Journal of
Mathematical
Inequalities
Volume 4, Number 4 (2010), 523–535

LINEAR INTEGRAL INEQUALITIES INVOLVING MAXIMA OF THE UNKNOWN SCALAR FUNCTIONS

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(*Communicated by A. Čižmešija*)

Abstract. This paper deals with linear integral inequalities that include the maximum of the unknown scalar function of one variable. The considered inequalities are generalizations of the classical integral inequality of Gronwall-Bellman. The importance of these integral inequalities is defined by their wide applications in qualitative investigations of differential equations with "maxima" and it is illustrated by some direct applications.

1. Introduction

Integral inequalities which provide explicit bounds of the unknown functions play a fundamental role in the development of the theory of differential and integral equations. In the past few years, a number of integral inequalities had been established by many scholars, which are motivated by certain applications such as existence, uniqueness, continuous dependence, comparison, perturbation, boundedness and stability of solutions of differential and integral equations (see for example [2] and the references cited therein). Among these integral inequalities, we cite the famous Gronwall inequality and its different generalizations ([3], [4], [7], [8], [9], [11]).

In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in the control theory correspond to the maximal deviation of the regulated quantity. For example, E. P. Popov ([12]) in 1966 considered the system for regulating the voltage of a generator of constant current. The object of the experiment was a generator of constant current with parallel simulation and the regulated quantity was the voltage at the source electric current. The equation describing the work of the regulator involves the maximum of the unknown function and it has the form ([12])

$$T_0 u'(t) + u(t) + q \max_{s\in[t-h,t]} u(s) = f(t),$$

where T_0 and q are constants characterizing the object, $u(t)$ is the regulated voltage and $f(t)$ is the perturbed effect.

Mathematics subject classification (2010): 26D10, 34D40.
Keywords and phrases: Integral inequalities, maxima, scalar functions of one variable, differential equations with "maxima".

Paper JMI-04-47

Later, N. Yoshida ([13]) used as a model differential equations with "maxima" when he studied the control of the temperature in a thermal system.

The purpose of this paper is to establish some new integral inequalities in the case when maxima of the unknown scalar function is involved in the integral. These inequalities are mathematical tools in the theory of differential equations with "maxima".

2. Main Results

Let $t_0 \geqslant 0$ and $T \geqslant t_0$ be fixed points. Note that T could be equal to ∞.

THEOREM 1. *Let the following conditions be fulfilled:*

1. The function $\alpha \in C^1([t_0,T),\mathbb{R}_+)$ is nondecreasing and $\alpha(t) \leqslant t$.

2. The functions $f,\ g \in C(\mathbb{R}_+,[1,\infty))$.

3. The functions $p,\ q \in C([t_0,T),\mathbb{R}_+)$ and $a,\ b \in C([\alpha(t_0),T),\mathbb{R}_+)$.

4. The function $\phi \in C([\alpha(t_0)-h,t_0],\mathbb{R}_+)$ and $M = \max_{s\in[\alpha(t_0)-h,t_0]} \phi(s)$.

5. The function $k \in C([t_0,T),(0,\infty))$ is nondecreasing and the inequality $M \geqslant k(t_0)$ holds.

6. The function $u \in C([\alpha(t_0)-h,T),\mathbb{R}_+)$ satisfies the inequalities

$$\begin{aligned} u(t) \leqslant k(t) + f(t)\int_{t_0}^{t}\Big[p(s)u(s)+q(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds \\ + g(t)\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)u(s)+b(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds \quad \text{for } t\in[t_0,T), \end{aligned} \tag{1}$$

$$u(t) \leqslant \phi(t) \qquad \text{for } t \in [\alpha(t_0)-h,t_0], \tag{2}$$

where $h = const \geqslant 0$.

Then for $t \in [t_0,T)$ the inequality

$$\begin{aligned} u(t) \leqslant f(t)g(t)\frac{k(t)}{k(t_0)}M\exp\Big(\int_{t_0}^{t}\Big[p(s)f(s)g(s)+q(s)\max_{\xi\in[s-h,s]}\big(f(\xi)g(\xi)\big)\Big]ds\Big) \\ \times\exp\Big(\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)f(s)g(s)+b(s)\max_{\xi\in[s-h,s]}\big(f(\xi)g(\xi)\big)\Big]ds\Big) \end{aligned} \tag{3}$$

holds.

