Journal of
Mathematical
Inequalities
Volume 4, Number 4 (2010), 581–586

SHARP BOUNDS FOR SEIFFERT MEANS IN TERMS OF LEHMER MEANS

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(*Communicated by E. Neuman*)

Abstract. In this paper, we establish two sharp inequalities as follows: $P(a,b) > L_{-\frac{1}{6}}(a,b)$ and $T(a,b) < L_{\frac{1}{3}}(a,b)$ for all $a,b > 0$ with $a \neq b$. Here, $L_r(a,b)$, $P(a,b)$ and $T(a,b)$ are the Lehmer, first and second Seiffert means of a and b, respectively.

1. Introduction

For $r \in \mathbb{R}$ and $a,b > 0$, the Lehmer mean $L_r(a,b)$ was introduced by Lehmer [1] as follows:

$$L_r(a,b) = \frac{a^{r+1} + b^{r+1}}{a^r + b^r}. \tag{1.1}$$

It is well known that $L_r(a,b)$ is increasing with respect to $r \in \mathbb{R}$ for fixed a and b. Many means are the special cases of Lehmer mean, for example,

$$\begin{aligned} A(a,b) &= \tfrac{a+b}{2} = L_0(a,b) && \text{is the arithmetic mean,} \\ G(a,b) &= \sqrt{ab} = L_{-\frac{1}{2}}(a,b) && \text{is the geometric mean,} \\ H(a,b) &= \tfrac{2ab}{a+b} = L_{-1}(a,b) && \text{is the harmonic mean.} \end{aligned}$$

Investigation of the inequalities between Lehmer and other means has attracted the attention of a considerable number of mathematicians [2–5].

The first and second Seiffert means $P(a,b)$ [6] and $T(a,b)$ [7] of two positive numbers a and b are defined by

$$P(a,b) = \begin{cases} \frac{a-b}{4\arctan\left(\sqrt{\frac{a}{b}}\right)-\pi}, & a \neq b, \\ a, & a = b \end{cases} \tag{1.2}$$

and

$$T(a,b) = \begin{cases} \frac{a-b}{2\arctan\left(\frac{a-b}{a+b}\right)}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.3}$$

Mathematics subject classification (2010): 26E60.
Keywords and phrases: Lehmer mean, first Seiffert mean, second Seiffert mean.

Paper JMI-04-51

respectively.

Recently, both means P and T have been the subject of intensive research. In particular, many remarkable inequalities for P and T can be found in the literature [7–11]. The first Seiffert mean $P(a,b)$ can be rewritten as (see [10, Eq.(2.4)])

$$P(a,b) = \begin{cases} \frac{a-b}{2\arcsin(\frac{a-b}{a+b})}, & a \neq b, \\ a, & a = b. \end{cases} \tag{1.4}$$

The power mean of order p of the positive real numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}$$

The main properties of the power mean M_p are given in [12]. In particular, $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed a and b with $a \neq b$.

Let

$$I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & b \neq a, \\ a, & b = a \end{cases}$$

and

$$L(a,b) = \begin{cases} \frac{b-a}{\log b - \log a}, & b \neq a, \\ a, & b = a \end{cases}$$

be the identric and logarithmic means of two positive numbers a and b, respectively. Then it is well known that

$$\begin{aligned} \min\{a,b\} &< H(a,b) = L_{-1}(a,b) = M_{-1}(a,b) < G(a,b) \\ &= L_{-\frac{1}{2}}(a,b) = M_0(a,b) < L(a,b) < I(a,b) < A(a,b) \\ &= L_0(a,b) = M_1(a,b) < \max\{a,b\} \end{aligned} \tag{1.5}$$

for all $a,b > 0$ with $a \neq b$.

In [6], Seiffert proved that

$$L(a,b) < P(a,b) < I(a,b) \tag{1.6}$$

for all $a,b > 0$ with $a \neq b$.

Alzer [4] established that

$$I(a,b) > L_{-\frac{1}{6}}(a,b)$$

for all $a,b > 0$ with $a \neq b$.

Seiffert [7] obtained the power mean bounds for the second Seiffert mean T as follows:

$$M_1(a,b) < T(a,b) < M_2(a,b) \tag{1.7}$$

for all $a,b > 0$ with $a \neq b$.

The following bounds for the first Seiffert mean P in terms of power means are proved by Hästö [8]:

$$M_{\frac{\log 2}{\log \pi}}(a,b) < P(a,b) < M_{\frac{2}{3}}(a,b)$$

for all $a,b > 0$ with $a \neq b$.

