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# ON CERTAIN SEQUENCES DERIVED FROM GENERALIZED EULER–MASCHERONI CONSTANTS

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*Abstract.* Let $0<\alpha<1$, and let

$$C_\alpha := \lim_{n\to\infty}\left(1+\frac{1}{2^\alpha}+\cdots+\frac{1}{n^\alpha}-\frac{n^{1-\alpha}}{1-\alpha}\right).$$

It is proved that there exists a unique sequence $(\omega_n)$ such that

$$1+\frac{1}{2^\alpha}+\cdots+\frac{1}{n^\alpha}=C_\alpha+\frac{(n+\omega_n)^{1-\alpha}}{1-\alpha}.$$

Moreover, the sequence $(\omega_n)$ is decreasing and satisfies $\frac{1}{2}\leqslant\omega_n\leqslant\frac{1}{4}\left[1+\left(1+\frac{1}{n}\right)^\alpha\right]$, whence $\lim_{n\to\infty}\omega_n=\frac{1}{2}$. This is only a special case of the more general results established in this paper. These results concern some sequences derived from generalized Euler–Mascheroni constants involving convex functions and complement similar ones obtained by V. Timofte [Integral estimates for convergent positive series. *J. Math. Anal. Appl.* **303** (2005), 90–102].

## 1. Introduction

Let $f:[1,\infty)\to(0,\infty)$ be a continuous function such that

$$S:=\sum_{n=1}^{\infty} f(n)<\infty. \tag{1}$$

V. Timofte [7, Proposition 1] proved that if the restriction of $f$ to $[3/2,\infty)$ is convex, then for every $n\in\mathbb{N}$ (the set of all positive integers) there exists a unique real number $\theta_n$ such that

$$f(1)+f(2)+\cdots+f(n)+\int_{n+\theta_n}^{\infty} f(t)dt=S \tag{2}$$

and

$$\frac{1}{2}\leqslant\theta_n\leqslant\frac{1}{4}\left[1+\frac{f(n)}{f(n+1)}\right]. \tag{3}$$

In particular, we have $\lim_{n\to\infty}\theta_n=\frac{1}{2}$ if $\lim_{n\to\infty}\frac{f(n)}{f(n+1)}=1$.

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In the present paper we are concerned with the case when the series $\sum_{n=1}^{\infty} f(n)$ diverges, i.e., (1) is not satisfied. In this case, for the asymptotic behavior of the sum $f(1)+f(2)+\cdots+f(n)$ the reader is referred to the paper by J. Sándor [6]. Let be given a continuous decreasing function $f:[1,\infty)\to(0,\infty)$, and let $(a_n)$ and $(b_n)$ be the sequences defined by

$$a_n := f(1)+f(2)+\cdots+f(n)-\int_1^{n+1} f(t)dt,$$
$$b_n := f(1)+f(2)+\cdots+f(n)-\int_1^{n} f(t)dt.$$

Then the chain of inequalities

$$a_n \leqslant a_{n+1} < b_{n+1} \leqslant b_n \tag{4}$$

holds for every positive integer $n$, whence the sequences $(a_n)$ and $(b_n)$ are both convergent. Under the additional assumption that

$$\lim_{x\to\infty} f(x)=0, \tag{5}$$

the two sequences have the same limit, say $\gamma_f$ (see [6, Theorem 1]). Moreover, due to (4) one has

$$a_n \leqslant \gamma_f \leqslant b_n \quad \text{for every } n\in\mathbb{N}.$$

Under the above assumptions ($f$ is a continuous positive decreasing function defined on $[1,\infty)$ which satisfies (5)) let $n$ be any positive integer, and let $F_n:[0,\infty)\to\mathbb{R}$ be the function defined by

$$F_n(x) := f(1)+f(2)+\cdots+f(n)-\int_1^{n+x} f(t)dt-\gamma_f. \tag{6}$$

Note that $F_n$ is continuous and strictly decreasing on $[0,\infty)$. Since

$$F_n(0)=b_n-\gamma_f \geqslant 0 \quad \text{and} \quad F_n(1)=a_n-\gamma_f \leqslant 0,$$

it follows that there exists a unique real number $\omega_n\in[0,1]$ such that $F_n(\omega_n)=0$, i.e.,

$$f(1)+f(2)+\cdots+f(n)-\int_1^{n+\omega_n} f(t)dt=\gamma_f. \tag{7}$$

The main purpose of the present paper is to investigate the sequence $(\omega_n)$, defined by (7). In section 2 we prove that although the equations (2) and (7) defining the sequences $(\theta_n)$ and $(\omega_n)$, respectively, are of completely different nature, in the presence of the convexity of $f$ the estimates provided for $\theta_n$ by (3) are valid for $\omega_n$, too. In section 3 we prove that also the monotonicity of the two sequences is the same under the additional assumption that $f$ is twice differentiable and $f''/f'$ is monotone.