Proof. From inequality (1) we obtain for $t \in [t_0,T)$ the inequality

$$\begin{aligned} \frac{u(t)}{k(t)} \leqslant 1 + f(t)\int_{t_0}^{t}\Big[p(s)\frac{u(s)}{k(t)}+q(s)\frac{\max_{\xi\in[s-h,s]}u(\xi)}{k(t)}\Big]ds \\ + g(t)\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)\frac{u(s)}{k(t)}+b(s)\frac{\max_{\xi\in[s-h,s]}u(\xi)}{k(t)}\Big]ds. \end{aligned} \tag{4}$$

From the monotonicity of $k(t)$ we obtain for $t \in [t_0,T)$ and $s \in [\alpha(t_0),t]$ the inequality

$$\frac{\max_{\xi\in[s-h,s]}u(\xi)}{k(t)} \leqslant \frac{\max_{\xi\in[s-h,s]}u(\xi)}{\hat{k}(s)} = \max_{\xi\in[s-h,s]}\frac{u(\xi)}{\hat{k}(s)} \leqslant \max_{\xi\in[s-h,s]}\frac{u(\xi)}{\hat{k}(\xi)}, \tag{5}$$

where the continuous nondecreasing function $\hat{k} : [\alpha(t_0) - h, T) \to \mathbb{R}_+$ is defined by

$$\hat{k}(t) = \begin{cases} k(t) & \text{for } t \in [t_0, T) \\ k(t_0) & \text{for } t \in [\alpha(t_0) - h, t_0]. \end{cases}$$

Define a function $\varphi : [\alpha(t_0) - h, T) \to \mathbb{R}_+$ by the equality $\varphi(t) = \dfrac{u(t)}{\hat{k}(t)}$.

From inequalities (4), (5) and the definition of the function $\varphi(t)$ follows that

$$\begin{aligned} \varphi(t) \leqslant 1 + f(t) \int_{t_0}^{t} \Big[p(s)\varphi(s) + q(s) \max_{\xi \in [s-h,s]} \varphi(\xi) \Big] ds \\ + g(t) \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)\varphi(s) + b(s) \max_{\xi \in [s-h,s]} \varphi(\xi) \Big] ds \quad \text{for } t \in [t_0, T), \end{aligned} \tag{6}$$

$$\varphi(t) = \frac{u(t)}{k(t_0)} \leqslant \frac{\phi(t)}{k(t_0)} \leqslant \frac{M}{k(t_0)} \qquad \text{for } t \in [\alpha(t_0) - h, t_0]. \tag{7}$$

Then for $t \in [t_0, T)$ the following inequality is valid

$$\begin{aligned} \varphi(t) \leqslant f(t)g(t) \Bigg(\frac{M}{k(t_0)} + \int_{t_0}^{t} \Big[p(s)\varphi(s) + q(s) \max_{\xi \in [s-h,s]} \varphi(\xi) \Big] ds \\ + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)\varphi(s) + b(s) \max_{\xi \in [s-h,s]} \varphi(\xi) \Big] ds \Bigg). \end{aligned} \tag{8}$$

Let us define a function $V : [\alpha(t_0) - h, T) \to \mathbb{R}_+$ by

$$V(t) = \begin{cases} \frac{M}{k(t_0)} + \int_{t_0}^{t} \Big[p(s)\varphi(s) + q(s) \max_{\xi \in [s-h,s]} \varphi(\xi) \Big] ds \\ \quad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)\varphi(s) + b(s) \max_{\xi \in [s-h,s]} \varphi(\xi) \Big] ds, & t \in [t_0, T) \\ \frac{M}{k(t_0)}, & t \in [\alpha(t_0) - h, t_0] \end{cases} \tag{9}$$

where M is defined by condition 4 of Theorem 1.

Note the function $V(t)$ is nondecreasing and

$$\varphi(t) \leqslant f(t)g(t)V(t) \quad \text{for } t \in [\alpha(t_0) - h, T). \tag{10}$$

Therefore, $\max_{s \in [t-h,t]} \varphi(s) \leqslant V(t) \max_{s \in [t-h,t]} \big(f(s)g(s) \big)$ for $t \in [\alpha(t_0), T)$. Then from the definition of the function $V(t)$ and inequality (8) we get for $t \in [t_0, T)$

$$\begin{aligned} V(t) \leqslant \frac{M}{k(t_0)} + \int_{t_0}^{t} \Big[p(s)f(s)g(s) + q(s) \max_{\xi \in [s-h,s]} \big(f(\xi)g(\xi) \big) \Big] V(s) ds \\ + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)f(s)g(s) + b(s) \max_{\xi \in [s-h,s]} \big(f(\xi)g(\xi) \big) \Big] V(s) ds. \end{aligned} \tag{11}$$

By a simple change of variable $s=\alpha(\eta)$ into the second integral of inequality (11) we obtain

$$\begin{aligned} V(t) \leqslant \frac{M}{k(t_0)} &+ \int_{t_0}^{t}\Big[p(s)f(s)g(s)+q(s)\max_{\xi\in[s-h,s]}\big(f(\xi)g(\xi)\big)\Big]V(s)ds \\ &+ \int_{t_0}^{t}\Big[a\big(\alpha(\eta)\big)f\big(\alpha(\eta)\big)g\big(\alpha(\eta)\big)\alpha'(\eta) \\ &+ b\big(\alpha(\eta)\big)\alpha'(\eta)\max_{\xi\in[\alpha(\eta)-h,\alpha(\eta)]}\big(f(\xi)g(\xi)\big)\Big]V\big(\alpha(\eta)\big)d\eta. \end{aligned} \tag{12}$$

We apply Gronwall inequality to (12) and obtain

$$\begin{aligned} V(t) \leqslant \frac{M}{k(t_0)} &\exp\Big(\int_{t_0}^{t}\Big[p(s)f(s)g(s)+q(s)\max_{\xi\in[s-h,s]}\big(f(\xi)g(\xi)\big)\Big]ds\Big) \\ &\times\exp\Big(\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)f(s)g(s)+b(s)\max_{\xi\in[s-h,s]}\big(f(\xi)g(\xi)\big)\Big]ds\Big). \end{aligned} \tag{13}$$

From inequalities (10), (13) and the definitions of the functions $\varphi(t)$ and $\hat{k}(t)$ we obtain the required inequality (3).