The purpose of this paper is to present the optimal upper and lower Lehmer mean bounds for the first and second Seiffert means.

2. Main Results

THEOREM 2.1. *Inequality $L_{-\frac{1}{6}}(a,b) < P(a,b) < L_0(a,b)$ holds for all $a,b > 0$ with $a \neq b$, and $L_{-\frac{1}{6}}(a,b)$ and $L_0(a,b)$ are the best possible lower and upper Lehmer mean bounds for the first Seiffert mean $P(a,b)$.*

Proof. From (1.1) and (1.4) we clearly see that both $L_r(a,b)$ and $P(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b = 1$. Let $t = \sqrt[6]{a} > 1$. Then (1.1) and (1.2) give

$$P(a,b) - L_{-\frac{1}{6}}(a,b) = -\frac{t(t^5+1)}{(4\arctan t^3 - \pi)(t+1)}\left[4\arctan t^3 - \frac{(t+1)(t^6-1)}{t(t^5+1)} - \pi\right]. \tag{2.1}$$

Let

$$f(t) = 4\arctan t^3 - \frac{(t+1)(t^6-1)}{t(t^5+1)} - \pi, \tag{2.2}$$

then simple computations yield that

$$\lim_{t\to 1} f(t) = 0 \tag{2.3}$$

and

$$f'(t) = -\frac{(t-1)^4(t+1)^2(t^2+t+1)}{t^2(t^5+1)^2(t^6+1)} f_1(t), \tag{2.4}$$

where

$$\begin{aligned} f_1(t) &= t^{10}+t^9+3t^8+4t^7-5t^6+3t^5-5t^4+4t^3+3t^2+t+1 \\ &= t^6(t^4+t^3+3t^2-5)+t^4(4t^3+3t-5)+4t^3+3t^2+t+1 \\ &> 0 \end{aligned} \tag{2.5}$$

for all $t > 0$.

Therefore, $P(a,b) > L_{-\frac{1}{6}}(a,b)$ follows from (2.1)–(2.5).

On the other hand, $P(a,b) < L_0(a,b)$ follows from (1.5) and (1.6).

Next, we prove that $L_{-\frac{1}{6}}(a,b)$ and $L_0(a,b)$ are the best possible lower and upper Lehmer mean bounds for $P(a,b)$.

For any $\varepsilon > 0$ and $x > 0$, from (1.1) and (1.2) one has

$$L_{-\frac{1}{6}+\varepsilon}(1,1+x) - P(1,1+x) = \frac{g_1(x)}{(4\arctan\sqrt{x+1}-\pi)[1+(x+1)^{-\frac{1}{6}+\varepsilon}]} \tag{2.6}$$

and

$$\lim_{x\to+\infty}\frac{P(1,x)}{L_{-\varepsilon}(1,x)}=\lim_{x\to+\infty}\frac{x-1}{\pi(x^{1-\varepsilon}+1)}=+\infty, \tag{2.7}$$

where $g_1(x)=[1+(x+1)^{\frac{5}{6}+\varepsilon}](4\arctan\sqrt{x+1}-\pi)-x[1+(x+1)^{-\frac{1}{6}+\varepsilon}]$.

Let $x\to 0$, making use of the Taylor expansion we get

$$\begin{aligned}g_1(x)=&\left[2+\left(\frac{5}{6}+\varepsilon\right)x+\left(\frac{5}{6}+\varepsilon\right)\left(\frac{\varepsilon}{2}-\frac{1}{12}\right)x^2+o(x^2)\right]\left[x-\frac{1}{2}x^2+\frac{7}{24}x^3+o(x^3)\right]\\&-x\left[2+\left(\varepsilon-\frac{1}{6}\right)x+\left(\varepsilon-\frac{1}{6}\right)\left(\frac{\varepsilon}{2}-\frac{7}{12}\right)x^2+o(x^2)\right]\\=&\frac{1}{2}\varepsilon x^3+o(x^3).\end{aligned} \tag{2.8}$$

Equations (2.6) and (2.8) imply that for any $\varepsilon>0$ there exists $\delta_1=\delta_1(\varepsilon)>0$, such that $L_{-\frac{1}{6}+\varepsilon}(1,1+x)>P(1,1+x)$ for $x\in(0,\delta_1)$.

