have $\Omega_2 = \Omega$, $\mu_2 = \mu$, $u \in L^1(\Omega_1,\mu_1)$ such that $|u|_1 > 0$, and k of the form $k(x,y) = l(x)m(y)$, for some positive measurable function $l : \Omega_1 \to \mathbb{R}$. Then $K(x) = |m|_1 l(x)$ and $A_kf(x) = A_mf \in I$, $x \in \Omega_1$, while $v(y) = \frac{|u|_1}{|m|_1} m(y)$, $y \in \Omega$. Thus, (2.1) reduces to (2.14) and it does not depend on Ω_1, l, and u. □

REMARK 2.4. Observe that, for $0 < |\Omega|_\mu < \infty$ and $m(y) \equiv 1$ on Ω, we have $|m|_1 = |\Omega|_\mu$, so (2.14) becomes classical refined Jensen's inequality

$$\frac{1}{|\Omega|_\mu} \int_{\Omega} \Phi(f(y))\,d\mu(y) - \Phi(Af) \\ \geqslant \frac{1}{|\Omega|_\mu} \int_{\Omega} \big|\,|\Phi(f(y)) - \Phi(Af)| - |\varphi(Af)| \cdot |f(y) - Af|\,\big|\,d\mu(y),$$

where

$$Af = \frac{1}{|\Omega|_\mu} \int_\Omega f(y)\, d\mu(y).$$

3. One-dimensional refined Hardy–type inequalities

In the sequel, the general results obtained in Section 2 are applied to particular measure spaces, convex functions, weights, and kernels. This enables us to refine and even generalize some important inequalities previously known from the literature.

First, we consider an one-dimensional setting, with intervals in $\mathbb{R}$ and the Lebesgue measure, to get refined Hardy and Pólya–Knopp–type inequalities, as well as related dual relations. In the following theorem, we state and prove a refinement of a Hardy–type inequality obtained by S. Kaijser et al. in [16].

THEOREM 3.1. *Let $0 < b \leqslant \infty$ and $k : (0,b)\times(0,b) \to \mathbb{R}$ be a non-negative measurable function, such that*

$$K(x) = \int_0^x k(x,y)\, dy > 0,\ x \in (0,b). \tag{3.1}$$

Let a weight $u : (0,b) \to \mathbb{R}$ be such that the function $x \mapsto \frac{u(x)}{x}\cdot\frac{k(x,y)}{K(x)}$ is integrable on (y,b) for each fixed $y \in (0,b)$, and let the function $w : (0,b) \to \mathbb{R}$ be defined by

$$w(y) = y\int_y^b \frac{k(x,y)}{K(x)} u(x)\, \frac{dx}{x}.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\begin{aligned}\int_0^b w(y)\Phi(f(y))\,\frac{dy}{y} - \int_0^b u(x)\Phi(A_k f(x))\,\frac{dx}{x} \geqslant \int_0^b \frac{u(x)}{K(x)} \int_0^x k(x,y)\,\times\\ \times \big|\,|\Phi(f(y)) - \Phi(A_k f(x))| - |\varphi(A_k f(x))|\cdot|f(y) - A_k f(x)|\,\big|\, dy\,\frac{dx}{x}\end{aligned} \tag{3.2}$$

holds for all measurable functions $f : (0,b) \to \mathbb{R}$ with values in I and for $A_k f$ defined by

$$A_k f(x) = \frac{1}{K(x)} \int_0^x k(x,y) f(y)\, dy,\ x \in (0,b). \tag{3.3}$$

If the function Φ is concave, the order of integrals on the left-hand side of (3.2) is reversed.

Proof. Denote $T_1 = \{(x,y) \in \mathbb{R}_+^2 : 0 < y \leqslant x < b\}$ and set $\Omega_1 = \Omega_2 = (0,b)$ in Theorem 2.1. Relation (3.2) follows from (2.1) by replacing $d\mu_1(x)$, $d\mu_2(y)$, $u(x)$, and $k(x,y)$ respectively with dx, dy, $\frac{u(x)}{x}$, and $k(x,y)\chi_{T_1}(x,y)$. In this case, (1.6) reduces to (3.3), while (1.7) becomes (3.1). Moreover, $w(y) = yv(y)$, $y \in (0,b)$. □

REMARK 3.1. Since the right-hand side of inequality (3.2) is non-negative, Theorem 3.1 can be seen as a refinement of Theorem 4.1 in [16]. In particular, for $k(x,y) \equiv 1$, $x,y \in (0,b)$, and the classical Hardy's operator

$$Hf(x) = \frac{1}{x}\int_0^x f(y)\,dy,\ x \in (0,b),$$

we get a refinement of Theorem 1 in [9], that is, the refined Hardy–type inequality for convex functions,

$$\begin{aligned}&\int_0^b w(y)\Phi(f(y))\,\frac{dy}{y} - \int_0^b u(x)\Phi(Hf(x))\,\frac{dx}{x}\\ &\geqslant \int_0^b \frac{u(x)}{x^2}\int_0^x \big|\,|\Phi(f(y)) - \Phi(Hf(x))| - |\varphi(Hf(x))|\cdot|f(y) - Hf(x)|\,\big|\,dy\,dx,\end{aligned}$$

where

$$w(y) = y\int_y^b \frac{u(x)}{x^2}\,dx,\ y \in (0,b).$$

Observing that the right-hand side of the above inequality is greater than

$$\begin{aligned}\Bigg|&\int_0^b u(x)\int_0^x |\Phi(f(y)) - \Phi(Hf(x))|\,dy\,\frac{dx}{x^2}\\ &- \int_0^b u(x)\,|\varphi(Hf(x))|\int_0^x |f(y) - Hf(x)|\,dy\,\frac{dx}{x^2}\Bigg|,\end{aligned}$$

we obtain Theorem 2.2 in [4]. Therefore, Theorem 3.1 generalizes the result mentioned.

Applying Theorem 3.1 to power and exponential functions, we get the following two corollaries.