## 2. Convergence of the sequence $(\omega_n)$

In order to prove that the estimates (3) are valid also for $(\omega_n)$, we are lead to consider (besides the sequences $(a_n)$ and $(b_n)$ introduced in section 1) the sequence $(c_n)$, defined by

$$c_n := f(1) + f(2) + \cdots + f(n) - \int_1^{n+\frac{1}{2}} f(t)dt. \tag{8}$$

In the special case when $f(x) := \frac{1}{x}$ it is known that $(c_n)$ converges faster than $(a_n)$ and $(b_n)$. In this case we have

$$\begin{aligned} b_n &:= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n, \\ c_n &:= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right), \end{aligned}$$

and $\gamma_f = \gamma$, the classical Euler–Mascheroni constant. It is known that

$$\frac{1}{2n+\frac{2}{5}} < b_n - \gamma < \frac{1}{2n+\frac{1}{3}} \quad \text{for all } n \in \mathbb{N} \quad \text{(see [8, 9])}$$

and that

$$\frac{1}{24(n+1)^2} < c_n - \gamma < \frac{1}{24n^2} \quad \text{for all } n \in \mathbb{N} \quad \text{(see [2])}. \tag{9}$$

On the other hand, in the general setting from section 1, J. Sándor [6, Theorem 2] proved that if $f : [1,\infty) \to (0,\infty)$ is a continuous decreasing convex function satisfying (5) and such that the function defined by $g(x) := xf(x)$ is concave, then

$$\frac{n}{2n+1} f(n) \leqslant b_n - \gamma_f \leqslant \frac{f(n)}{2} \quad \text{for all } n \in \mathbb{N},$$

whence

$$\lim_{n\to\infty} \frac{1}{f(n)} (b_n - \gamma_f) = \frac{1}{2}.$$

In what follows we prove that the sequence $(c_n)$, defined for an arbitrary function $f$ by (8), possesses similar properties with the particular sequence $(c_n)$ obtained by specializing $f(x) := \frac{1}{x}$. More precisely, we prove that $(c_n)$ is decreasing and converges to $\gamma_f$ whenever $f$ is convex and satisfies (5). Moreover, $(c_n)$ converges faster than $(a_n)$ and $(b_n)$ if, in addition, $f$ satisfies

$$\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1. \tag{10}$$

THEOREM 1. *Let $f : [1,\infty) \to (0,\infty)$ be a continuous convex function. Then the inequalities*

$$c_{n+1} \leqslant c_n \tag{11}$$

*and*

$$a_n < c_n < b_n \tag{12}$$

*hold for all* $n \in \mathbb{N}$. *If, in addition,* $f$ *is decreasing and satisfies* (5)*, then* $(c_n)$ *converges to* $\gamma_f$*, the common limit of* $(a_n)$ *and* $(b_n)$.

*Proof.* Let $n$ be any positive integer. Since $f$ takes only positive values, (12) is obvious. The inequality (11) is equivalent to

$$f(n+1) \leqslant \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} f(t)dt,$$

which holds by virtue of the famous Hermite–Hadamard inequality (see, for instance, [3, pp. 150–152], [5, p. 15], [4, Section 1.9] or [1, Section 3.7])

$$(b-a)f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(t)dt \leqslant (b-a)\frac{f(a)+f(b)}{2},$$

valid for every convex function $f : [a,b] \to \mathbb{R}$.