Equation (2.7) implies that for any $\varepsilon>0$ there exists $X_1=X_1(\varepsilon)>1$, such that $P(1,x)>L_{-\varepsilon}(1,x)$ for $x\in(X_1,\infty)$. □

THEOREM 2.2. *Inequality $L_0(a,b)<T(a,b)<L_{\frac{1}{3}}(a,b)$ holds for all $a,b>0$ with $a\neq b$, and $L_0(a,b)$ and $L_{\frac{1}{3}}(a,b)$ are the best possible lower and upper Lehmer mean bounds for the second Seiffert mean $T(a,b)$.*

Proof. From (1.1) and (1.3) we clearly see that both $L_r(a,b)$ and $T(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b=1$. Let $t=\sqrt[3]{a}>1$. Then (1.1) and (1.3) give

$$T(a,b)-L_{\frac{1}{3}}(a,b)=\frac{t^4+1}{2(t+1)\arctan\frac{t^3-1}{t^3+1}}\left[\frac{(t^3-1)(t+1)}{t^4+1}-2\arctan\frac{t^3-1}{t^3+1}\right]. \tag{2.9}$$

Let

$$g(t)=\frac{(t^3-1)(t+1)}{t^4+1}-2\arctan\frac{t^3-1}{t^3+1}, \tag{2.10}$$

then simple computations yield that

$$\lim_{t\to 1}g(t)=0, \tag{2.11}$$

$$g'(t)=-\frac{(t-1)^4(t^2+t+1)}{(t^4+1)^2(t^6+1)}(t^6+3t^5+9t^4+12t^3+9t^2+3t+1)<0 \tag{2.12}$$

for $t>1$.

Therefore, $T(a,b)<L_{\frac{1}{3}}(a,b)$ follows from (2.9)–(2.12).

On the other hand, $T(a,b)>L_0(a,b)$ follows from (1.5) and (1.7).

Next we prove that $L_0(a,b)$ and $L_{\frac{1}{3}}(a,b)$ are the best possible lower and upper Lehmer mean bounds for $T(a,b)$.

For any $\varepsilon > 0$ and $x > 0$, from (1.1) and (1.3) one has

$$T(1,1+x) - L_{\frac{1}{3}-\varepsilon}(1,1+x) = \frac{g_2(x)}{2[1+(1+x)^{\frac{1}{3}-\varepsilon}]\arctan\frac{x}{2+x}} \tag{2.13}$$

and

$$\lim_{x\to+\infty}\frac{L_\varepsilon(1,x)}{T(1,x)} = \lim_{x\to+\infty}\frac{\pi(x^{\varepsilon+1}+1)}{2(x^\varepsilon+1)(x-1)} = \frac{\pi}{2} > 1, \tag{2.14}$$

where $g_2(x) = x[1+(1+x)^{\frac{1}{3}-\varepsilon}] - 2[1+(1+x)^{\frac{4}{3}-\varepsilon}]\arctan\frac{x}{2+x}$.

Let $x \to 0$, making use of the Taylor expansion we get

$$\begin{aligned} g_2(x) &= x\left[2+\left(\frac{1}{3}-\varepsilon\right)x - \frac{(1-3\varepsilon)(2+3\varepsilon)}{18}x^2 + o(x^2)\right] \\ &\quad - x\left[1-\frac{1}{2}x+\frac{1}{6}x^2+o(x^2)\right]\left[2+\left(\frac{4}{3}-\varepsilon\right)x + \frac{(4-3\varepsilon)(1-3\varepsilon)}{18}x^2 + o(x^2)\right] \\ &= \frac{1}{2}\varepsilon x^3 + o(x^3). \end{aligned} \tag{2.15}$$

Equations (2.13) and (2.15) imply that for any $0 < \varepsilon < \frac{1}{3}$, there exists $\delta_2 = \delta_2(\varepsilon) > 0$ such that $T(1,1+x) > L_{\frac{1}{3}-\varepsilon}(1,1+x)$ for $x \in (0,\delta_2)$.

Equation (2.14) implies that for any $\varepsilon > 0$ there exists $X_2 = X_2(\varepsilon) > 1$, such that $L_\varepsilon(1,x) > T(1,x)$ for $x \in (X_2,\infty)$. □

Acknowledgements. This research is partially supported by N S Foundation of China under Grant 60850005 and the N S Foundation of Zhejiang province under Grants D7080080 and Y607128.

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(Received December 30, 2009)

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