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SOME NONLINEAR DYNAMIC INEQUALITIES ON TIME SCALES AND APPLICATIONS

S. H. SAKER

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Abstract. Some nonlinear dynamic inequalities on a time scale $\mathbb{T}$ are formulated in this paper. Some sufficient conditions for global existence and an estimate of the rate of decay of solutions are obtained. The results not only unify the results of differential and difference inequalities but can be applied on different types of time scales. These inequalities are of interest in a study of continuous and discrete dynamical systems and nonlinear evolution equations as well as in oscillation theory of dynamic equations on time scales and can be applied to the study of global existence of nonlinear PDE. Some applications illustrating the main results are given.

1. Introduction

One of the most useful methods for studying linear and nonlinear dynamic equations on time scales is the use of linear and nonlinear dynamic inequalities which provide explicit bounds on the unknown functions. During the past decade a number of dynamic inequalities on time scales has been established by some authors which are motivated by some applications, for example, when studying the behavior of solutions of certain class of dynamic equations on time scales, the bounds provided by earlier inequalities are inadequate in applications and we need some new and specific type of dynamic inequalities on time scales. For contributions, we refer the reader to [1], [2], [3], [4], [5], [6], [7], [8], [12], [13], [17], [19], and the references cited therein. In [5, Theorem 6.1] it is proved that if $y,\ a$ and $p \in C_{rd}$ and $p \in \mathscr{R}^+$, then

$$y^{\Delta}(t) \leqslant f(t) + p(t)y(t), \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

implies

$$y(t) \leqslant y(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(s))f(s)\Delta s, \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}, \tag{1.2}$$

where $\mathscr{R}^+ := \{a \in \mathscr{R} : 1+\mu(t)a(t) > 0,\ t \in \mathbb{T}\}$ and $\mathscr{R}$ is the class of rd-continuous and regressive functions.

Since (1.2) provides an explicit bound to the unknown function and a tool to the study of many qualitative as well as quantitative properties of solutions of dynamic

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equations, it has become one of the very few classic and most influential results in the theory and applications of inequalities. Following this trend in this paper, we consider the general nonlinear dynamic inequality

$$g^{\Delta}(t) \leqslant \beta(t) - \gamma(t)g^{\sigma}(t) + \alpha(t)g^{p}(t), \text{ for } t \geqslant t_0, \tag{1.3}$$

on a time scale $\mathbb{T}$ which is unbounded above, where $g(t) \geqslant 0$. Our concern in this paper is to establish some sufficient conditions for global existence and an estimate of the rate of decay of solutions to this inequality which is different from the usual task that has been considered in the above mentioned papers. We note that (1.1) is a special case of (1.3) when $\gamma(t) = 0$ and $p = 1$. Since we are interested in the asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$. For completeness, we recall the following concepts related to the notion of time scales. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd-continuous) provided f is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$ where $\sigma(t)$ is the forward jump operator defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two differentiable functions f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \text{ and } \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{1.4}$$

An integration by parts formula reads

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t)g^{\sigma}\Delta t. \tag{1.5}$$

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0,\ t \in \mathbb{T}$. The set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by $p \oplus q := p + q + \mu pq$. We denote the set of all $f : \mathbb{T} \to \mathbb{R}$ which are rd–continuous and regressive by $\mathscr{R}$. If $p \in \mathscr{R}$, then we can define the exponential function by $e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$, for $t \in \mathbb{T},\ s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Alternately, for $p \in \mathscr{R}$ one can define the exponential function $e_p(\cdot,t_0)$, to be the unique solution of the IVP $x^{\Delta} = p(t)x$, with $x(t_0) = 1$. We define $\mathscr{R}^+ := \{f \in \mathscr{R} : 1 + \mu(t)f(t) > 0,\ t \in \mathbb{T}\}$. From the properties of the exponential function, see Bohner and Peterson [5], we will use the following properties $e_a^{\Delta}(t,t_0) = a(t)e_a(t,t_0)$ and $e_p(\sigma(t),t_0) = [1 + \mu(t)p(t)]e_p(t,t_0)$. Also if $a \in \mathscr{R}$, then $e_a(t,s)$ is real-valued and nonzero on $\mathbb{T}$. If $a \in \mathscr{R}^+$, then $e_a(t,t_0)$ always positive and $e_p(t,t) = 1$ and $e_0(t,s) =$

1. Note that if $\mathbb{T}=\mathbb{R}$, then $e_a(t,t_0)=\exp(\int_{t_0}^{t} a(s)ds)$, if $\mathbb{T}=\mathbb{N}$, then $e_a(t,t_0)=\prod_{s=t_0}^{t-1}(1+a(s))$, and if $\mathbb{T}=q^{\mathbb{N}_0}$, then $e_a(t,t_0)=\prod_{s=t_0}^{t-1}(1+(q-1)sa(s))$.

The study of dynamic equations and inequalities on time scales, which goes back to its founder Stefan Hilger [10], is an area of mathematics that has recently received a lot of attention. The general idea is to prove a result for a dynamic equation or inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. The book on the subject of time scale, i.e., measure chain, by Bohner and Peterson [5] summarizes and organizes much of time scale calculus. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [11]), i.e, when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$ where $q>1$. There are applications of dynamic equations on time scales to *quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics*. A cover story article in New Scientist [20] discusses several possible applications.

Throughout the paper we will assume the following hypotheses:

$$(H)\quad \begin{cases} \alpha,\ \beta \text{ and } \gamma \text{ are real valued rd-continuous functions defined on } [t_0,\infty)_{\mathbb{T}}, \\ \qquad \text{such that } \gamma\in\mathscr{R}, \\ \alpha(t)\geqslant 0, \text{for all } t\geqslant t_0, \text{where } t_0\geqslant 0 \text{ is a fixed number and } p>0 \text{ is a constant.} \end{cases}$$

Also we consider the inequality

$$g^{\Delta}(t)\geqslant \beta(t)-\gamma(t)g^{\sigma}(t)+\alpha(t)g^{p}(t),\ \ \text{for } t\geqslant t_0, \tag{1.6}$$

on a time scale $\mathbb{T}$ which is unbounded above, and establish the lower bound of the solution $g(t)$ when $g(t_0)\neq 0$.

Our motivations for considering this type of inequalities are their applications of continuous and discrete dynamical systems. Also in studying the large time behavior of solutions to nonlinear evolution equations and oscillation of dynamic equations on time scales. Moreover these inequalities can be used in studying the stability of dynamical system method of the ill-posed operator equations and in the study of global existence of nonlinear PDE [15, 16, 18].

We note that, the inequalities (1.3) and (1.6) cover several different types of differential and difference inequalities depending on the choice of the time scale $\mathbb{T}$. We consider here (1.3) and give different types of inequalities and the reader can similarly construct the others for (1.6). For example, if $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$, $\mu(t)=0$, $g^{\Delta}(t)=g'(t)$ and (1.3) becomes the differential inequality

$$g'(t)\leqslant \beta(t)-\gamma(t)g(t)+\alpha(t)g^{p}(t),\ \text{for } t\geqslant t_0. \tag{1.7}$$

If $\mathbb{T}=\mathbb{N}$, then $\sigma(n)=n+1$, $\mu(n)=1$, $g^{\Delta}(n)=\Delta g(n)=g(n+1)-g(n)$ and (1.3) becomes the difference inequality

$$\Delta g(n)\leqslant \beta(n)-\gamma(n)g(n+1)+\alpha(n)g^{p}(n),\ \text{for } n\geqslant n_0. \tag{1.8}$$

If $\mathbb{T} = h\mathbb{N}$, $h>0$, then $\sigma(n)=n+h,\ \mu(n)=h,\ g^{\Delta}(n)=\Delta_h g(n)=(g(n+h)-g(n))/h,$ and (1.3) becomes the generalized difference inequality

$$\Delta_h g(n) \leqslant \beta(n)-\gamma(n)g(n+h)+\alpha(n)g^p(n),\ \text{for } n \geqslant n_0. \tag{1.9}$$