COROLLARY 3.1. *Let $0 < b \leqslant \infty$ and k, K, u, and w be as in Theorem 3.1. Let $p \in \mathbb{R}$ be such that $p \neq 0$, f be a non-negative measurable function on $(0,b)$ (f positive for $p < 0$), and let $A_k f$ and $R_{p,k} f$ be defined by (3.3) and (2.9) respectively. If $p > 1$ or $p < 0$, then*

$$\int_0^b w(y)f^p(y)\,\frac{dy}{y} - \int_0^b u(x)A_k^p f(x)\,\frac{dx}{x} \geqslant \int_0^b \frac{u(x)}{K(x)}\int_0^x k(x,y)R_{p,k}f(x,y)\,dy\,\frac{dx}{x}, \quad (3.4)$$

while for $p \in (0,1)$ the order of integrals on the left-hand side of (3.4) is reversed. If $p = 1$, then both-hand sides of (3.4) are equal to 0.

COROLLARY 3.2. *Let $0 < b \leqslant \infty$, k, K, u, and w be as in Theorem 3.1, and let $p \in \mathbb{R}$ be such that $p \neq 0$. If f is a positive measurable function on $(0,b)$,*

$$G_k f(x) = \exp\left(\frac{1}{K(x)}\int_0^x k(x,y)\ln f(y)\,dy\right),\ x \in (0,b),$$

and $S_{p,k}f$ is defined by (2.12)*, then*

$$\int_0^b w(y)f^p(y)\,\frac{dy}{y} - \int_0^b u(x)G_k^p f(x)\,\frac{dx}{x} \geqslant \int_0^b \frac{u(x)}{K(x)} \int_0^x k(x,y)S_{p,k}f(x,y)\,dy\,\frac{dx}{x}. \quad (3.5)$$

Moreover, for $p=1$ we have

$$\int_0^b w(y)f(y)\,\frac{dy}{y} - \int_0^b u(x)G_k f(x)\,\frac{dx}{x} \geqslant \int_0^b \frac{u(x)}{K(x)} \int_0^x k(x,y) \times \\ \times \left| |f(y)-G_k f(x)| - G_k(x) \left| \ln \frac{f(y)}{G_k f(x)} \right| \right| dy\,\frac{dx}{x} \quad (3.6)$$

and relations (3.5) *and* (3.6) *are equivalent.*

The above results can be applied to some important particular kernels. Namely, in the following example we discuss refined Hardy and Pólya–Knopp–type inequalities related to the Riemann–Liouville operator

$$R_\gamma f(x) = \frac{\gamma}{x^\gamma} \int_0^x (x-y)^{\gamma-1} f(y)\,dy, \quad (3.7)$$

where $\gamma \in \mathbb{R}_+$. Of course, for $\gamma = 1$ we have $R_1 = H$, that is, the classical Hardy's integral operator.

EXAMPLE 3.1. Suppose $0 < b \leqslant \infty$, $\gamma \in \mathbb{R}_+$, and T_1 is as in the proof of Theorem 3.1. If $u(x) \equiv 1$, $k(x,y) = \frac{\gamma}{x^\gamma}(x-y)^{\gamma-1}\chi_{T_1}(x,y)$, and $R_\gamma f(x)$ is as in (3.7), then inequality (3.2) reads

$$\int_0^b \left(1-\frac{y}{b}\right)^\gamma \Phi(f(y))\,\frac{dy}{y} - \int_0^b \Phi(R_\gamma f(x))\,\frac{dx}{x} \geqslant \gamma \int_0^b \int_0^x (x-y)^{\gamma-1} \times \\ \times \left| \left|\Phi(f(y)) - \Phi(R_\gamma f(x))\right| - \left|\varphi(R_\gamma f(x))\right| \cdot \left|f(y)-R_\gamma f(x)\right| \right| dy\,\frac{dx}{x^{\gamma+1}}, \quad (3.8)$$

so we obtained a refinement of Example 4.2 in [16].

As in Corollaries 3.1 and 3.2, relation (3.8) can be considered with Φ being a power or exponential function. In particular, let $p,k \in \mathbb{R}$ be such that $\frac{k-1}{p} > 0$, f be a non-negative function on $(0,b)$ (positive for $p<0$), and

$$Rf(x) = \int_0^x \left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1} f(y)\,dy,\ x \in (0,b).$$

Rewrite (3.8) for $\Phi(x) = x^p$ and substitute $b^{\frac{k-1}{p}}$ and $f\left(y^{\frac{p}{k-1}}\right) y^{\frac{p}{k-1}-1}$ instead of b and

$f(y)$ respectively. After suitable variable changes, for $p \geqslant 1$ and $p < 0$ we get

$$\begin{aligned}
&\left(\frac{p}{\gamma(k-1)}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right]^\gamma x^{p-k} f^p(x)\,dx - \int_0^b x^{-k} R^p f(x)\,dx \\
&\geqslant \left| \left(\frac{p}{\gamma(k-1)}\right)^{p-1} \int_0^b x^{\frac{1-k}{p}-1} \int_0^x \left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1} y^{\frac{k-1}{p}-1} \times \right. \\
&\quad \times \left| y^{p-k+1} f^p(y) - \left(\frac{\gamma(k-1)}{p}\right)^p x^{1-k} R^p f(x) \right| dy\,dx \\
&\quad - |p| \int_0^b x^{-k} R^{p-1} f(x) \int_0^x \left[1-\left(\frac{y}{x}\right)^{\frac{k-1}{p}}\right]^{\gamma-1} \times \\
&\quad \left. \times \left| f(y) - \frac{k-1}{p} \cdot \frac{\gamma}{y} \left(\frac{y}{x}\right)^{\frac{k-1}{p}} R f(x) \right| dy\,dx \right|, \qquad (3.9)
\end{aligned}$$

while for $p \in (0,1)$ the order of integrals on the left-hand side of (3.9) is reversed. Note that for $\gamma = 1$ inequality (3.9) reduces to the refined strengthened Hardy's inequality from Corollary 3.1 in [4]. Moreover, for $b = \infty$ and $p = k$ we obtain a refinement of classical Hardy's inequality (1.1).