□

As a special case of Theorem 1 we obtain the following result:

THEOREM 2. *Let the following conditions be satisfied:*
1. The function $\alpha\in C^1([t_0,T),\mathbb{R}_+)$ *is nondecreasing and* $\alpha(t)\leqslant t$.
2. The functions $p,\ q\in C([t_0,T),\mathbb{R}_+)$ *and* $a,\ b\in C([\alpha(t_0),T),\mathbb{R}_+)$.
3. The function $\phi\in C([\alpha(t_0)-h,t_0],\mathbb{R}_+)$.
4. The function $u\in C([\alpha(t_0)-h,T),\mathbb{R}_+)$ *satisfies the inequalities*

$$\begin{aligned} u(t)\leqslant k &+ \int_{t_0}^{t}\Big[p(s)u(s)+q(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)u(s)+b(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds \quad \text{for } t\in[t_0,T), \end{aligned} \tag{14}$$

$$u(t)\leqslant\phi(t) \qquad \text{for } t\in[\alpha(t_0)-h,t_0], \tag{15}$$

where $h=const\geqslant 0,\ k=const>0$ *such that* $k\leqslant\max_{s\in[t_0-h,t_0]}\phi(s)$.

Then for $t\in[t_0,T)$ *inequality*

$$u(t)\leqslant\Big(\max_{s\in[t_0-h,t_0]}\phi(s)\Big)\exp\Big(\int_{t_0}^{t}\Big[p(s)+q(s)\Big]ds+\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)+b(s)\Big]ds\Big) \tag{16}$$

holds.

REMARK 1. As a special case of Theorem 1 we obtain a result for integral inequality without maximum ([11], Theorem 1).

In the case when the function $k(t)$ involved into the right part of inequality (1) is not a monotonic function, we obtain the following result:

THEOREM 3. *Let the following conditions be fulfilled:*
1. The conditions 1, 2, 3 of Theorem 1 are satisfied.
2. The function $k \in C([\alpha(t_0)-h,T),\mathbb{R}_+)$, $max_{s\in[\alpha(t_0)-h,t_0]}k(s) > 0$.
3. The function $u \in C([\alpha(t_0)-h,T),\mathbb{R}_+)$ *satisfies the inequalities*

$$\begin{aligned} u(t) \leqslant k(t) + f(t)\int_{t_0}^{t}\Big[p(s)u(s) + q(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds \\ + g(t)\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)u(s) + b(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds \quad \text{for } t\in[t_0,T), \end{aligned} \tag{17}$$

$$u(t) \leqslant k(t) \qquad \text{for } t\in[\alpha(t_0)-h,t_0], \tag{18}$$

where $h = const \geqslant 0$.

Then for $t\in[t_0,T)$ *the inequality*

$$\begin{aligned} u(t) \leqslant k(t) + f(t)g(t)e(t)\exp\left(\int_{t_0}^{t}\Big[p(s)f(s)g(s) + q(s)\max_{\xi\in[s-h,s]}\big(f(\xi)g(\xi)\big)\Big]ds\right) \\ \times\exp\left(\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)f(s)g(s) + b(s)\max_{\xi\in[s-h,s]}\big(f(\xi)g(\xi)\big)\Big]ds\right) \end{aligned} \tag{19}$$

holds, where the function $e:\ [t_0,T)\to(0,\infty)$ *is defined by*

$$\begin{aligned} e(t) = \max_{s\in[\alpha(t_0)-h,t_0]}k(s) + \int_{t_0}^{t}\Big[p(s)k(s) + q(s)\max_{\xi\in[s-h,s]}k(\xi)\Big]ds \\ + \int_{\alpha(t_0)}^{\max(\alpha(t),t_0)}\Big[a(s)k(s) + b(s)\max_{\xi\in[s-h,s]}k(\xi)\Big]ds. \end{aligned} \tag{20}$$

Proof. From inequality (17) for $t\in[t_0,T)$ we obtain

$$\begin{aligned} u(t) \leqslant k(t) + f(t)g(t)\Bigg(\int_{t_0}^{t}\Big[p(s)u(s) + q(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds \\ + \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)u(s) + b(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds\Bigg). \end{aligned} \tag{21}$$