If $\mathbb{T}=\{t : t=q^n,\ n\in\mathbb{N},\ q>1\}$, then $\sigma(t)=qt,\ \mu(t)=(q-1)t,\ g^{\Delta}(t)=\Delta_q g(t)=(g(qt)-g(t))/((q-1)t)$, and (1.3) becomes the quantum inequality

$$\Delta_q g(t) \leqslant \beta(t)-\gamma(t)g(qt)+\alpha(t)g^p(t),\ \text{for } t \geqslant t_0. \tag{1.10}$$

Our aim in this paper is to establish some sufficient conditions which yield the global existence and estimate of the rate of decay of solutions to (1.3) and find the upper bound of its solutions when $p>1$ and when $p=\frac{1}{q}<1$. Also we find the lower bound of solutions of the inequality (1.6). The results not only unify the results for (1.7) and (1.8) but also can be applied to (1.9) and (1.10). We will apply the results on nonlinear evolution dynamic equation, Sine-Gordon nonhomogeneous equation, iterative process of nonlinear evolution equations and a delay hyperbolic equation.

2. Main Results

In this section, we state and prove the main results. First, we consider (1.3).

THEOREM 2.1. *Assume that (H) holds, $p \geqslant 1$, and there exists a positive rd-continuous function $\pi(t)$ such that $\pi \in C_r^1[t_0,\infty)_{\mathbb{T}}$ and*

$$\beta(t)+\frac{\alpha(t)}{\pi^p(t)} \leqslant \frac{1}{\pi^{\sigma}(t)}\left[\gamma(t)-\frac{\pi^{\Delta}(t)}{\pi(t)}\right], \textit{for } t\in[t_0,\infty)_{\mathbb{T}}. \tag{2.1}$$

Let $g(t)\geqslant 0$ be a solution to inequality (1.3) such that

$$\pi(t_0)g(t_0)<1. \tag{2.2}$$

Then $g(t)$ exists globally and the following estimate holds:

$$0 \leqslant g(t) < \frac{1}{\pi(t)},\quad \textit{for } t \geqslant t_0.$$

Consequently, if $\lim_{t\to\infty}\pi(t)=\infty$, *then* $\lim_{t\to\infty} g(t)=0$.

Proof. Let $w(t) := g(t)e_{\gamma}(t,t_0)$. Using the product rule in (1.4), we have

$$w^{\Delta}(t)=g^{\Delta}(t)e_{\gamma}(t,t_0)+\gamma(t)g^{\sigma}(t)e_{\gamma}(t,t_0).$$

This and (1.3) imply that

$$\begin{aligned} w^{\Delta}(t) &\leqslant e_{\gamma}(t,t_0)\left[\beta(t)-\gamma(t)g^{\sigma}(t)+\alpha(t)g^p(t)\right]+\gamma(t)g^{\sigma}(t)e_{\gamma}(t,t_0) \\ &= e_{\gamma}(t,t_0)\beta(t)-\gamma(t)e_{\gamma}(t,t_0)g^{\sigma}(t)+\alpha(t)e_{\gamma}(t,t_0)g^p(t)+\gamma(t)g^{\sigma}(t)e_{\gamma}(t,t_0) \\ &= e_{\gamma}(t,t_0)\beta(t)+\alpha(t)e_{\gamma}(t,t_0)g^p(t) \\ &= b(t)+a(t)w^p(t), \end{aligned} \tag{2.3}$$

where

$$b(t) := e_\gamma(t,t_0)\beta(t) \quad \text{and} \quad a(t) := \alpha(t)\left(e_\gamma(t,t_0)\right)^{1-p} > 0. \tag{2.4}$$

Define

$$\eta(t) := \frac{e_\gamma(t,t_0)}{\pi(t)}. \tag{2.5}$$

From (2.2) and (2.5), we have

$$w(t_0) = g(t_0) < \frac{1}{\pi(t_0)} = \eta(t_0), \tag{2.6}$$

where $e_\gamma(t_0,t_0) = 1$. It follows from the inequalities (2.1), (2.3), and (2.6) that

$$\begin{aligned} w^\Delta(t_0) &\leqslant \beta(t_0) + \alpha(t_0)g^p(t_0) \leqslant \beta(t_0) + \frac{\alpha(t_0)}{\pi^p(t_0)} \leqslant \frac{1}{\pi^\sigma(t_0)}\left[\gamma(t_0) - \frac{\pi^\Delta(t_0)}{\pi(t_0)}\right] \\ &= \frac{e_\gamma(t_0,t_0)}{\pi^\sigma(t_0)}\left[\gamma(t_0) - \frac{\pi^\Delta(t_0)}{\pi(t_0)}\right]. \end{aligned} \tag{2.7}$$

Using the quotient rule in (1.4), we note that

$$\begin{aligned} \frac{e_\gamma(t,t_0)}{\pi^\sigma(t)}\left[\gamma(t) - \frac{\pi^\Delta(t)}{\pi(t)}\right] &= \frac{\gamma(t)e_\gamma(t,t_0)}{\pi^\sigma(t)} - \frac{e_\gamma(t,t_0)\pi^\Delta(t)}{\pi(t)\pi^\sigma(t)} \\ &= \frac{\gamma(t)\pi(t)e_\gamma(t,t_0) - e_\gamma(t,t_0)\pi^\Delta(t)}{\pi(t)\pi^\sigma(t)} \\ &= \left(\frac{e_\gamma(t,t_0)}{\pi(t)}\right)^\Delta. \end{aligned} \tag{2.8}$$

This, (2.5) and (2.7) imply that

$$w^\Delta(t_0) \leqslant \left(\frac{e_\gamma(t,t_0)}{\pi(t)}\right)^\Delta\Bigg|_{t=t_0} = \eta^\Delta(t_0). \tag{2.9}$$

From (2.6) and (2.9), it follows that there exists $\varepsilon > 0$, such that

$$w(t) \leqslant \eta(t), \quad \text{for } t_0 \leqslant t \leqslant T, \tag{2.10}$$

where ε is chosen so that $T = t + \varepsilon \in \mathbb{T}$. Now, we prove that if (2.10) holds, then

$$w^\Delta(t) \leqslant \eta^\Delta(t), \quad \text{for } t \in [t_0, T_1], \quad \text{for } T_1 > t_0. \tag{2.11}$$

From (2.3), (2.4) and (2.10), we see that

$$\begin{aligned} w^\Delta(t) &\leqslant e_\gamma(t,t_0)\beta(t) + \alpha(t)\left(e_\gamma(t,t_0)\right)^{1-p} w^p(t) \\ &\leqslant e_\gamma(t,t_0)\beta(t) + \alpha(t)\left(e_\gamma(t,t_0)\right)^{1-p} \eta^p(t) \\ &= e_\gamma(t,t_0)\beta(t) + \alpha(t)\left(e_\gamma(t,t_0)\right)^{1-p}\left(\frac{e_\gamma(t,t_0)}{\pi(t)}\right)^p \\ &= e_\gamma(t,t_0)\beta(t) + \frac{\alpha(t)e_\gamma(t,t_0)}{\pi^p(t)}. \end{aligned}$$

This, (2.8) and (2.1) imply that

$$w^{\Delta}(t) \leqslant e_{\gamma}(t,t_0)\left[\beta(t)+\frac{\alpha(t)}{\pi^{p}(t)}\right] \leqslant \frac{e_{\gamma}(t,t_0)}{\pi^{\sigma}(t)}\left[\gamma(t)-\frac{\pi^{\Delta}(t)}{\pi(t)}\right]$$
$$= \left(\frac{e_{\gamma}(t,t_0)}{\pi(t)}\right)^{\Delta} = \eta^{\Delta}(t), \quad \text{for } t \geqslant T_1.$$