On the other hand, for $\gamma = 1$, $\Phi(x) = e^x$, a positive function f on $(0,b)$, $f(y)$ replaced with $\ln(yf(y))$, and

$$Gf(x) = \exp\left(\frac{1}{x}\int_0^x \ln f(y)\,dy\right),\ x \in (0,b),$$

relation (3.8) becomes

$$\begin{aligned}
e\int_0^b \left(1-\frac{y}{b}\right) f(y)\,dy - \int_0^b Gf(x)\,dx &\geqslant \left| \int_0^b \int_0^x |eyf(y) - xGf(x)|\,dy\,\frac{dx}{x^2} \right. \\
&\quad \left. - \int_0^b Gf(x) \int_0^x \left|\ln\left(\frac{eyf(y)}{xGf(x)}\right)\right| dy\,\frac{dx}{x} \right|, \qquad (3.10)
\end{aligned}$$

so we obtained the refined strengthened Pólya–Knopp's inequality from Corollary 3.3 in [4]. In the case when $b = \infty$, we get a refinement of classical Pólya–Knopp's inequality (1.3). □

We continue by formulating results dual to Theorem 3.1 and its corollaries. They are derived from Theorem 2.1 by similar arguments. The following theorem is dual to Theorem 3.1.

THEOREM 3.2. *For $0 \leqslant b < \infty$, let $k : (b,\infty) \times (b,\infty) \to \mathbb{R}$ be a non-negative measurable function, such that*

$$\tilde{K}(x) = \int_x^\infty k(x,y)\,dy > 0,\ x \in (b,\infty), \qquad (3.11)$$

and a weight $u:(b,\infty)\to\mathbb{R}$ *be such that the function* $x\mapsto \frac{u(x)}{x}\cdot\frac{k(x,y)}{\tilde{K}(x)}$ *is integrable on* (b,y) *for each fixed* $y\in(b,\infty)$. *Let the function* $\tilde{w}:(b,\infty)\to\mathbb{R}$ *be defined by*

$$\tilde{w}(y)=y\int_b^y\frac{k(x,y)}{\tilde{K}(x)}\,u(x)\,\frac{dx}{x}.$$

If Φ *is a convex function on an interval* $I\subseteq\mathbb{R}$ *and* $\varphi:I\to\mathbb{R}$ *is such that* $\varphi(x)\in\partial\Phi(x)$ *for all* $x\in\operatorname{Int}I$, *then the inequality*

$$\int_b^\infty\tilde{w}(y)\Phi(f(y))\,\frac{dy}{y}-\int_b^\infty u(x)\Phi(\tilde{A}_kf(x))\,\frac{dx}{x}\geqslant\int_b^\infty\frac{u(x)}{\tilde{K}(x)}\int_x^\infty k(x,y)\times$$
$$\times\left|\left|\Phi(f(y))-\Phi(\tilde{A}_kf(x))\right|-\left|\varphi(\tilde{A}_kf(x))\right|\cdot\left|f(y)-\tilde{A}_kf(x)\right|\right|dy\,\frac{dx}{x}\tag{3.12}$$

holds for all measurable functions $f:(b,\infty)\to\mathbb{R}$ *with values in* I *and for* $\tilde{A}_kf$ *defined by*

$$\tilde{A}_kf(x)=\frac{1}{\tilde{K}(x)}\int_x^\infty k(x,y)f(y)\;dy,\;x\in(b,\infty).\tag{3.13}$$

If the function Φ *is concave, the order of integrals on the left-hand side of* (3.13) *is reversed.*

Proof. Let $T_2=\{(x,y)\in\mathbb{R}_+^2: b<x\leqslant y<\infty\}$. Inequality (3.12) follows directly from Theorem 2.1, rewritten with $\Omega_1=\Omega_2=(b,\infty)$, $d\mu_1(x)=dx$, $d\mu_2(y)=dy$, and with $\frac{u(x)}{x}$ and $k(x,y)\chi_{T_2}(x,y)$ instead of $u(x)$ and $k(x,y)$. Note that (1.6) and (1.7) respectively become (3.13) and (3.11), while $\tilde{w}(y)=yv(y)$, $y\in(b,\infty)$. □

REMARK 3.2. Note that Theorem 3.2 provides a refinement of Theorem 4.3 in [16]. Furthermore, by setting $k(x,y)=\frac{1}{y^2}$, $x,y\in(b,\infty)$, and denoting

$$\tilde{H}f(x)=x\int_x^\infty f(y)\,\frac{dy}{y^2},\;x\in(b,\infty),$$

inequality (3.12) reduces to the following refined dual Hardy–type inequality for convex functions:

$$\int_b^\infty\tilde{w}(y)\Phi(f(y))\,\frac{dy}{y}-\int_b^\infty u(x)\Phi(\tilde{H}f(x))\,\frac{dx}{x}$$
$$\geqslant\int_b^\infty u(x)\int_x^\infty\left|\left|\Phi(f(y))-\Phi(\tilde{H}f(x))\right|-\left|\varphi(\tilde{H}f(x))\right|\cdot\left|f(y)-\tilde{H}f(x)\right|\right|\frac{dy}{y^2}\,dx,$$

where

$$\tilde{w}(y)=\frac{1}{y}\int_b^y u(x)\,dx,\,y\in(b,\infty).$$

Since the right-hand side of this inequality is not less than

$$\left|\int_b^\infty u(x)\int_x^\infty\left|\Phi(f(y))-\Phi(\tilde{H}f(x))\right|\frac{dy}{y^2}\,dx\right.$$
$$\left.-\int_b^\infty u(x)\left|\varphi(\tilde{H}f(x))\right|\int_x^\infty\left|f(y)-\tilde{H}f(x)\right|\frac{dy}{y^2}\,dx\right|,$$

as a consequence of our result we get Theorem 2.3 in [4].

The next two corollaries are dual to Corollary 3.1 and Corollary 3.2.

COROLLARY 3.3. *Let* $0 \leqslant b < \infty$ *and let* k, $\tilde{K}$, u, *and* $\tilde{w}$ *be as in Theorem* 3.2. *For* $p \in \mathbb{R}$, $p \neq 0$, *and a non-negative measurable function* f *on* (b,∞) (f *positive for* $p<0$), *let* $\tilde{A}_k f$ *be defined by* (3.13) *and*

$$\tilde{R}_{p,k}f(x,y) = \left| \left|f^p(y) - \tilde{A}_k^p f(x)\right| - |p| \cdot \left|\tilde{A}_k f(x)\right|^{p-1} \left|f(y) - \tilde{A}_k f(x)\right| \right|,$$

for $x,y \in (b,\infty)$. *Then the inequality*

$$\int_b^\infty \tilde{w}(y) f^p(y)\,\frac{dy}{y} - \int_b^\infty u(x)\tilde{A}_k^p f(x)\,\frac{dx}{x} \geqslant \int_b^\infty \frac{u(x)}{\tilde{K}(x)} \int_x^\infty k(x,y)\tilde{R}_{p,k}f(x,y)\,dy\,\frac{dx}{x} \tag{3.14}$$

holds for $p>1$ *and* $p<0$. *For* $p \in (0,1)$ *the order of integrals on the left-hand side of* (3.14) *is reversed, while for* $p=1$ *its both-hand sides are equal to* 0.