Let us define a function $z:[\alpha(t_0)-h,T)\to\mathbb{R}_+$ by

$$z(t) = \begin{cases} \int_{t_0}^{t}\Big[p(s)u(s) + q(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds \\ \quad + \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)u(s) + b(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds, & t\in[t_0,T) \\ 0, & t\in[\alpha(t_0)-h,t_0). \end{cases} \tag{22}$$

The function $z(t)$ is an increasing function and from inequality (21) it follows that

$$u(t) \leqslant k(t) + f(t)g(t)z(t) \quad \text{for } [\alpha(t_0) - h, T). \tag{23}$$

Let $t \in [t_0, T)$ be such that $\alpha(t) \geqslant t_0$. Then from inequality (23) it follows the validity of the inequality

$$\begin{aligned}
&\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)u(s) + b(s) \max_{\xi\in[s-h,s]} u(\xi)\Big] ds \\
&\quad \leqslant \int_{\alpha(t_0)}^{\max(\alpha(t),t_0)} \Big[a(s)k(s) + b(s) \max_{\xi\in[s-h,s]} k(\xi)\Big] ds \\
&\qquad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)f(s)g(s) + b(s) \max_{\xi\in[s-h,s]} \big(f(\xi)g(\xi)\big)\Big] z(s) ds.
\end{aligned} \tag{24}$$

Let $t \in [t_0, T)$ be such that $\alpha(t) < t_0$. Then from the definition of function $z(t)$ we get

$$\begin{aligned}
&\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)u(s) + b(s) \max_{\xi\in[s-h,s]} u(\xi)\Big] ds \\
&\quad = \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)k(s) + b(s) \max_{\xi\in[s-h,s]} k(\xi)\Big] ds \\
&\quad \leqslant \int_{\alpha(t_0)}^{\max(\alpha(t),t_0)} \Big[a(s)k(s) + b(s) \max_{\xi\in[s-h,s]} k(\xi)\Big] ds \\
&\qquad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)f(s)g(s) + b(s) \max_{\xi\in[s-h,s]} \big(f(\xi)g(\xi)\big)\Big] z(s) ds.
\end{aligned} \tag{25}$$

From the definition of the function $z(t)$ and inequalities (24), (25) follows that

$$\begin{aligned}
z(t) \leqslant e(t) &+ \int_{t_0}^{t} \Big[p(s)f(s)g(s) + q(s) \max_{\xi\in[s-h,s]} \big(f(\xi)g(\xi)\big)\Big] z(s) ds \\
&+ \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)f(s)g(s) + b(s) \max_{\xi\in[s-h,s]} \big(f(\xi)g(\xi)\big)\Big] z(s) ds, \qquad t \in [t_0, T),
\end{aligned} \tag{26}$$

$$z(t) = 0, \qquad t \in [\alpha(t_0) - h, t_0], \tag{27}$$

where function $e(t)$ is defined by equality (20). Note that function $e : [t_0, T) \to (0, \infty)$ is nondecreasing for $t \in [t_0, T)$ and $e(t_0) = \max_{s\in[t_0-h,t_0]} k(s)$.

From inequalities (26), (27) as in the proof of Theorem 1 we get

$$\begin{aligned}
z(t) \leqslant e(t) &\exp\bigg(\int_{t_0}^{t} \Big[p(s)f(s)g(s) + q(s) \max_{\xi\in[s-h,s]} \big(f(\xi)g(\xi)\big)\Big] ds \bigg) \\
&\times \exp\bigg(\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)f(s)g(s) + b(s) \max_{\xi\in[s-h,s]} \big(f(\xi)g(\xi)\big)\Big] ds \bigg).
\end{aligned} \tag{28}$$

From inequalities (28) and (23) we obtain inequality (19).

□

Now we will consider an inequality in which the unknown function into the left part is presented in a power.

THEOREM 4. *Let the following conditions be fulfilled:*

1. The conditions 1, 2, 3, 4 of Theorem 1 are satisfied.

2. The function $k \in C([t_0,T),(0,\infty))$ *is nondecreasing and the inequality* $M = \max_{s\in[\alpha(t_0)-h,t_0]} \phi(s) \geqslant \sqrt[n]{k(t_0)}$ *holds.*

3. The function $u \in C([\alpha(t_0)-h,T),\mathbb{R}_+)$ *satisfies the inequalities*

$$\begin{aligned} u^n(t) \leqslant k(t) + f(t)\int_{t_0}^{t}\Big[p(s)u(s)+q(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds \\ + g(t)\int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)u(s)+b(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds \quad \text{for } t\in[t_0,T), \end{aligned} \tag{29}$$

$$u(t) \leqslant \phi(t) \qquad \text{for } t \in [\alpha(t_0)-h,t_0], \tag{30}$$

where $h = const \geqslant 0$, $n = const > 1$.