Denote

$$T_1 := \sup\{\delta \in \mathbb{R} : w(t) < \eta(t), \quad \text{for } t \in [t_0, t_0+\delta]_{\mathbb{T}}\}.$$

Now, we claim that $T_1 = \infty$, which says that every nonnegative solution $g(t)$ to inequality (1.3) satisfying assumption (2.2) is defined globally. Assume the contrary, i.e., $T_1 < \infty$. It follows from this and (2.11) that

$$w^{\Delta}(t) \leqslant \eta^{\Delta}(t), \quad \text{for } t \in [t_0, T_1]_{\mathbb{T}}. \tag{2.12}$$

This implies, after integrating from t_0 to T_1, that

$$w(T_1) - w(t_0) = \int_{t_0}^{T_1} w^{\Delta}(s)\Delta s \leqslant \int_{t_0}^{T_1} \eta^{\Delta}(s)\Delta s = \eta(T_1) - \eta(t_0).$$

Since $w(t_0) < \eta(t_0)$ by assumption (2.2), we see that

$$w(T_1) < \eta(T_1). \tag{2.13}$$

It follows from (2.12) and (2.13), as above with $t_0 = T_1$, there exists $\varepsilon_1 > 0$ such that $w(t) < \eta(t)$, for $T_1 \leqslant t \leqslant T_1 + \varepsilon_1$,where ε is chosen so that $T_1 + \varepsilon \in \mathbb{T}$. This contradicts the definition of T_1 and the contradiction proves the desired conclusion $T_1 = \infty$. It follows from the definitions of $w(t)$ and $\eta(t)$, and from the relation $T_1 = \infty$, that

$$g(t) = \frac{w(t)}{e_{\gamma}(t,t_0)} < \frac{\eta(t)}{e_{\gamma}(t,t_0)} = \frac{1}{\pi(t)}, \quad \text{for } t \geqslant t_0.$$

From this we see that if $\lim_{t\to\infty} \pi(t) = 0$, then $\lim_{t\to\infty} g(t) = 0$. The proof is complete. □

REMARK 2.1. In Theorem 2.1, if we assume that there exists a positive function $f(t)$ such that $\gamma(t) - \left(\pi^{\Delta}(t)/\pi(t)\right) = f(t)$, then $\pi(t) = e_{\gamma-f}(t,t_0)$, with $\pi(t_0) = 1$. Using this in Theorem 2.1 and applying the property $e_a(\sigma(t),t_0) = [1+\mu(t)a(t)]e_a(t,t_0)$ of the generalized exponential function, we get the following result.

THEOREM 2.2. *Assume that (H) holds, $p \geqslant 1$ and there exists a positive rd-continuous function $f(t)$ such that $\gamma(t) - f(t) \in \mathscr{R}$ and*

$$\beta(t)e_{\gamma-f}(t,t_0) + \alpha(t) \leqslant f(t)(1+\mu(t)(\gamma(t)-f(t))).$$

Let $g(t) \geqslant 0$ be a solution to inequality (1.3) such that $g(t_0) < 1$. Then $g(t)$ exists globally and the following estimate holds: $0 \leqslant g(t) < \left(1/e_{\gamma-f}(t,t_0)\right)$, for $t \geqslant t_0$. Consequently, if $\gamma(t) > f(t)$, then $\lim_{t\to\infty} g(t) = 0$.

REMARK 2.2. In Theorem 2.1, we assumed that there exists a positive *rd*-continuous function $\pi(t)$, $\pi \in C_r^1[t_0,\infty)_{\mathbb{T}}$ such that (2.2) holds. The question now is: if it is possible to find new conditions without (2.2). This will be left to the interested reader and also it will be of our interest in future.

With an appropriate choice of the functions α, β, γ, f and π one can derive a number of results. For example we have the following results.

COROLLARY 2.1. *Assume that* (H) *holds,* $p \geqslant 1$ *and there exists a positive rd-continuous function* $\pi(t) > 0$, $\pi \in C_r^1[t_0,\infty)_{\mathbb{T}}$, *and* $\theta \in (0,1)$ *such that*

$$0 \leqslant \alpha(t) \leqslant \theta \frac{\pi^p(t)}{\pi^\sigma(t)} \left[\gamma(t) - \frac{\pi^\Delta(t)}{\pi(t)}\right], \text{ and } \beta(t) \leqslant \frac{1-\theta}{\pi^\sigma(t)} \left[\gamma(t) - \frac{\pi^\Delta(t)}{\pi(t)}\right].$$

Let $g(t) \geqslant 0$ *be a solution to inequality (*1.3*) such that* $\pi(t_0)g(t_0) < 1$. *Then* $g(t)$ *exists globally and* $0 \leqslant g(t) < 1/\pi(t)$, *for* $t \geqslant t_0$. *Consequently, if* $\lim_{t\to\infty}\pi(t) = \infty$, *then* $\lim_{t\to\infty} g(t) = 0$.

COROLLARY 2.2. *Assume that* (H) *holds,* $p \geqslant 1$ *and there exists a positive rd-continuous function* $f(t)$ *and* $\theta \in (0,1)$ *such that* $\gamma(t) - f(t) \in R$, $\alpha(t) \leqslant \theta f(t)(1+\mu(t)(\gamma(t)-f(t)))$, *and* $\beta(t)e_{\gamma-f}(t,t_0) \leqslant (1-\theta)f(t)(1+\mu(t)(\gamma(t)-f(t))$. *Let* $g(t) \geqslant 0$ *be a solution to inequality (*1.3*) such that* $g(t_0) < 1$. *Then* $g(t)$ *exists globally and the following estimate holds:* $0 \leqslant g(t) < 1/e_{\gamma-f}(t,t_0)$, *for* $t \geqslant t_0$. *Consequently, if* $\gamma(t) > f(t)$, *then* $\lim_{t\to\infty} g(t) = 0$.

In the following, we apply Theorem 2.1 and Corollary 2.1 on different time scales and the application of Theorem 2.2 and Corollary 2.2 is left to the interested reader.

First, we consider the case when $\mathbb{T} = \mathbb{R}$. In this case we have the results established by Hoang and Ramm [9].

THEOREM 2.3. [9, Theorem 1]. *Let* $\alpha(t)$, $\beta(t)$ *and* $\gamma(t)$ *be continuous functions defined on* $[t_0,\infty)$ *and* $\alpha(t) \geqslant 0$ *for all* $t \geqslant t_0$ *and* $p \geqslant 1$. *Suppose there exists a function* $\pi(t) > 0$, $\pi \in C^1[t_0,\infty)$, *such that*

$$\beta(t) + \frac{\alpha(t)}{\pi^p(t)} \leqslant \frac{1}{\pi(t)} \left[\gamma(t) - \frac{\pi'(t)}{\pi(t)}\right].$$

Let $g(t) \geqslant 0$ *be a solution to inequality (*1.7*) such that* $\pi(t_0)g(t_0) < 1$. *Then* $g(t)$ *exists globally and* $0 \leqslant g(t) < 1/\pi(t)$, *for* $t \geqslant t_0$. *Consequently, if* $\lim_{t\to\infty}\pi(t) = \infty$, *then* $\lim_{t\to\infty} g(t) = 0$.