COROLLARY 3.4. *Suppose that* $p \in \mathbb{R}\setminus\{0\}$, $0 \leqslant b < \infty$, *and that* k, $\tilde{K}$, u, *and* $\tilde{w}$ *are as in Theorem* 3.2. *If* f *is a positive measurable function on* (b,∞),

$$\tilde{G}_k f(x) = \exp\left(\frac{1}{\tilde{K}(x)} \int_x^\infty k(x,y) \ln f(y)\,dy\right),\ x \in (b,\infty),$$

and

$$\tilde{S}_{p,k}f(x,y) = \left| \left|f^p(y) - \tilde{G}_k^p f(x)\right| - |p|\,\tilde{G}_k^p(x) \left| \ln \frac{f(y)}{\tilde{G}_k f(x)} \right| \right|,\ x,y \in (b,\infty),$$

then the inequality

$$\int_b^\infty \tilde{w}(y) f^p(y)\,\frac{dy}{y} - \int_b^\infty u(x)\tilde{G}_k^p f(x)\,\frac{dx}{x} \geqslant \int_b^\infty \frac{u(x)}{\tilde{K}(x)} \int_x^\infty k(x,y)\tilde{S}_{p,k}f(x,y)\,dy\,\frac{dx}{x} \tag{3.15}$$

holds. In particular, for $p=1$ *we have*

$$\begin{aligned} \int_b^\infty \tilde{w}(y) f(y)\,\frac{dy}{y} - \int_b^\infty u(x)\tilde{G}_k f(x)\,\frac{dx}{x} &\geqslant \int_b^\infty \frac{u(x)}{\tilde{K}(x)} \int_x^\infty k(x,y) \times \\ &\times \left| \left|f(y) - \tilde{G}_k f(x)\right| - \tilde{G}_k(x) \left| \ln \frac{f(y)}{\tilde{G}_k f(x)} \right| \right| dy\,\frac{dx}{x} \end{aligned} \tag{3.16}$$

and relations (3.15) *and* (3.16) *are equivalent.*

We conclude this section by giving results dual to those from Example 3.1, that is, by explicating refined Hardy and Pólya–Knopp–type inequalities related to Weyl's fractional integral operator

$$W_\gamma f(x) = \gamma x \int_x^\infty (y-x)^{\gamma-1} f(y)\,\frac{dy}{y^{\gamma+1}}, \tag{3.17}$$

where $\gamma \in \mathbb{R}_+$. Note that $W_1 = \tilde{H}$, that is, for $\gamma = 1$ we get the classical dual Hardy's integral operator and related inequalities.

EXAMPLE 3.2. Let $0 \leqslant b < \infty$, $\gamma \in \mathbb{R}_+$, and T_2 be as in the proof of Theorem 3.2. For $u(x) \equiv 1$, $k(x,y) = \gamma \frac{x}{y^{\gamma+1}}(y-x)^{\gamma-1}\chi_{T_2}(x,y)$, and $W_\gamma f(x)$ as in (3.17), inequality (3.12) becomes

$$\int_b^\infty \left(1-\frac{b}{y}\right)^\gamma \Phi(f(y))\,\frac{dy}{y} - \int_b^\infty \Phi(W_\gamma f(x))\,\frac{dx}{x} \geqslant \gamma \int_b^\infty \int_x^\infty (y-x)^{\gamma-1} \times$$
$$\times \Big| \big|\Phi(f(y)) - \Phi(W_\gamma f(x))\big| - \big|\varphi(W_\gamma f(x))\big| \cdot \big|f(y) - W_\gamma f(x)\big| \Big| \,\frac{dy}{y^{\gamma+1}}\,dx. \tag{3.18}$$

Now, we apply (3.18) to power and exponential functions. Namely, let $p,k \in \mathbb{R}$ be such that $\frac{p}{1-k} > 0$, f be a non-negative measurable function on (b,∞) (f positive for $p < 0$),

$$Wf(x) = \int_x^\infty \left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1} f(y)\,dy,\ x \in (b,\infty),$$

and $\Phi(x) = x^p$. Rewrite (3.18) with $b^{\frac{1-k}{p}}$ and $f\left(y^{\frac{p}{1-k}}\right) y^{\frac{p}{1-k}+1}$ instead of b and $f(y)$ respectively. After some variable substitutions, for $p \geqslant 1$ and $p < 0$ we obtain the inequality

$$\left(\frac{p}{\gamma(1-k)}\right)^p \int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right]^\gamma x^{p-k} f^p(x)\,dx - \int_b^\infty x^{-k} W^p f(x)\,dx$$
$$\geqslant \Bigg| \left(\frac{p}{\gamma(1-k)}\right)^{p-1} \int_b^\infty x^{\frac{1-k}{p}-1} \int_x^\infty \left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1} y^{\frac{k-1}{p}-1} \times$$
$$\times \left| y^{p-k+1} f^p(y) - \left(\frac{\gamma(1-k)}{p}\right)^p x^{1-k} W^p f(x)\right| dy\,dx$$
$$- |p| \int_b^\infty x^{-k} W^{p-1} f(x) \int_x^\infty \left[1-\left(\frac{x}{y}\right)^{\frac{1-k}{p}}\right]^{\gamma-1} \times$$
$$\times \left| f(y) - \frac{(1-k)}{p}\cdot\frac{\gamma}{y}\left(\frac{x}{y}\right)^{\frac{1-k}{p}} Wf(x)\right| dy\,dx \Bigg|. \tag{3.19}$$

For $p \in (0,1)$, relation (3.19) holds with integrals on its left-hand side written in the reverse order. Moreover, if $\gamma = 1$, then (3.19) becomes the refined strengthened dual Hardy's inequality from Corollary 3.2 in [4].