Then for $t \in [t_0,T)$ *the inequality*

$$u(t) \leqslant \sqrt[n]{k(t)} + f(t)g(t)\left(M + \frac{e(t)}{n\,\big(k(t_0)\big)^{1-\frac{1}{n}}}\right)\exp\Big(A(t)+B(t)\Big) \tag{31}$$

holds, where

$$\begin{aligned} e(t) = \int_{t_0}^{t}\Big[p(s)\omega(s)+q(s)\max_{\xi\in[s-h,s]}\omega(\xi)\Big]ds \\ + \int_{\alpha(t_0)}^{\max(\alpha(t),t_0)}\Big[a(s)\omega(s)+b(s)\max_{\xi\in[s-h,s]}\omega(\xi)\Big]ds, \end{aligned} \tag{32}$$

$$A(t) = \frac{1}{n}\int_{t_0}^{t}\Big(k(s)\Big)^{\frac{1-n}{n}}\Big[p(s)f(s)g(s)+q(s)\max_{\xi\in[s-h,s]}\big(f(\xi)g(\xi)\big)\Big]ds, \tag{33}$$

$$B(t) = \frac{1}{n}\int_{\alpha(t_0)}^{\alpha(t)}\Big(K(s)\Big)^{\frac{1-n}{n}}\Big[a(s)f(s)g(s)+b(s)\max_{\xi\in[s-h,s]}\big(f(\xi)g(\xi)\big)\Big]ds, \tag{34}$$

$$K(t) = \begin{cases} k(t), \; t\in[t_0,T) \\ k(t_0), \; t\in[\alpha(t_0),t_0), \end{cases} \qquad \omega(t) = \begin{cases} \sqrt[n]{k(t)}, \; t\in(t_0,T) \\ M, \quad t\in[\alpha(t_0)-h,t_0]. \end{cases}$$

Proof. From inequality (29) for $t \in [t_0,T)$ we get

$$\begin{aligned} u^n(t) \leqslant k(t) + f(t)g(t)\Bigg(\int_{t_0}^{t}\Big[p(s)u(s)+q(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds \\ + \int_{\alpha(t_0)}^{\alpha(t)}\Big[a(s)u(s)+b(s)\max_{\xi\in[s-h,s]}u(\xi)\Big]ds\Bigg). \end{aligned} \tag{35}$$

Let us define a function $z:[\alpha(t_0)-h,T)\to\mathbb{R}_+$ by

$$z(t)=\begin{cases}\frac{\sqrt[n]{k(t)}}{n\,k(t)}\Big(\int_{t_0}^{t}\Big(p(s)u(s)+q(s)\max_{\xi\in[s-h,s]}u(\xi)\Big)ds\\ \quad+\int_{\alpha(t_0)}^{\alpha(t)}\Big(a(s)u(s)+b(s)\max_{\xi\in[s-h,s]}u(\xi)\Big)ds\Big), & t\in[t_0,T)\\ 0, & t\in[\alpha(t_0)-h,t_0).\end{cases} \tag{36}$$

From inequality (35) we have for $t\in[t_0,T)$

$$u^n(t)\leqslant k(t)\left(1+nf(t)g(t)\frac{z(t)}{\sqrt[n]{k(t)}}\right).$$

or

$$u(t)\leqslant\sqrt[n]{k(t)}\left(1+nf(t)g(t)\frac{z(t)}{\sqrt[n]{k(t)}}\right)^{\frac{1}{n}}.$$

Apply Bernoulli's inequality $(1+x)^a\leqslant 1+ax$, where $0<a<1$ and $-1<x$, and observe that

$$\begin{aligned}u(t)&\leqslant\sqrt[n]{k(t)}\left(1+f(t)g(t)\frac{z(t)}{\sqrt[n]{k(t)}}\right)\\ &=\sqrt[n]{k(t)}+f(t)g(t)z(t)=\omega(t)+f(t)g(t)z(t),\quad t\in[t_0,T),\end{aligned} \tag{37}$$

and

$$u(t)\leqslant\phi(t)\leqslant\omega(t),\qquad t\in[\alpha(t_0)-h,t_0]. \tag{38}$$

Therefore,

$$\max_{\xi\in[s-h,s]}u(\xi)\leqslant\max_{\xi\in[s-h,s]}\omega(\xi)+z(s)\max_{\xi\in[s-h,s]}\big(f(\xi)g(\xi)\big)\quad s\in[\alpha(t_0),T). \tag{39}$$

Let $t\in[t_0,T)$ be such that $\alpha(t)\geqslant t_0$. Then from inequalities (37), (38) we get

$$\begin{aligned}&\int_{\alpha(t_0)}^{\alpha(t)}\left[a(s)u(s)+b(s)\max_{\xi\in[s-h,s]}u(\xi)\right]ds\\ &\quad\leqslant\int_{\alpha(t_0)}^{\max(\alpha(t),t_0)}\left[a(s)\omega(s)+b(s)\max_{\xi\in[s-h,s]}\omega(\xi)\right]ds\\ &\qquad+\int_{\alpha(t_0)}^{\alpha(t)}\left[a(s)f(s)g(s)+b(s)\max_{\xi\in[s-h,s]}\big(f(\xi)g(\xi)\big)\right]z(s)ds.\end{aligned} \tag{40}$$