COROLLARY 2.3. [9, Corollary 1]. *Let* $\alpha(t)$, $\beta(t)$ *and* $\gamma(t)$ *be continuous functions defined on* $[t_0,\infty)$ *and* $\alpha(t) \geqslant 0$ *for all* $t \geqslant t_0$ *and* $p \geqslant 1$. *Suppose there exists a function* $\pi(t) > 0$, $\pi \in C_r^1[t_0,\infty)$, *and* $\theta \in (0,1)$ *such that*

$$0 \leqslant \alpha(t) \leqslant \theta \pi^{p-1}(t) \left[\gamma(t) - \frac{\pi'(t)}{\pi(t)}\right], \text{ and } \beta(t) \leqslant \frac{1-\theta}{\pi(t)} \left[\gamma(t) - \frac{\pi'(t)}{\pi(t)}\right].$$

Let $g(t) \geqslant 0$ *be a solution to inequality (*1.7*) such that* $\pi(t_0)g(t_0) < 1$. *Then* $g(t)$ *exists globally and* $0 \leqslant g(t) < 1/\pi(t)$, *for* $t \geqslant t_0$. *Consequently, if* $\lim_{t\to\infty}\pi(t) = \infty$, *then* $\lim_{t\to\infty} g(t) = 0$.

Now, we consider the case when $\mathbb{T}=\mathbb{N}$, and establish some criteria for the inequality (1.8). Before we do this we should note that if we use the substitution $\Delta g(n)=g(n+1)-g(n)$, then (1.8) becomes

$$g(n+1)\leqslant \frac{\gamma_n}{1+\gamma(n)}g(n)+\frac{\beta(n)}{1+\gamma(n)}+\frac{\alpha(n)}{1+\gamma(n)}g^p(n),\ for\ \ n\geqslant n_0, \tag{2.14}$$

which is the same inequality that has been considered in [9, inequality (40)], with $1-\gamma(n)$ is replaced by $\frac{\gamma_n}{1+\gamma(n)}$, $\beta(n)$ is replaced by $\frac{\beta(n)}{1+\gamma(n)}$ and $\alpha(n)$ is replaced by $\frac{\alpha(n)}{1+\gamma(n)}$. Also if we put $\Delta g(n)=(g(n+1)-g(n))/h_n$, then (1.8) becomes

$$g(n+1)\leqslant \frac{\gamma_n h_n}{1+\gamma(n)}g(n)+\frac{h_n\beta(n)}{1+\gamma(n)}+\frac{h_n\alpha(n)}{1+\gamma(n)}g^p(n),\ for\ \ n\geqslant n_0, \tag{2.15}$$

which is the same inequality that has been considered in [9, inequality (30)]. The following results are different from the results established in [9] for the difference inequalities (2.14) and (2.15) in the sense that our results do not require that $0<\gamma(n)<1$ and $0<h_n\gamma(n)<1$ respectively.

THEOREM 2.4. *Let $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ be nonnegative sequences and $p\geqslant 1$. Suppose there exists a sequence $\pi(n)>0$, for $n\geqslant n_0$ such that*

$$\beta(n)+\frac{\alpha(n)}{\pi^p(n)}\leqslant \frac{1}{\pi(n+1)}\left[1+\gamma(n)-\frac{\pi(n+1)}{\pi(n)}\right].$$

Let $g(n)\geqslant 0$ be a solution to inequality (1.8) such that $\pi(n_0)g(n_0)<1$. Then $g(n)$ exists globally and $0\leqslant g(n)<1/\pi(n)$, for $n\geqslant n_0$. Consequently, if $\lim_{n\to\infty}\pi(n)=\infty$, then $\lim_{n\to\infty}g(n)=0$.

COROLLARY 2.4. *Let $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ be nonnegative sequences and $p\geqslant 1$. Suppose there exists a sequence $\pi(n)>0$, for $n\geqslant n_0$ such that*

$$0\leqslant \alpha(n)\leqslant \theta\frac{\pi^p(n)}{\pi(n+1)}\left[1+\gamma(n)-\frac{\pi(n+1)}{\pi(t)}\right],$$

and

$$\beta(t)\leqslant \frac{1-\theta}{\pi(n+1)}\left[1+\gamma(n)-\frac{\pi(n+1)}{\pi(n)}\right],\ where\ \theta\in(0,1).$$

Let $g(n)\geqslant 0$ be a solution to inequality (1.8) such that $\pi(n_0)g(n_0)<1$. Then $g(n)$ exists globally and $0\leqslant g(n)<1/\pi(n)$, for $n\geqslant n_0$. Consequently, if $\lim_{n\to\infty}\pi(n)=\infty$, then $\lim_{n\to\infty}g(n)=0$.

Next, we consider the case when $\mathbb{T}=q^{\mathbb{N}}$, and establish some criteria for the quantum inequality (1.9).

THEOREM 2.5. *Let $\alpha(n)$, $\beta(n)$ and $\gamma(n)$ be nonnegative sequences defined on $\mathbb{T}=q^{\mathbb{N}}$ and $p\geqslant 1$. Suppose there exists a sequence $\pi(n)>0$, for $n\geqslant n_0$ such that*

$$\beta(n)+\frac{\alpha(n)}{\pi^p(n)}\leqslant \frac{1}{\pi(qn)}\left[1+\gamma(n)-\frac{\pi(qn)}{\pi(n)}\right].$$

Let $g(n) \geqslant 0$ *be a solution to inequality* (1.9) *such that* $\pi(n_0)g(n_0) < 1$. *Then* $g(n)$ *exists globally and* $0 \leqslant g(n) < 1/\pi(n)$, *for* $n \geqslant n_0$. *Consequently, if* $\lim_{n\to\infty} \pi(n) = \infty$, *then* $\lim_{n\to\infty} g(n) = 0$.

COROLLARY 2.5. *Let* $\alpha(n)$, $\beta(n)$ *and* $\gamma(n)$ *be nonnegative sequences defined on* $\mathbb{T} = q^{\mathbb{N}}$ *and* $p \geqslant 1$. *Suppose there exists a positive sequence* $\pi(n) > 0$, *for* $n \geqslant n_0$, *such that*

$$0 \leqslant \alpha(n) \leqslant \theta \frac{\pi^p(n)}{\pi(qn)} \left[1 + \gamma(n) - \frac{\pi(qn)}{\pi(n)}\right],$$

and

$$\beta(t) \leqslant \frac{1-\theta}{\pi(qn)} \left[1 + \gamma(n) - \frac{\pi(qn)}{\pi(n)}\right], \text{ where } \theta \in (0,1).$$

Let $g(n) \geqslant 0$ *be a solution to inequality* (1.9) *such that* $\pi(n_0)g(n_0) < 1$. *Then* $g(n)$ *exists globally and* $0 \leqslant g(n) < 1/\pi(n)$, *for* $n \geqslant n_0$. *Consequently, if* $\lim_{n\to\infty} \pi(n) = \infty$, *then* $\lim_{n\to\infty} g(n) = 0$.

REMARK 1. The other results on the time scales $\mathbb{T} = h\mathbb{T}$, $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$, $\mathbb{T}_2 = \{\sqrt{n} : n \in \mathbb{N}_0\}$, $\mathbb{T}_3 = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$, and when $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}_0\}$ can similarly be stated. There are, however, no new principles involved. Note that if $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$, then $\sigma(t) = (\sqrt{t}+1)^2$ and $\mu(t) = 1 + 2\sqrt{t}$, $g^{\Delta}(t) = \Delta_2 g(t) = \left(g((\sqrt{t}+1)^2) - g(t)\right) / \left(1 + 2\sqrt{t}\right)$, and (1.3) becomes the difference inequality

$$\Delta_2 g(t) \leqslant \beta(t) - \gamma(t) g((\sqrt{t}+1)^2) + \alpha(t) g^p(t), \text{ for } t \geqslant t_0. \tag{2.16}$$

If $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}\}$ where H_n be the so-called harmonic numbers defined by $t_0 = 0$, $t_n = \sum_{k=1}^{n} (1/k)$, $n \in \mathbb{N}_0$, then $\sigma(t_n) = t_{n+1}$, $\mu(t_n) = \frac{1}{n+1}$, $g^{\Delta}(t) = \Delta_{t_n} g(t_n) = (n+1)\Delta g(t_n)$ and (1.3) becomes the inequality

$$\Delta_{t_n} g(t) \leqslant \beta(t_n) - \gamma(t_n) g(t_{n+1}) + \alpha(t_n) g^p(t_n). \tag{2.17}$$