In the case when $\gamma = 1$, $\Phi(x) = e^x$, f is a positive function on (b,∞), and

$$\tilde{G}f(x) = \exp\left(x \int_x^\infty \ln f(y)\,\frac{dy}{y^2}\right),\ x \in (b,\infty),$$

after substituting $\ln(yf(y))$ instead of $f(y)$, relation (3.18) reads

$$\frac{1}{e}\int_b^\infty\left(1-\frac{b}{x}\right)f(x)\,dx-\int_b^\infty \tilde{G}f(x)\,dx \geqslant \left|\int_b^\infty\int_x^\infty\left|\frac{1}{e}yf(y)-x\tilde{G}f(x)\right|\frac{dy}{y^2}\,dx\right.$$
$$\left.-\int_b^\infty x\tilde{G}f(x)\int_x^\infty\left|\ln\left(\frac{yf(y)}{ex\tilde{G}f(x)}\right)\right|\frac{dy}{y^2}\,dx\right|,$$

that is, it is reduced to the refined strengthened dual Pólya–Knopp's inequality given in Corollary 3.4 in [4]. □

4. One-dimensional refined Hardy–Hilbert–type inequalities

We continue the above analysis by considering some important kernels related to $\Omega_1=\Omega_2=\mathbb{R}_+$ and by assuming that $d\mu_1(x)=dx$, $d\mu_2(y)=dy$, and that $\Phi:\mathbb{R}_+\to\mathbb{R}$ is given by $\Phi(x)=x^p$, where $p\in\mathbb{R}$, $p\neq 0$. In this setting, Corollary 2.1 provides new refinements of some well-known one-dimensional Hardy–Hilbert–type inequalities.

First, we obtain a generalization and a refinement of classical Hardy–Hilbert's inequality (1.2). It is given in the following example.

EXAMPLE 4.1. For $p\in\mathbb{R}\setminus\{0\}$, let $s\in\mathbb{R}$ be such that $\frac{s-2}{p},\frac{s-2}{p'}>-1$ and the kernel $k:\mathbb{R}_+^2\to\mathbb{R}$ be defined by $k(x,y)=\left(\frac{y}{x}\right)^{\frac{s-2}{p}}(x+y)^{-s}$. Let $\alpha\in\left(-\frac{s-2}{p'}-1,\frac{s-2}{p}+1\right)$ be arbitrary and the weight $u:\mathbb{R}_+\to\mathbb{R}$ be given by $u(x)=x^{\alpha-1}$. Set

$$C_1=B\left(\frac{s-2}{p}-\alpha+1,\frac{s-2}{p'}+\alpha+1\right)\quad\text{and}\quad C_2=B\left(\frac{s-2}{p}+1,\frac{s-2}{p'}+1\right),$$

where $B(\cdot,\cdot)$ denotes the usual Beta function. Suppose f is a non-negative function on $\mathbb{R}_+$ (positive for $p<0$) and

$$Af(x)=\int_0^\infty\frac{f(y)}{(x+y)^s}\,dy,\ x\in\mathbb{R}_+,$$

is its generalized Stieltjes transform (see e.g. [2] and [27] for further information). Corollary 2.1, rewritten with $f(y)y^{\frac{2-s}{p}}$ instead of $f(y)$, implies that the inequality

$$C_1C_2^{p-1}\int_0^\infty y^{\alpha-s+1}f^p(y)\,dy-\int_0^\infty x^{\alpha+(s-1)(p-1)}A^pf(x)\,dx$$
$$\geqslant\left|C_2^{p-1}\int_0^\infty x^{\alpha+\frac{s-2}{p'}}\int_0^\infty\frac{y^{\frac{s-2}{p}}}{(x+y)^s}\left|f^p(y)y^{2-s}-\frac{x^{(s-1)(p-1)+1}}{C_2^p}A^pf(x)\right|dy\,dx\right.$$
$$\left.-|p|\int_0^\infty x^{\alpha+(s-1)(p-1)}\int_0^\infty\frac{A^{p-1}f(x)}{(x+y)^s}\left|f(y)-\frac{1}{C_2}x^{\frac{s-2}{p'}+1}y^{\frac{s-2}{p}}Af(x)\right|dy\,dx\right| \tag{4.1}$$

holds for $p\geqslant 1$ and $p<0$, while for $p\in(0,1)$ it holds with the reverse order of the integrals on its left-hand side. In particular, for $\alpha=0$ we get a refinement of the general Hardy–Hilbert–type inequality from [28], with the best possible constant $C = C_2^p = B^p\left(\frac{s-2}{p}+1, \frac{s-2}{p'}+1\right)$. Moreover, for $p > 1$, $\alpha = 0$, and $s = 1$, we have $C_1 = C_2 = B\left(\frac{1}{p}, \frac{1}{p'}\right) = \frac{\pi}{\sin\frac{\pi}{p}}$, so relation (4.1) provides a new refinement of classical Hardy–Hilbert's inequality (1.2). □

In the next example, we generalize and refine classical Hardy–Littlewood–Pólya's inequality.

EXAMPLE 4.2. Let the real parameters p, s, α, and the weight function u be as in Example 4.1. Define the kernel $k:\mathbb{R}_+^2 \to \mathbb{R}$ by $k(x,y) = \left(\frac{y}{x}\right)^{\frac{s-2}{p}} \max\{x,y\}^{-s}$ and for a non-negative function f on $\mathbb{R}_+$ (positive for $p < 0$) set

$$Lf(x) = \int_0^\infty \frac{f(y)}{\max\{x,y\}^s}\, dy,\ x \in \mathbb{R}_+.$$

Finally, denote

$$D_1 = \frac{pp's}{(p-p\alpha+s-2)(\alpha p'+p'+s-2)} \quad \text{and} \quad D_2 = \frac{pp's}{(p+s-2)(p'+s-2)}.$$