Let $t \in [t_0, T)$ be such that $\alpha(t) < t_0$. Then from the definition of function $z(t)$ and inequalities (37), (38) we get

$$\begin{aligned}
&\int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)u(s) + b(s) \max_{\xi\in[s-h,s]} u(\xi)\Big] ds \\
&\quad = \int_{\alpha(t_0)}^{\max(\alpha(t),t_0)} \Big[a(s)\omega(s) + b(s) \max_{\xi\in[s-h,s]} \omega(\xi)\Big] ds. \qquad (41)\\
&\qquad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)f(s)g(s) + b(s) \max_{\xi\in[s-h,s]} \big(f(\xi)g(\xi)\big)\Big] z(s)ds.
\end{aligned}$$

Let $C = Mn\,k(t_0)^{1-\frac{1}{n}} > 0$. Note the function $v : [t_0, T) \to (0,\infty)$, $v(t) = \frac{1}{n\,k(t_0)^{1-\frac{1}{n}}} \Big(C + e(t)\Big)$ is nondecreasing and the equality $v(t_0) = \frac{1}{n\,k(t_0)^{1-\frac{1}{n}}} \Big(C + e(t_0)\Big) = M$ holds, where function $e(t)$ is defined by (32). From the definition of the function $z(t)$ and inequalities (40), (41) follows that

$$\begin{aligned}
z(t) \leqslant & v(t) + \frac{1}{n\,k(t)^{1-\frac{1}{n}}} \int_{t_0}^{t} \Big[p(s)f(s)g(s)z(s) + q(s) \max_{\xi\in[s-h,s]} \big(f(\xi),g(\xi)\big)z(s)\Big] ds \\
& + \frac{1}{n\,k(t)^{1-\frac{1}{n}}} \int_{\alpha(t_0)}^{\alpha(t)} \Big[a(s)f(s)g(s)z(s) + b(s) \max_{\xi\in[s-h,s]} \big(f(\xi),g(\xi)\big)z(s)\Big] ds \\
\leqslant & v(t) + \int_{t_0}^{t} \frac{1}{n} \left[\frac{p(s)f(s)g(s)}{\big(k(s)\big)^{1-\frac{1}{n}}} z(s) + \frac{q(s)\max_{\xi\in[s-h,s]}\big(f(\xi),g(\xi)\big)}{\big(k(s)\big)^{1-\frac{1}{n}}} z(s)\right] ds \\
& + \int_{\alpha(t_0)}^{\alpha(t)} \frac{1}{n} \left[\frac{a(s)f(s)g(s)}{\big(K(s)\big)^{1-\frac{1}{n}}} z(s) + \frac{b(s)\max_{\xi\in[s-h,s]}\big(f(\xi),g(\xi)\big)}{\big(K(s)\big)^{1-\frac{1}{n}}} z(s)\right] ds, \\
& \qquad \text{for} \quad t \in [t_0, T),
\end{aligned} \qquad (42)$$

$$z(t) \leqslant v(t_0) = M, \qquad t \in [\alpha(t_0) - h, t_0]. \qquad (43)$$

From inequalities (42), (43) according to Theorem 1 we get

$$z(t) \leqslant \left(M + \frac{e(t)}{n\,\big(k(t_0)\big)^{1-\frac{1}{n}}}\right) \exp\Big(A(t) + B(t)\Big), \qquad (44)$$

where $A(t)$ and $B(t)$ are defined by (33) and (34), respectively.

Substituting the bound (44) for $z(t)$ into the right part of (37) we obtain the required inequality (31).

□

REMARK 2. As special cases of Theorem 3 and Theorem 4 we obtain results for integral inequality without maximum ([9], Theorem 2.1 and Theorem 2.2).

3. Applications

We will apply some of the proved above inequalities to study properties of solutions of differential equations with "maxima". Note that differential equations with "maxima" are adequate models of real processes which present state depends on the maximal its deviation in the past. Such kind of problems are, for example, Hausrath equation ([13]), the model describing the vision process in the compound eye ([5]), the model of generator ([12]). On the other hand, it is relevant to mention here the opinion of Myshkis that "the specific character of these equations is not yet sufficiently clear". In his survey [10] he also distinguishes the equations with maxima as differential equations with deviating argument of complex structure.

EXAMPLE 1. Consider the system of differential equations with "maximum"

$$x^{'} = F\big(t, x(t), \max_{s\in[\beta(t),\alpha(t)]} x(s)\big) \text{ for } t \geqslant t_0 \tag{45}$$

with initial condition

$$x(t) = \phi(t) \quad \text{for} \quad t \in [\alpha(t_0) - h, t_0], \tag{46}$$

where $x \in \mathbb{R}^n, \phi : [t_0 - h, t_0] \to \mathbb{R}^n, F : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $h > 0$ is a constant.

THEOREM 5. (Uniqueness). *Let the following conditions be fulfilled:*

1. The functions $\alpha, \beta \in C([t_0,\infty), \mathbb{R}_+)$ are such that $\alpha(t)$ is an increasing function, $\beta(t) \leqslant \alpha(t) \leqslant t$, and there exists a constant $h > 0$: $0 < \alpha(t) - \beta(t) \leqslant h$ for $t \geqslant t_0$.