If $\mathbb{T} = \mathbb{T}_2 = \{\sqrt{n} : n \in \mathbb{N}_0\}$, then $\sigma(t) = \sqrt{n^2+1}$ and $\mu(n) = \sqrt{n^2+1} - n$, $g^{\Delta}(t) = \Delta^* g(n) = \left(g(\sqrt{n^2+1}) - g(n)\right) / \left(\sqrt{n^2+1} - n\right)$, and (1.3) becomes

$$\Delta^* g(n) \leqslant \beta(n) - \gamma(n) g(\sqrt{n^2+1}) + \alpha(n) g^p(n). \tag{2.18}$$

If $\mathbb{T} = \mathbb{T}_3 = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$, then $\sigma(n) = \sqrt[3]{n^3+1}$ and $\mu(n) = \sqrt[3]{n^3+1} - n$, $g^{\Delta}(n) = \Delta_3 g(n) = \left(g(\sqrt[3]{n^3+1}) - g(n)\right) / \left(\sqrt[3]{n^3+1} - n\right)$, and (1.3) becomes

$$\Delta_3 g(n) \leqslant \beta(n) - \gamma(n) g(\sqrt[3]{n^3+1}) + \alpha(n) g^p(n). \tag{2.19}$$

In the following, we consider the case when $p = (\lambda+1)/\lambda$ where λ is a ratio of odd positive integers, $\alpha(t) < 0$, and assume that the solution of (1.3) exists for all $t \geqslant t_0$. In this case the inequality (1.3) reduces to

$$w^{\Delta}(t) \leqslant b(t) - \delta(t) w^{\frac{\lambda+1}{\lambda}}(t), \tag{2.20}$$

where $w(t) := g(t) e_{\gamma}(t, t_0)$ and $b(t) := e_{\gamma}(t, t_0)\beta(t)$, and $\delta(t) := -\alpha(t) e_{\gamma}^{-\frac{1}{\lambda}}(t, t_0) > 0$.

THEOREM 2.6. *Assume that* (H) *holds,* $p=(\lambda+1)/\lambda$ *where* λ *is a ratio of odd positive integers and* $\alpha(t)<0$. *Assume that there exists a positive* rd*-continuous function* ϕ *such that* $\phi^{\Delta}(t)\geqslant 0$. *If* $\phi(t_0)g(t_0)\leqslant 1$, *then a solution* $g(t)$ *of* (1.3) *satisfies*

$$\begin{aligned} g(t) \leqslant\ & \frac{1}{\phi(t)e_{\gamma}(t,t_0)}+\frac{1}{\phi(t)e_{\gamma}(t,t_0)}\int_{t_0}^{t}\phi^{\sigma}b(s)\Delta s \\ & +\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}\phi(t)e_{\gamma}(t,t_0)}\int_{t_0}^{t}\frac{\left(\phi^{\Delta}(s)\right)^{\lambda+1}}{(\delta(s)\phi^{\sigma})^{\lambda}}\Delta s. \end{aligned}$$

Proof. Multiplying (2.20) by $\phi^{\sigma}(t)$, and integrating from t_0 to t, we have

$$\int_{t_0}^{t}\phi^{\sigma}(s)w^{\Delta}(s)\Delta s\leqslant\int_{t_0}^{t}\phi^{\sigma}(s)b(s)\Delta s-\int_{t_0}^{t}\delta(s)\phi^{\sigma}(s)w^{\frac{\lambda+1}{\lambda}}(s)\Delta s.$$

Integrating the left hand side by parts, we have

$$\begin{aligned} \phi(t)w(t) &\leqslant \phi(t_0)w(t_0)+\int_{t_0}^{t}\phi^{\sigma}(s)b(s)\Delta s \\ &\quad +\int_{t_0}^{t}\phi^{\Delta}(s)w(s)\Delta s-\int_{t_0}^{t}\delta(s)\phi^{\sigma}(s)w^{\frac{\lambda+1}{\lambda}}(s)\Delta s \\ &= \phi(t_0)g(t_0)+\int_{t_0}^{t}\phi^{\sigma}b(s)\Delta s \\ &\quad +\int_{t_0}^{t}\left[\phi^{\Delta}(s)w(s)-\delta(s)\phi^{\sigma}w^{\frac{\lambda+1}{\lambda}}(s)\right]\Delta s \\ &\leqslant \phi(t_0)g(t_0)+\int_{t_0}^{t}\phi^{\sigma}(s)b(s)\Delta s \\ &\quad +\int_{t_0}^{t}\left[\phi^{\Delta}(s)w(s)-\delta(s)\phi^{\sigma}(s)w^{\frac{\lambda+1}{\lambda}}(s)\right]\Delta s. \end{aligned}$$

Applying the inequality (with $A>0$) $Bu-Au^{\frac{\lambda+1}{\lambda}}\leqslant\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}\frac{B^{\lambda+1}}{A^{\lambda}}$, by setting $B=\phi^{\Delta}(s)$, $A=\delta(s)\phi^{\sigma}(s)$ and $u=w$, we have

$$\phi(t)w(t)\leqslant 1+\int_{t_0}^{t}\phi^{\sigma}(s)b(s)\Delta s+\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}\int_{t_0}^{t}\frac{\left(\phi^{\Delta}(s)\right)^{\lambda+1}}{(\delta(s)\phi^{\sigma})^{\lambda}}\Delta s,$$

where $\phi(t_0)g(t_0)\leqslant 1$. This implies that

$$w(t)\leqslant\frac{1}{\phi(t)}+\frac{1}{\phi(t)}\int_{t_0}^{t}\phi^{\sigma}(s)b(s)\Delta s+\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}\phi(t)}\int_{t_0}^{t}\frac{\left(\phi^{\Delta}(s)\right)^{\lambda+1}}{(\delta(s)\phi^{\sigma}(s))^{\lambda}}\Delta s.$$

So that

$$\begin{aligned} g(t) \leqslant\ & \frac{1}{\phi(t)e_{\gamma}(t,t_0)}+\frac{1}{\phi(t)e_{\gamma}(t,t_0)}\int_{t_0}^{t}\phi^{\sigma}(s)b(s)\Delta s \\ & +\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}\phi(t)e_{\gamma}(t,t_0)}\int_{t_0}^{t}\frac{\left(\phi^{\Delta}(s)\right)^{\lambda+1}}{(\delta(s)\phi^{\sigma}(s))^{\lambda}}\Delta s, \end{aligned}$$

which is the desired inequality. The proof is complete. □

As a special case of Theorem 2.6, if $\phi(t) = 1$, we have the following result.

COROLLARY 2.6. *Assume that (H) holds, $p = (\lambda+1)/\lambda$ where λ is a ratio of odd positive integers and $\alpha(t) < 0$. If $g(t_0) \leqslant 1$, then a solution $g(t)$ of (1.3) (if it is global) satisfies*

$$g(t) \leqslant \frac{1}{e_\gamma(t,t_0)} + \frac{1}{e_\gamma(t,t_0)} \int_{t_0}^{t} e_\gamma(s,t_0)\beta(s)\Delta s.$$

In the following, we consider the inequality

$$g^\Delta(t) \leqslant \beta(t) - \gamma(t) g^\sigma(t) + \alpha(t) g^{\frac{1}{q}}(t), \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}, \tag{2.21}$$

where $q > 1$ and provides an explicit bound to the unknown function $g(t)$. Note that when $\gamma(t) = 0$, then (2.21) becomes

$$g^\Delta(t) \leqslant \beta(t) + \alpha(t) g^{\frac{1}{q}}(t), \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}, \tag{2.22}$$

which is different from (1.1) and has not been considered before in the literature.