Applying the same procedure as in Example 4.1, we obtain that the inequality

$$\begin{aligned} &D_1 D_2^{p-1}\int_0^\infty y^{\alpha-s+1} f^p(y)\,dy - \int_0^\infty x^{\alpha+(s-1)(p-1)} L^p f(x)\,dx \\ &\geqslant \left| D_2^{p-1}\int_0^\infty x^{\alpha+\frac{s-2}{p'}} \int_0^\infty \frac{y^{\frac{s-2}{p}}}{\max\{x,y\}^s} \left| f^p(y) y^{2-s} - \frac{x^{(s-1)(p-1)+1}}{D_2^p} L^p f(x)\right| dy\,dx \right. \\ &\left. -|p|\int_0^\infty x^{\alpha+(s-1)(p-1)} \int_0^\infty \frac{L^{p-1}f(x)}{\max\{x,y\}^s} \left| f(y) - \frac{1}{D_2} x^{\frac{s-2}{p'}+1} y^{\frac{s-2}{p}} Lf(x)\right| dy\,dx \right| \end{aligned} \tag{4.2}$$

holds for $p \geqslant 1$ and $p < 0$, while for $p \in (0,1)$ it holds with integrals on its left-hand side given in the reverse order. Note that the constant $C = D_2^p = \left[\frac{pp's}{(p+s-2)(p'+s-2)}\right]^p$ is the best possible for the Hardy–Littlewood–Pólya–type inequalities with $\alpha = 0$. As a special case, for $p > 1$, $\alpha = 0$, and $s = 1$, we get $D_1 = D_2 = pp'$, that is, relation (4.2) is a new refinement of the classical Hardy–Littlewood–Pólya's inequality (see [12] for further details). □

REMARK 4.1. A similar approach can be applied to obtain refined inequalities involving the generalized Stieltjes transformation defined by

$$Sf(x) = \int_a^b \frac{f(y)}{\rho(x)+\rho(y)}\, dy,\ x \in (a,b),$$

where $-\infty \leqslant a < b \leqslant \infty$ and ρ is a positive, continuous and strictly increasing function on (a,b) (see Remark 4.6. in [10] for further details).

To calculate integrals in our last example of refined Hardy–Hilbert–type inequalities, we used the well-known reflection formula for the Digamma function ψ,

$$\int_0^\infty \frac{\ln x}{x-1} x^{-\alpha} dx = \psi'(1-\alpha) + \psi'(\alpha) = \frac{\pi^2}{\sin^2 \pi\alpha},$$

where $\alpha \in (0,1)$ (for details on ψ see [1]).

EXAMPLE 4.3. As in previous examples, let $p \in \mathbb{R}$, $p \neq 0$. For $\alpha \in (0,1)$, let the kernel k be defined on $\mathbb{R}_+^2$ by $k(x,y) = \frac{\ln y - \ln x}{y-x} \left(\frac{x}{y}\right)^\alpha$ and the weight $u : \mathbb{R}_+ \to \mathbb{R}$ by $u(x) = x^\beta$, where $\beta \in (-\alpha-1,-1)$. For a non-negative function f on $\mathbb{R}_+$ (positive for $p < 0$), let

$$Mf(x) = \int_0^\infty \frac{\ln y - \ln x}{y-x} f(y)\, dy, \ x \in \mathbb{R}_+.$$

Corollary 2.1, applied with the function $y \mapsto f(y)y^\alpha$ instead of f, then implies the inequality

$$\begin{aligned}
&\frac{\pi^{2p}}{\sin^{2(p-1)} \pi\alpha \cdot \sin^2 \pi(\alpha+\beta)} \int_0^\infty y^{p\alpha+\beta} f^p(y)\, dy - \int_0^\infty x^{p\alpha+\beta} M^p f(x)\, dx \\
&\geqslant \Bigg| \left(\frac{\pi}{\sin \pi\alpha}\right)^{2(p-1)} \int_0^\infty x^{\alpha+\beta} \int_0^\infty \frac{\ln y - \ln x}{y-x} y^{-\alpha} \times \\
&\quad \times \left| f^p(y) y^{p\alpha} - \left(\frac{\sin \pi\alpha}{\pi}\right)^{2p} x^{p\alpha} M^p f(x) \right| dy\, dx \\
&\quad - |p| \int_0^\infty x^{p\alpha+\beta} M^{p-1} f(x) \int_0^\infty \frac{\ln y - \ln x}{y-x} \times \\
&\quad \times \left| f(y) - \frac{\sin^2 \pi\alpha}{\pi^2} \left(\frac{x}{y}\right)^\alpha Mf(x) \right| dy\, dx \Bigg|
\end{aligned} \tag{4.3}$$

for $p \geqslant 1$ and $p < 0$, while for $p \in (0,1)$ the order of integrals on the left-hand side of (4.3) is reversed. Especially, for $p > 1$, $\alpha = \frac{1}{p}$, and $\beta = -1$, the left-hand side of (4.3) becomes

$$\left(\frac{\pi}{\sin \frac{\pi}{p}}\right)^{2p} \int_0^\infty f^p(y)\, dy - \int_0^\infty M^p f(x)\, dx.$$

Since the above expression is positive (unless $f \equiv 0$) and bounded from below by a positive constant, relation (4.3) provides a generalization and a refinement of another classical Hardy–Hilbert–type inequality. □

5. Refined Godunova–type inequalities

In previous two sections, Theorem 2.1 was considered only in various one-dimensional settings. Since it covers much more general situations, we conclude this paper by applying it to n-dimensional cells in $\mathbb{R}^n_+$. As a consequence, a generalization and a refinement of Godunova's inequality (1.5) is derived.

Before presenting our results, it is necessary to introduce some further notation. For $\mathbf{u},\mathbf{v} \in \mathbb{R}^n_+$, $\mathbf{u} = (u_1,u_2,\ldots,u_n)$, $\mathbf{v} = (v_1,v_2,\ldots,v_n)$, let

$$\frac{\mathbf{u}}{\mathbf{v}} = \left(\frac{u_1}{v_1},\frac{u_2}{v_2},\ldots,\frac{u_n}{v_n}\right) \text{ and } \mathbf{u}^{\mathbf{v}} = u_1^{v_1}u_2^{v_2}\cdots u_n^{v_n}.$$

In particular, $\mathbf{u}^{\mathbf{1}} = \prod_{i=1}^n u_i$, $\mathbf{u}^{\mathbf{2}} = (\prod_{i=1}^n u_i)^2$, and $\mathbf{u}^{-\mathbf{1}} = (\prod_{i=1}^n u_i)^{-1}$, where $\mathbf{n} = (n,n,\ldots,n)$ for $n \in \{-2,-1,0,1,2,\infty\}$. We write $\mathbf{u} < \mathbf{v}$ if componentwise $u_i < v_i$, $i = 1,\ldots,n$. Relations $\leqslant$, $>$, and $\geqslant$ are defined analogously. Finally, we denote $(\mathbf{0},\mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{0} < \mathbf{x} < \mathbf{b}\}$ and $(\mathbf{b},\infty) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{b} < \mathbf{x} < \infty\}$.

Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = \mathbb{R}^n_+$, the Lebesgue measure $d\mu_1(\mathbf{x}) = d\mathbf{x}$ and $d\mu_2(\mathbf{y}) = d\mathbf{y}$, and the kernel $k : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ of the form $k(\mathbf{x},\mathbf{y}) = l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)$, where $l : \mathbb{R}^n_+ \to \mathbb{R}$ is a non-negative measurable function, we obtain the following theorem.

THEOREM 5.1. *Let l and u be non-negative measurable functions on $\mathbb{R}^n_+$, such that $0 < L(\mathbf{x}) = \mathbf{x}^{\mathbf{1}} \int_{\mathbb{R}^n_+} l(\mathbf{y})\,d\mathbf{y} < \infty$ for all $\mathbf{x} \in \mathbb{R}^n_+$, and that the function $\mathbf{x} \mapsto u(\mathbf{x})\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}$ is integrable on $\mathbb{R}^n_+$ for each fixed $\mathbf{y} \in \mathbb{R}^n_+$. Let the function v be defined on $\mathbb{R}^n_+$ by*

$$v(\mathbf{y}) = \int_{\mathbb{R}^n_+} u(\mathbf{x})\frac{l\left(\frac{\mathbf{y}}{\mathbf{x}}\right)}{L(\mathbf{x})}\,d\mathbf{x}.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\begin{aligned} &\int_{\mathbb{R}^n_+} v(\mathbf{y})\Phi(f(\mathbf{y}))\,d\mathbf{y} - \int_{\mathbb{R}^n_+} u(\mathbf{x})\Phi(A_l f(\mathbf{x}))\,d\mathbf{x} \\ &\geqslant \int_{\mathbb{R}^n_+} \frac{u(\mathbf{x})}{L(\mathbf{x})} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) R_{\Phi,l} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\,d\mathbf{x} \end{aligned} \tag{5.1}$$

holds for all measurable functions $f : \mathbb{R}^n_+ \to \mathbb{R}$ with values in I, where $A_l f(\mathbf{x})$ and $R_{\Phi,l} f(\mathbf{x},\mathbf{y})$ are for $\mathbf{x},\mathbf{y} \in \mathbb{R}^n_+$ respectively defined by

$$A_l f(\mathbf{x}) = \frac{1}{L(\mathbf{x})} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y})\,d\mathbf{y}$$

and

$$R_{\Phi,l} f(\mathbf{x},\mathbf{y}) = \big|\,|\Phi(f(\mathbf{y})) - \Phi(A_l f(\mathbf{x}))| - |\varphi(A_l f(\mathbf{x}))| \cdot |f(\mathbf{y}) - A_l f(\mathbf{x})|\,\big|. \tag{5.2}$$

If the function Φ is concave, the order of integrals on the left-hand side of (5.1) *is reversed.*

Especially, for $\int_{\mathbb{R}^n_+} l(\mathbf{t})\,d\mathbf{t} = 1$ and $u(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}}$, Theorem 5.1 becomes the following refinement of Godunova's inequality (1.5).

COROLLARY 5.1. *Let* $l : \mathbb{R}^n_+ \to \mathbb{R}$ *be a non-negative measurable function and* $\int_{\mathbb{R}^n_+} l(\mathbf{t})\,d\mathbf{t} = 1$. *If* Φ *is a convex function on an interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \operatorname{Int} I$*, then the inequality*

$$\int_{\mathbb{R}^n_+} \Phi(f(\mathbf{y}))\,\frac{d\mathbf{y}}{\mathbf{y}} - \int_{\mathbb{R}^n_+} \Phi(A_l f(\mathbf{x}))\,\frac{d\mathbf{x}}{\mathbf{x}} \geqslant \int_{\mathbb{R}^n_+} \mathbf{x}^{-2} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) R_{\Phi,l} f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\,d\mathbf{x} \tag{5.3}$$

holds for all measurable functions $f : \mathbb{R}^n_+ \to \mathbb{R}$ *with values in* I*, where*

$$A_l f(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}} \int_{\mathbb{R}^n_+} l\left(\frac{\mathbf{y}}{\mathbf{x}}\right) f(\mathbf{y})\,d\mathbf{y},\ \mathbf{x} \in \mathbb{R}^n_+,$$

and $R_{\Phi,l} f$ *is defined by* (5.2)*. If* Φ *is concave, the integrals on the left-hand side of* (5.3) *are given in the reverse order.*

To conclude the paper, we give n-dimensional analogues of some results from Section 3, that is, some new multidimensional refined general Hardy–type inequalities. These results can be regarded as refinements of those obtained in [25]. Namely, the following theorem is a refinement of Lemma 2.1 in [25].

THEOREM 5.2. *Suppose that* $\mathbf{0} < \mathbf{b} \leqslant \infty$*, that* u *is a weight on* $(\mathbf{0},\mathbf{b})$*, such that the function* $\mathbf{x} \mapsto \frac{u(\mathbf{x})}{\mathbf{x}^{\mathbf{2}}}$ *is locally integrable in* $(\mathbf{0},\mathbf{b})$*, and that the weight* w *is defined by*

$$w(\mathbf{y}) = \mathbf{y}^{\mathbf{1}} \int_{(\mathbf{y},\mathbf{b})} u(\mathbf{x})\,\frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{2}}},\ \mathbf{y} \in (\mathbf{0},\mathbf{b}).$$