2. The function $F \in C([t_0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and satisfies for $t \geqslant t_0$ and $x_i, y_i \in \mathbb{R}^n$, $i = 1,2$ the condition

$$\|F(t,x_1,y_1) - F(t,x_2,y_2)\| \leqslant P(t)\,\|x_1 - x_2\| + r(t)\,\|y_1 - y_2\|,$$

where $P(t), r(t) \in C([t_0,\infty), \mathbb{R}_+)$.

3. For any initial function $\phi \in C([\alpha(t_0) - h, t_0], \mathbb{R}^n)$ the initial value problem (45),(46) has at least one solution $x(t;t_0,\phi)$ defined for $t \geqslant \alpha(t_0) - h$.

Then the initial problem (45),(46) has exactly one solution.

Proof. Let $\phi \in C([\alpha(t_0) - h, t_0], \mathbb{R}^n)$ be a fixed initial function. Assume there exist two different solutions $u(t) = u(t;t_0,\phi)$ and $v(t) = v(t;t_0,\phi)$ of the initial value problem (45),(46), defined for $t \geqslant \alpha(t_0) - h$. Both functions $u(t)$ and $v(t)$ satisfy the integral equations

$$u(t) = \phi(t_0) + \int_{t_0}^{t} F(s, u(s), \max_{\xi\in[\beta(s),\alpha(s)]} u(\xi))ds \text{ for } t \geqslant t_0,$$

$$v(t) = \phi(t_0) + \int_{t_0}^{t} F(s, v(s), \max_{\xi\in[\beta(s),\alpha(s)]} v(\xi))ds \text{ for } t \geqslant t_0$$

and $u(t) = v(t) = \phi(t)$ for $t \in [\alpha(t_0) - h, t_0]$.

Then the norm of the difference of the solutions $u(t)$ and $v(t)$ satisfies the inequalities

$$\begin{aligned} ||u(t)-v(t)|| &\leqslant \int_{t_0}^{t} ||F(s,u(s), \max_{\xi\in[\beta(s),\alpha(s)]} u(\xi))-F(s,v(s), \max_{\xi\in[\beta(s),\alpha(s)]} v(\xi))||ds \\ &\leqslant \int_{t_0}^{t} \Big(P(s)||u(s)-v(s)||ds+r(s)|| \max_{\xi\in[\beta(s),\alpha(s)]} u(\xi)- \max_{\xi\in[\beta(s),\alpha(s)]} v(\xi)||\Big)ds \\ &\leqslant \int_{t_0}^{t} P(s)\,\|u(s)-v(s)\|\,ds+\int_{t_0}^{t} r(s) \max_{\xi\in[\beta(s),\alpha(s)]} \|u(\xi)-v(\xi)\|ds, \qquad t\geqslant t_0, \end{aligned} \tag{47}$$

Set $w(t) = \|u(t) - v(t)\|$ for $t \in [\alpha(t_0) - h, \infty)$, change the variable $\eta = \alpha(s)$ in the second integral of (47), use the inequality $\max_{\xi\in[\beta(t),\,\alpha(t)]} w(\xi) \leqslant \max_{\xi\in[\alpha(t)-h,\,\alpha(t)]} w(\xi)$ that follows from condition 1 of Theorem 5 and obtain the inequality

$$w(t) \leqslant \int_{t_0}^{t} P(\eta)w(\eta)d\eta + \int_{\alpha(t_0)}^{\alpha(t)} r(\alpha^{-1}(\eta))(\alpha^{-1}(\eta))' \max_{\xi\in[\eta-h,\eta]} w(\xi)d\eta, \ \ t \geqslant t_0. \tag{48}$$

According to Theorem 2 from inequality (48) and $w(t) \equiv 0$, $t \in [\alpha(t_0) - h, t_0]$ for $k = 0$, $p(t) \equiv P(t)$, $q(t) \equiv 0$ on $[t_0, \infty)$, $a(t) \equiv 0$, $b(t) \equiv r(\alpha^{-1}(\eta))(\alpha^{-1}(\eta))'$ on $[\alpha(t_0), \infty)$ we obtain $w(t) \leqslant 0$ for $t \geqslant t_0$, that proves the validity of inequality $\|u(t) - v(t)\| = 0$ for $t \geqslant t_0$ or $u(t) \equiv v(t)$.

□

EXAMPLE 2. Consider the scalar differential equations with “maxima“

$$x\,x' = F\big(t, x(t), \max_{s\in[\beta(t),\alpha(t)]} x(s)\big) \ \text{ for } \ t \geqslant t_0 \tag{49}$$

with initial condition

$$x(t) = \phi(t) \quad \text{for} \quad t \in [\alpha(t_0) - h, t_0], \tag{50}$$

where $x \in \mathbb{R}$, $\phi : [\alpha(t_0) - h, t_0] \to \mathbb{R}$, $F : [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h > 0$ is a constant.