THEOREM 2.7. *Assume that α, β and γ are real valued rd-continuous positive functions defined on $[t_0,\infty)_{\mathbb{T}}$. If $g(t)$ is a solution of (2.21), then $g(t)$ satisfies*

$$g(t) \leqslant g(t_0) e_{B\ominus\gamma}(t,t_0) + \frac{1}{e_\gamma(t,t_0)} \int_{t_0}^{t} e_B(t,\sigma(s)) K(s)\Delta s, \textit{ for all } t \in [t_0,\infty)_{\mathbb{T}},$$

where $B \ominus \gamma = \frac{B-\gamma}{1+\mu\gamma}$,

$$B(t) := b(t) + d(t)\frac{q-1}{q} k^{\frac{1}{q}}, \quad K(t) := \frac{1}{q} d(t) k^{\frac{1-q}{q}}, \textit{for any } k > 0,$$

$$b(t) := e_\gamma(t,t_0)\beta(t) \quad \textit{and} \quad d(t) := \alpha(t)\left(e_\gamma(t,t_0)\right)^{1-\frac{1}{q}} > 0.$$

Proof. Let $w(t) := g(t) e_\gamma(t,t_0)$ and proceed as in the proof of Theorem 2.1 to get

$$w^\Delta(t) \leqslant b(t) + d(t) w^{\frac{1}{q}}(t). \tag{2.23}$$

Using the inequality ([14, Lemma 2]), since $q > 1$,

$$w^{\frac{1}{q}} \leqslant \frac{1}{q} k^{\frac{1-q}{q}} w + \frac{q-1}{q} k^{\frac{1}{q}}, \text{ for any } k > 0,$$

we see that

$$w^\Delta(t) \leqslant b(t) + d(t)\left(\frac{1}{q} k^{\frac{1-q}{q}} w + \frac{q-1}{q} k^{\frac{1}{q}}\right) = B(t) + K(t) w(t).$$

Using the inequality (1.2), we have

$$w(t) \leqslant w(t_0) e_B(t,t_0) + \int_{t_0}^{t} e_B(t,\sigma(s)) K(s)\Delta s, \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}.$$

From this and $w(t) := g(t)e_\gamma(t,t_0)$, we have

$$g(t) \leqslant g(t_0)\frac{e_B(t,t_0)}{e_\gamma(t,t_0)} + \frac{1}{e_\gamma(t,t_0)}\int_{t_0}^{t} e_B(t,\sigma(s))K(s)\Delta s, \text{ for all } t \in [t_0,\infty)_{\mathbb{T}},$$

which is the desired inequality. The proof is complete. □

COROLLARY 2.7. *Assume that α, β and γ are real valued rd-continuous positive functions defined on $[t_0,\infty)_{\mathbb{T}}$. If $g(t)$ is a solution of (2.22), then $g(t)$ satisfies*

$$g(t) \leqslant g(t_0)e_B(t,t_0) + \frac{k^{\frac{1-q}{q}}}{q}\int_{t_0}^{t} e_B(t,\sigma(s))\alpha(s)\Delta s, \textit{ for all } t \in [t_0,\infty)_{\mathbb{T}},$$

where

$$B(t) = \beta(t) + \left(\frac{q-1}{q}k^{\frac{1}{q}}\right)\alpha(t), \textit{ for any } k > 0.$$

In the following we consider the inequality (1.6) on a time scale $\mathbb{T}$ which is unbounded above, and establish the lower bound of its solution $g(t)$ when $g(t_0) \neq 0$.

THEOREM 2.8. *Assume that α, β and γ are real valued rd-continuous positive functions defined on $[t_0,\infty)_{\mathbb{T}}$, and $p > 1$ is a constant and $\gamma \in \mathscr{R}$. If $g(t)$ is a solution of (1.6) then $g(t)$ satisfies*

$$g(t) \geqslant \frac{g(t_0)e_Q(t,t_0)}{e_\gamma(t,t_0)},$$

where

$$Q(t) = p(p-1)^{1/p-1}a^{1/p}(t)b^{1-1/p}(t).$$

Proof. Let $w(t) := g(t)e_\gamma(t,t_0)$. Using the product rule in (1.4), we have

$$w^\Delta(t) = g^\Delta(t)e_\gamma(t,t_0) + \gamma(t)g^\sigma e_\gamma(t,t_0).$$

This and (1.6) imply that

$$\begin{aligned} w^\Delta(t) &\geqslant e_\gamma(t,t_0)\left[\beta(t) - \gamma(t)g^\sigma(t) + \alpha(t)g^p(t)\right] + \gamma(t)g^\sigma e_\gamma(t,t_0) \\ &= e_\gamma(t,t_0)\beta(t) - \gamma(t)e_\gamma(t,t_0)g^\sigma(t) + \alpha(t)e_\gamma(t,t_0)g^p(t) + \gamma(t)g^\sigma e_\gamma(t,t_0) \\ &= e_\gamma(t,t_0)\beta(t) + \alpha(t)e_\gamma(t,t_0)g^p(t) \\ &= b(t) + a(t)w^p(t), \end{aligned} \tag{2.24}$$

where

$$b(t) := e_\gamma(t,t_0)\beta(t) \quad \text{and} \quad a(t) := \alpha(t)\left(e_\gamma(t,t_0)\right)^{1-p} > 0. \tag{2.25}$$

Using the fact that the function (where a and b be as defined in (2.24))

$$G(w) := aw^{\gamma-1} + \frac{b}{w},$$

satisfies

$$G(w) \geqslant \gamma(\gamma-1)^{\frac{1}{\gamma}-1} a^{\frac{1}{\gamma}} b^{1-1/\gamma} \quad \text{for} \quad \gamma > 1 \text{ and } w > 0,$$

we see that

$$aw^p + b \geqslant p(p-1)^{1/p-1} a^{1/p} b^{1-1/p} w, \text{ where } p > 1.$$

This and (2.24) imply that

$$w^{\Delta}(t) \geqslant p(p-1)^{1/p-1} a^{1/p}(t) b^{1-1/p}(t) w = Q(t)w(t).$$

This implies that

$$w^{\Delta}(t) - Q(t)w(t) \geqslant 0.$$

Using the product rule and the properties $p \ominus q = \frac{p-q}{1+\mu q}$, and $\ominus(\ominus q) = q$, we see that

$$\begin{aligned}
(w(t)e_{\ominus Q}(t,t_0))^{\Delta} &= w^{\Delta}(t)e_{\ominus Q}(\sigma(t),t_0) + (\ominus Q(t))w(t)e_{\ominus Q}(t,t_0) \\
&= w^{\Delta}(t)e_{\ominus Q}(\sigma(t),t_0) + w(t)\frac{\ominus Q(t)}{1+\mu(t)(\ominus Q(t))}e_{\ominus Q}(\sigma(t),t_0) \\
&= \left(w^{\Delta}(t) - w(t)(\ominus(\ominus Q(t)))\right) e_{\ominus Q}(\sigma(t),t_0) \\
&= \left(w^{\Delta}(t) - Q(t)w(t)\right) e_{\ominus Q}(\sigma(t),t_0) \geqslant 0,
\end{aligned}$$

where $e_{\ominus Q}(\sigma(t),t_0) > 0$. This implies after using the fact that $\frac{1}{e_{\ominus Q}(t,t_0)} = e_Q(t,t_0)$ ([5, Theorem 2.36]) that

$$w(t) \geqslant \frac{w(t_0)}{e_{\ominus Q}(t,t_0)} = w(t_0)e_Q(t,t_0).$$

From this we see that

$$g(t) \geqslant \frac{g(t_0)e_Q(t,t_0)}{e_{\gamma}(t,t_0)},$$

which is the desired inequality. The proof is complete. □

3. Applications

In this section, we apply the results on some different continuous and discrete inequalities which has been used in studying the large time behavior of solutions to evolution equations.