Let $\Phi : I \to \mathbb{R}$ *be a convex function and* $\varphi : I \to \mathbb{R}$ *be any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \operatorname{Int} I$*. If* $f : (\mathbf{0},\mathbf{b}) \to \mathbb{R}$ *is a measurable function, such that* $f(\mathbf{y}) \in I$ *for all* $y \in (\mathbf{0},\mathbf{b})$*, and* $Hf(\mathbf{x})$ *and* $R_\Phi f(\mathbf{x},\mathbf{y})$ *are for* $\mathbf{x},\mathbf{y} \in (\mathbf{0},\mathbf{b})$ *respectively defined by*

$$Hf(\mathbf{x}) = \mathbf{x}^{-\mathbf{1}} \int_{(\mathbf{0},\mathbf{x})} f(\mathbf{y})\,d\mathbf{y}$$

and

$$R_\Phi f(\mathbf{x},\mathbf{y}) = \big|\,|\Phi(f(\mathbf{y})) - \Phi(Hf(\mathbf{x}))| - |\varphi(Hf(\mathbf{x}))| \cdot |f(\mathbf{y}) - Hf(\mathbf{x})|\,\big|,$$

then

$$\begin{aligned}\int_{(\mathbf{0},\mathbf{b})} w(\mathbf{y})\Phi(f(\mathbf{y}))\,\frac{d\mathbf{y}}{\mathbf{y}^{\mathbf{1}}} &- \int_{(\mathbf{0},\mathbf{b})} u(\mathbf{x})\Phi(Hf(\mathbf{x}))\,\frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{1}}} \\ &\geqslant \int_{(\mathbf{0},\mathbf{b})} u(\mathbf{x}) \int_{(\mathbf{0},\mathbf{b})} R_\Phi f(\mathbf{x},\mathbf{y})\,d\mathbf{y}\,\frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{2}}}.\end{aligned} \tag{5.4}$$

If Φ *is concave, the order of integrals on the left-hand side of* (5.4) *is reversed.*

Proof. Let $S_1 = \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \mathbf{0} < \mathbf{y} \leqslant \mathbf{x} < \mathbf{b}\}$ and $\Omega_1 = \Omega_2 = (\mathbf{0},\mathbf{b})$. The proof follows directly from Theorem 2.1, applied with $d\mu_1(\mathbf{x}) = d\mathbf{x}$, $d\mu_2(\mathbf{y}) = d\mathbf{y}$, $k = \chi_{S_1}$, and with $u(\mathbf{x})$ replaced with $\frac{u(\mathbf{x})}{\mathbf{x}^{\mathbf{1}}}$. Note that $w(\mathbf{y}) = \mathbf{y}^{\mathbf{1}} v(\mathbf{y})$. □

REMARK 5.1. Observe that for $u(\mathbf{x}) \equiv 1$ we have $w(\mathbf{y}) = \left(\mathbf{1} - \frac{\mathbf{y}}{\mathbf{b}}\right)^{\mathbf{1}}$.

Our last result is dual to Theorem 5.2 and provides a refinement of Lemma 2.3 in [25].

THEOREM 5.3. *For $\mathbf{0} \leqslant \mathbf{b} < \infty$, let $u : (\mathbf{b}, \infty) \to \mathbb{R}$ be a locally integrable weight in $(\mathbf{b}, \infty)$, and the weight w be given by*

$$w(\mathbf{y}) = \mathbf{y}^{-\mathbf{1}} \int_{(\mathbf{b},\mathbf{y})} u(\mathbf{x})\, d\mathbf{x},\ \mathbf{y} \in (\mathbf{b}, \infty).$$

Suppose $\Phi : I \to \mathbb{R}$ is a convex function and $\varphi : I \to \mathbb{R}$ is any function, such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$. If $f : (\mathbf{b}, \infty) \to \mathbb{R}$ is a measurable function, such that $f(\mathbf{y}) \in I$ for all $y \in (\mathbf{b}, \infty)$, and $\tilde{H} f(\mathbf{x})$ and $\tilde{R}_\Phi f(\mathbf{x}, \mathbf{y})$ are for $\mathbf{x}, \mathbf{y} \in (\mathbf{b}, \infty)$ respectively defined by

$$\tilde{H} f(\mathbf{x}) = \mathbf{x}^{\mathbf{1}} \int_{(\mathbf{x},\infty)} f(\mathbf{y})\, \frac{d\mathbf{y}}{\mathbf{y}^{\mathbf{2}}}$$

and

$$\tilde{R}_\Phi f(\mathbf{x}, \mathbf{y}) = \left| \left|\Phi(f(\mathbf{y})) - \Phi(\tilde{H} f(\mathbf{x}))\right| - \left|\varphi(\tilde{H} f(\mathbf{x}))\right| \cdot \left|f(\mathbf{y}) - \tilde{H} f(\mathbf{x})\right| \right|,$$

then the inequality

$$\int_{(\mathbf{b},\infty)} w(\mathbf{y}) \Phi(f(\mathbf{y}))\, \frac{d\mathbf{y}}{\mathbf{y}^{\mathbf{1}}} - \int_{(\mathbf{b},\infty)} u(\mathbf{x}) \Phi(\tilde{H} f(\mathbf{x}))\, \frac{d\mathbf{x}}{\mathbf{x}^{\mathbf{1}}} \geqslant \int_{(\mathbf{b},\infty)} u(\mathbf{x}) \int_{(\mathbf{x},\infty)} \tilde{R}_\Phi f(\mathbf{x}, \mathbf{y})\, \frac{d\mathbf{y}}{\mathbf{y}^{\mathbf{2}}}\, d\mathbf{x} \tag{5.5}$$

holds. If the function Φ is concave, the order of integrals on the left-hand side of (5.5) *is reversed.*

Proof. Let $S_2 = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \mathbf{b} < \mathbf{x} \leqslant \mathbf{y} < \infty\}$ and $\Omega_1 = \Omega_2 = (\mathbf{b}, \infty)$. The proof follows directly from Theorem 2.1, rewritten with the Lebesgue measures $d\mu_1(\mathbf{x}) = d\mathbf{x}$, $d\mu_2(\mathbf{y}) = d\mathbf{y}$, the kernel $k(\mathbf{x}, \mathbf{y}) = \mathbf{y}^{-\mathbf{2}} \chi_{S_2}(\mathbf{x}, \mathbf{y})$, and with the weight $\frac{u(\mathbf{x})}{\mathbf{x}^{\mathbf{1}}}$ instead of $u(\mathbf{x})$. Note that $w(\mathbf{y}) = \mathbf{y}^{\mathbf{1}} v(\mathbf{y})$. □

REMARK 5.2. Observe that for $u(\mathbf{x}) \equiv 1$ we get $w(\mathbf{y}) = \left(\mathbf{1} - \frac{\mathbf{b}}{\mathbf{y}}\right)^{\mathbf{1}}$.

Of course, all results obtained in this section can be rewritten with particular convex (or concave) functions, for example, with power and exponential functions. This leads to multidimensional analogues of corollaries and examples from Section 3. Due to the lack of space, they are omitted.

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