THEOREM 6. (Upper bound for solutions). *Let the following conditions be fulfilled:*

1. The functions $\alpha, \beta \in C([t_0, \infty), \mathbb{R}_+)$ are such that $\alpha(t)$ is an increasing function, $\beta(t) \leqslant \alpha(t) \leqslant t$ and there exists a constant $h > 0$: $0 < \alpha(t) - \beta(t) \leqslant h$ for $t \geqslant t_0$.

2. The function $F \in C([t_0, \infty) \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $F(t, 0, 0) \equiv 0$, $t \in \mathbb{R}$ and satisfies for $t \geqslant t_0$ and $x, y \in \mathbb{R}$ the condition $|F(t, x, y)| \leqslant P(t)|x| + r(t)|y|$, where $P(t), r(t) \in C([t_0, \infty), \mathbb{R}_+)$.

3. The function $\phi \in C([\alpha(t_0) - h, t_0], \mathbb{R})$ and $\max_{t\in[\alpha(t_0)-h,t_0]} |\phi(t)| = M > 0$.

4. The function $x(t;t_0,\phi)$ is the solution of the initial value problem (49),(50), defined for $t \geqslant \alpha(t_0)-h$.

Then the solution of initial value problem problem (49),(50) satisfies

$$|u(t)| \leqslant |\phi(t_0)| + M\Big(1+\frac{1}{|\phi(t_0)|}\int_{t_0}^{t}(P(s)+r(s))ds\Big)e^{\frac{1}{|\phi(t_0)|}\int_{t_0}^{t}(P(s)+r(s))ds}, \quad t \geqslant t_0. \tag{51}$$

Proof. Let $\phi \in C([\alpha(t_0)-h,t_0],\mathbb{R})$ be an initial function and $x(t)=x(t;t_0,\phi)$ be a solution of the initial value problem (49),(50), that is defined for $t \geqslant t_0-h$. Function $x(t)$ satisfies the integral equation

$$\begin{aligned}&\Big(x(t)\Big)^2 = \Big(\phi(t_0)\Big)^2 + \int_{t_0}^{t} 2F(s,u(s),\max_{s\in[\beta(t),\alpha(t)]}u(s))ds \ \text{ for } t \geqslant t_0,\\ &x(t)=\phi(t), \qquad t\in[\alpha(t_0)-h,t_0],\end{aligned}$$

Then we obtain

$$\begin{aligned}|x(t)|^2 &\leqslant |\phi(t_0)|^2 + 2\int_{t_0}^{t}|F(s,x(s),\max_{\xi\in[\beta(s),\alpha(s)]}x(\xi))|ds\\ &\leqslant |\phi(t_0)|^2 + 2\int_{t_0}^{t}P(s)|x(s)|ds + 2\int_{t_0}^{t}r(s)\max_{\xi\in[\beta(s),\alpha(s)]}|x(\xi)|ds, \ t\geqslant t_0.\end{aligned} \tag{52}$$

Set $u(t)=|x(t)|$ for $t\in[\alpha(t_0)-h,\infty)$, change the variable $\eta=\alpha(s)$ in the second integral of (52), use the inequality $\max_{\xi\in[\beta(t),\,\alpha(t)]}u(\xi) \leqslant \max_{\xi\in[\alpha(t)-h,\,\alpha(t)]}u(\xi)$ that follows from condition 1 of Theorem 6 and obtain for $t\geqslant t_0$ the following inequality

$$\Big(u(t)\Big)^2 \leqslant |\phi(t_0)|^2 + \int_{t_0}^{t}2P(\eta)u(\eta)d\eta + \int_{\alpha(t_0)}^{\alpha(t)}2r(\alpha^{-1}(\eta))(\alpha^{-1}(\eta))'\max_{\xi\in[\eta-h,\eta]}u(\xi)d\eta. \tag{53}$$

According to Theorem 4 for $n=2$,

$$f(t)=g(t)\equiv 1, \quad p(t)=2P(t), \quad b(t)=2r(\alpha^{-1}(t))(\alpha^{-1}(t))', \quad q(t)=a(t)=0$$

from inequality (53) we obtain

$$u(t) \leqslant |\phi(t_0)| + \Big(M+\frac{1}{2|\phi(t_0)|}e(t)\Big)e^{A(t)+B(t)}, \tag{54}$$

where

$$e(t) \leqslant 2M\int_{t_0}^{t}\Big(2P(s)+r(s)\Big)ds,$$

$$\begin{aligned}A(t)&=\frac{1}{|\phi(t_0)|}\int_{t_0}^{t}P(s)ds,\\ B(t)&=\frac{1}{2|\phi(t_0)|}\int_{\alpha(t_0)}^{\alpha(t)}\Big[2r(\alpha^{-1}(s))(\alpha^{-1}(s))'\Big]ds = \frac{1}{|\phi(t_0)|}\int_{t_0}^{t}r(s)ds.\end{aligned}$$

Inequality (54) proves the validity of inequality (51).

□

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(Received August 7, 2009)

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