EXAMPLE 1. Consider the nonlinear dynamic evolution equation

$$u^{\Delta}(t) = A(t)u^{\sigma} + h(t,u) + f, \quad u(t_0) = u_0,\, t \in [t_0,\infty)_{\mathbb{T}}, \tag{3.1}$$

where $\mathbb{T}$ is a time scale, and $u : \mathbb{T} \to \mathbb{H}$, $\mathbb{H}$ is a Hilbert space, A is a linear dynamic operator function in $\mathbb{H}$, and $h : \mathbb{T} \times \mathbb{H} \to \mathbb{H}$ is a nonlinear dynamic operator, and f :

$\mathbb{T} \to \mathbb{H}$. Without specifying here all the assumptions on A, h and f, let us assume that $p > 1$,

$$\text{Re}\,\langle Au^{\sigma}, u\rangle \leqslant -\gamma(t)\,\|u^{\sigma}\|^{2},\ \text{Re}\,\langle h(t,u), u\rangle \leqslant \alpha(t)\,\|u\|^{p+1},\ \|f(t)\| \leqslant \beta(t). \tag{3.2}$$

We note that if $\mathbb{T} = \mathbb{R}$, the equation (3.1) becomes the evolution equation

$$u'(t) = A(t)u + h(t,u) + f, \quad u(t_0) = u_0,\, t \geqslant t_0. \tag{3.3}$$

If $\mathbb{T} = \mathbb{N}$, then (3.1) becomes the discrete evolution equation

$$\Delta u(n) = A(n)u(n) + h(n,u(n)) + f(n), \quad u(n_0) = u_0,\, n \geqslant n_0. \tag{3.4}$$

Multiplying (3.1) by u, we have

$$\left\langle u^{\Delta}, u\right\rangle = \langle Au^{\sigma}, u\rangle + \langle h, u\rangle + \langle f, u\rangle.$$

Using the assumptions in (3.2), denoting $\|u(t)\| = g(t)$ and assuming that $g^{\Delta}(t) = \left\langle u^{\Delta}, u\right\rangle$, we have

$$g^{\Delta}(t) \leqslant -\gamma(t)g^{\sigma}(t) + \alpha(t)g^{p}(t) + \beta(t),\ p > 1.$$

Theorem 2.1 is directly applicable, and under the assumptions of Theorem 2.1 by different choice of the functions one can obtain the global existence of solutions and an estimate of its large time behavior.

EXAMPLE 2. Consider the iterative process (see [16, Theorem 2])

$$u_{n+1} = u_n - h_n T_{a_n}^{-1}[A^*(u_n)F(u_n) + a_n(u_n - z)], \quad u(0) = u_0,$$

where $h_n > 0$ and $a_n > 0$. We apply Theorem 2.4 and establish a sufficient condition on a_n and h_n which guarantee that the sequence u_n converging to y, where y is a solution of the equation $F(u) = 0$, where $F : \mathbb{H} \to \mathbb{H}$ be a map in a Hilbert space, such that $F'(y) \neq 0$ and $\sup_{u \in B(u_0,R)} \left\|F^{(j)}(u)\right\| \leqslant M_j(R)$, $j = 0,1,2$ and $R > 0$. Let $w_n = u_n - y$ and $g(n) := \|w_n\|$ and proceeding as in [16, Theorem 2] to get

$$g(n+1) \leqslant \left(1 - \frac{h_n}{2}\right) g(n) + \frac{c_0 h_n}{\sqrt{a_n}} g^2(n) + h_n a_n \|\upsilon\|, \quad g_0 = \|u_0 - y\|. \tag{3.5}$$

Putting $g(n) = g(n+1) - \Delta g(n)$, we have form (3.5) that

$$\Delta g(n) \leqslant -\gamma(n)g(n+1) + \alpha(n)g^2(n) + \beta(n), \tag{3.6}$$

where

$$\gamma(n) = \frac{h_n}{2 - h_n}, \quad \alpha(n) = \frac{2c_0 h_n}{(2 - h_n)\sqrt{a_n}}, \quad \beta(n) = \frac{2h_n a_n \|\upsilon\|}{2 - h_n}.$$

Assume that there exists a positive sequence $\pi(n)$ such that $g(n_0)\pi(n_0) < 1$ and

$$\frac{2h_n a_n \|\upsilon\|}{2-h_n} + \frac{2c_0 h_n}{\pi^2(n)(2-h_n)\sqrt{a_n}} \leqslant \frac{1}{\pi(n+1)}\left[\frac{2}{(2-h_n)} - \frac{\pi(n+1)}{\pi(n)}\right]. \tag{3.7}$$

Then $g(n)$ exists globally and $0 \leqslant g(n) < 1/\pi(n)$, for $n \geqslant n_0$. Consequently, if $\lim_{n\to\infty}\pi(n) = \infty$, then $\lim_{n\to\infty} g(n) = 0$. So by Theorem 2.4, we see that if there exists a positive sequence $\pi(n)$ such that $g(n_0)\pi(n_0) < 1$, (3.7) holds and $\lim_{n\to\infty}\pi(n) = \infty$, then $\lim_{n\to\infty}\|u_n - y\| = 0$.

EXAMPLE 3. Consider the iterative process of ill-posed equation with monotone operators ([15, Theorem 3.1])

$$u_{n+1} = u_n - h_n[A(u_n) + a_n]^{-1}[B(u_n) + a_n u_n - f], \quad u(0) = u_0,$$

where $h_n > 0$ and $a_n > 0$. We apply Theorem 2.4 and establish a sufficient condition on a_n and h_n which guarantee that the sequence u_n converging to the minimal norm solution y, where y is a solution of the equation $B(u) - f = 0$, where $f \in \mathbb{H}$ and B is a monotone, nonlinear C^2_{loc} operator in a real Hilbert space $\mathbb{H}$, $\sup_{u\in B(u_0,R)}\left\|B^{(j)}(u)\right\| \leqslant M_j(R)$, $j = 0,1,2$ and $R > 0$ is arbitrary, $B(u_0,R) = \{u : \|u - u_0\| \leqslant R\}$, $B^{(j)}$ is the Fréchet derivative and the set $N := \{z : B(z) - f = 0\}$ is non-empty, and y is its minimal-norm element. Let $w_n = u_n - y$ and $g(n) := \|w_n\|$ and proceeding as in [15, Theorem 3.1] to get

$$g(n+1) \leqslant (1-h_n)g(n) + \frac{ch_n}{a_n}g^2(n) + b(n), \quad g_0 = \|u_0 - y\|, \tag{3.8}$$

where $0 < h(n) < 1$, $b(n) = \|V_{n+1} - V_n\|$, where V_n is a solution of the equation $B(V_n) + a_n V_n - f = 0$, $a_n > 0$. Putting $g(n) = g(n+1) - \Delta g(n)$, we have form (3.8) that

$$\Delta g(n) \leqslant -\frac{h_n}{1-h_n}g(n+1) + \frac{ch_n}{a_n(1-h_n)}g^2(n) + \frac{b(n)}{1-h_n}. \tag{3.9}$$

Assume that there exists a positive sequence $\pi(n)$ such that $g(n_0)\pi(n_0) < 1$, and

$$\frac{b(n)}{1-h_n} + \frac{ch_n}{a_n\pi^2(n)(1-h_n)} \leqslant \frac{1}{\pi(n+1)}\left[\frac{1}{1-h_n} - \frac{\pi(n+1)}{\pi(n)}\right]. \tag{3.10}$$

Then $g(n)$ exists globally and $0 \leqslant g(n) < 1/\pi(n)$, for $n \geqslant n_0$. Consequently, if $\lim_{n\to\infty}\pi(n) = \infty$, then $\lim_{n\to\infty} g(n) = 0$. So by Theorem 2.4, we see that if there exists a positive sequence $\pi(n)$ such that $g(n_0)\pi(n_0) < 1$, (3.10) holds and $\lim_{n\to\infty}\pi(n) = \infty$, then $\lim_{n\to\infty}\|u_n - y\| = 0$.

EXAMPLE 4. Consider the iterative process (see [16, Section 4])

$$\upsilon_{n+1} = \upsilon_n - h_n T_{a_n}^{-1}[A^*(\upsilon_n)(F(\upsilon_n) - f_\delta) + a_n(\upsilon_n - z)], \quad \upsilon_0 = u_0,$$

which is the iterative of the dynamical equation $F(u)=f$, where f is unknown and the noisy datum f_δ is known such that $\|f_\delta - f\| \leqslant \delta$. Letting $w_n := \upsilon_n - y$, $\|w(n)\| := g_n$, and proceeding as in [16], we have the inequality

$$g(n+1) \leqslant \gamma(n)g(n) + pg^2(n) + \beta(n), \quad g_0 = \|u_0 - y\|, \tag{3.11}$$

where $0<\gamma(n)<1$, $p>0$ and $\beta(n) = \frac{h(n)\delta}{2\sqrt{a(n)}}$. Putting $g(n) = g(n+1) - \Delta g(n)$, we have form (3.11) that

$$\Delta g(n) \leqslant -\frac{(1-\gamma(n))}{\gamma(n)}g(n+1) + \frac{p}{\gamma(n)}g^2(n) + \frac{\beta(n)}{\gamma(n)}. \tag{3.12}$$

Theorem 2.4 is directly applicable on (3.12), and show that if there exists a sequence $\pi(n)>0$, for $n \geqslant n_0$ such that

$$\frac{h(n)\delta}{2\sqrt{a(n)}} + \frac{p}{\pi^2(n)} \leqslant \frac{\gamma(n)}{\pi(n+1)}\left[\frac{1}{\gamma(n)} - \frac{\pi(n+1)}{\pi(n)}\right], \tag{3.13}$$

and $g(n_0) < 1/\pi(n_0)$, then $g(n)$ exists globally and $0 \leqslant g(n) < 1/\pi(n)$, for $n \geqslant n_0$. Consequently, if $\lim_{n\to\infty}\pi(n)=\infty$, then $\lim_{n\to\infty} g(n) = 0$. Note that the results do not require that $h(n)$ is a constant as proposed in [16]. This proves that if (3.13) holds, then $\lim_{n\to\infty}\|\upsilon_n - y\| = 0$. Note that the result is different from the result in [16, Theorem 4] in the sense that we do not assume that the function $a(n) = 16c_0^2 g_n^2$, which depends on the solution g_n.

EXAMPLE 5. Consider the Sine-Gordon nonhomogeneous equation of the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} - a(t)\Delta u(x,t) + q\sin u = \phi(t), \quad (x,t) \in \Omega \times \mathbb{R}_+ = \mathbb{G}, \tag{3.14}$$

where $\Omega = (a,b)$, a bounded interval, q is a constant, $u(t)$ satisfies the Dirichlet boundary conditions and initial condition

$$u=0, \ \text{on}\,(x,t) \in (a,b)\times\mathbb{R}_+, \ (u(x,0), \ \frac{\partial u}{\partial t}(x,0)) = w_0(x,0), \ x\in[a,b].$$

Set $H = L^2(\Omega)$ and assume that N is globally bounded and globally Lipschitz continuous. Define the operators A, R and N by

$$A = \begin{pmatrix} 0 & I \\ -a\Delta u & 0 \end{pmatrix}, \ R\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -q\sin u \end{pmatrix}, M\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \phi \end{pmatrix}.$$

So that the evolution equation of (3.14) can be written as

$$\frac{dw}{dt} = Aw(t) + R(w(t)) + \phi(t),$$

where $w = (u, u')$. Again Theorem 2.2 is directly applicable if one can choose the coefficients as in Example 1.

EXAMPLE 6. Consider the nonlinear delay hyperbolic equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a(t)\Delta u(x,t) - q(t)f(u(x,g(t))), (x,t) \in \Omega \times \mathbb{R}_+ = \mathbb{G}, \tag{3.15}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geqslant 1,$ with a piecewise smooth boundary $\partial\Omega$, and Δu is the Laplacian in $\mathbb{R}^n$, $a \in C(\mathbb{R}_+,\mathbb{R}_+)$, $q \in C(\bar{\mathbb{G}},\mathbb{R}_+)$, $g(t) \leqslant t$, such that $g'(t) \leqslant 0$, and $f \in C(\mathbb{R},\mathbb{R})$ is convex in $\mathbb{R}_+$, $uf(u) > 0$ and $f(u)/u \geqslant k > 0$ for $u \neq 0$. We consider (3.15) with the boundary condition

$$\frac{\partial u(x,t)}{\partial N} = 0, \qquad \text{on } (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{3.16}$$

where N is the unit exterior normal vector to $\partial\Omega$ and γ is a nonnegative continuous function on $\partial\Omega \times R_+$. Integrating (3.15) with respect to x over the domain Ω, we have

$$\frac{d^2}{dt^2}\int_\Omega u(x,t)dx = a(t)\int_\Omega \Delta u(x,t)dx - \int_\Omega q(x,t)f(u(x,g(t)))dx \tag{3.17}$$

Using Green's formula and (3.16), we have

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial N} dS = 0, \quad t \geqslant t_1, \tag{3.18}$$

where dS is the surface element on $\partial\Omega$. Using Jensen's inequality (where f is a convex function), we have

$$\int_\Omega q(t)f(u(x,g(t)))dx \geqslant q(t)\int_\Omega dx f\left(\int_\Omega u(x,g(t))dx \Big/ \int_\Omega dx\right), \ t \geqslant t_1. \tag{3.19}$$

Therefore, from (3.17)-(3.19), we have

$$U''(t) + q(t)f(U(g(t))) \leqslant 0, \quad t \geqslant t_1, \tag{3.20}$$

where

$$U(t) = \int_\Omega u(x,t)dx \Big/ \int_\Omega dx, \quad t \geqslant t_1. \tag{3.21}$$

Without loss of generality, we assume that $u(x,t) > 0$, and $u(x,t) \in \Omega \times (t_0,\infty)$, $(t_0 \geqslant 0)$. It follows that $U(t) > 0$, $U'(t) > 0$ and $U''(t) \leqslant 0$ for $t \geqslant t_1$. Set

$$w(t) = \rho(t)\frac{U'(t)}{U(g(t))} > 0, \qquad \text{for } t \geqslant t_1. \tag{3.22}$$

From (3.20) and (3.22), we have

$$w'(t) \leqslant \beta(t) - \gamma(t)w(t) + \alpha(t)w^2(t), \qquad \text{for } t \geqslant t_1, \tag{3.23}$$

where $\beta(t) = -k\rho(t)q(t)$, $\gamma(t) = -\frac{\rho'(t)}{\rho(t)}$ and $\alpha(t) = -g'(t)/\rho(t) > 0$. Suppose there exist two function $\rho(t)$ and $\pi(t)$, ρ, $\pi \in C^1[t_0,\infty)$, such that

$$-k\rho(t)q(t) - \frac{g'(t)}{\rho(t)\pi^2(t)} \leqslant \frac{1}{\pi(t)}\left[\frac{\rho'(t)}{\rho(t)} - \frac{\pi'(t)}{\pi(t)}\right],$$

assume that such that $\pi(t_0)w(t_0) < 1$. Then by Theorem 2.3, $w(t)$ exists globally, $0 \leqslant w(t) < 1/\pi(t)$, and consequently, if $\lim_{t\to\infty}\pi(t) = \infty$, then $\lim_{t\to\infty} w(t) = 0$.

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S. H. Saker
Department of Mathematics Skills, PYD
King Saud University
Riyadh 11451
Saudi Arabia
and
Department of Math., Faculty of Science
Mansoura University
Mansoura 35516
Egypt
e-mail: mathcoo@py.ksu.edu.sa, shsaker@mans.edu.eg