If, in addition, $f$ is decreasing and satisfies (5), then $(a_n)$ and $(b_n)$ are both convergent and have the same limit $\gamma_f$. By (12) we deduce that $(c_n)$ converges to $\gamma_f$, too. $\square$

THEOREM 2. *Let* $f : [1,\infty) \to (0,\infty)$ *be a continuous decreasing convex function satisfying* (5)*, and let* $\omega_n$ *be the unique real number in* $[0,1]$ *defined by* (7)*. Then*

$$\frac{1}{2} \leqslant \omega_n \leqslant \frac{1}{4}\left[1+\frac{f(n)}{f(n+1)}\right] \quad \textit{for all } n \in \mathbb{N}. \tag{13}$$

*In particular, we have* $\lim_{n\to\infty} \omega_n = \frac{1}{2}$ *if* $f$ *satisfies* (10).

*Proof.* Let $(c_n)$ be the sequence defined by (8). By Theorem 1 it follows that

$$\gamma_f \leqslant c_n \quad \text{for every } n \in \mathbb{N}.$$

Further, let $F_n : [0,\infty) \to \mathbb{R}$ be the strictly decreasing function defined by (6). Since $F_n(\omega_n) = 0$ and

$$F_n\left(\frac{1}{2}\right) = c_n - \gamma_f \geqslant 0,$$

it follows that $\omega_n \geqslant \frac{1}{2}$.

In order to derive the upper estimate for $\omega_n$ in (13), note first that

$$F_n\left(\frac{1}{2}\right) = F_n\left(\frac{1}{2}\right) - F_n(\omega_n) = \int_{n+\frac{1}{2}}^{n+\omega_n} f(t)dt.$$

Since $f$ is convex and decreasing, by the Hermite–Hadamard inequality we deduce that

$$F_n\left(\frac{1}{2}\right) \geqslant \left(\omega_n - \frac{1}{2}\right) f\left(n+\frac{2\omega_n+1}{4}\right) \geqslant \left(\omega_n - \frac{1}{2}\right) f(n+1). \tag{14}$$

Next we claim that

$$\frac{f(n)-f(n+1)}{4} \geqslant F_n\left(\frac{1}{2}\right). \tag{15}$$

Indeed, taking into account that $F_n\left(\frac{1}{2}\right) = c_n - \gamma_f$, inequality (15) is equivalent to

$$c_n - \frac{f(n)-f(n+1)}{4} \leqslant \gamma_f. \tag{16}$$

Let $(c_n')$ be the sequence defined by

$$c_n' := c_n - \frac{f(n)-f(n+1)}{4}.$$

Since $f$ satisfies (5) and $(c_n)$ converges to $\gamma_f$, it follows that $(c_n')$ converges to $\gamma_f$, too. So, in order to establish (16) it suffices to prove that $(c_n')$ is increasing. Note that

$$\begin{aligned} c_{n+1}' - c_n' &= \frac{f(n+2)+2f(n+1)+f(n)}{4} - \int_{n+\frac{1}{2}}^{n+\frac{3}{2}} f(t)dt \\ &\geqslant \frac{f(n+2)+2f(n+1)+f(n)}{4} - \frac{f\left(n+\frac{3}{2}\right)+f\left(n+\frac{1}{2}\right)}{2} \end{aligned}$$

by virtue of the Hermite–Hadamard inequality. Thus, in order to complete the proof of (15) it remains to show that

$$f(n+2)+2f(n+1)+f(n) \geqslant 2f\left(n+\frac{3}{2}\right)+2f\left(n+\frac{1}{2}\right). \tag{17}$$

But inequality (17) is an immediate consequence of the celebrated Hardy–Littlewood-Pólya majorization inequality (see, for instance, [3, pp. 89–91], [5, p. 259, Theorem B], [4, Theorem 1.5.4] or [1, Section 3.4]): given a nonempty interval $I \subseteq \mathbb{R}$, a convex function $f: I \to \mathbb{R}$, and a positive integer $m$, let $x_1, \ldots, x_m, y_1, \ldots, y_m \in I$ be such that

(i) $x_1 \geqslant \cdots \geqslant x_m$ and $y_1 \geqslant \cdots \geqslant y_m$;

(ii) $x_1 + \cdots + x_k \geqslant y_1 + \cdots + y_k$ for $1 \leqslant k \leqslant m-1$;

(iii) $x_1 + \cdots + x_m = y_1 + \cdots + y_m$.

Then the following inequality holds:

$$f(x_1)+\cdots+f(x_m) \geqslant f(y_1)+\cdots+f(y_m). \tag{18}$$

Let $m = 4$ and consider the numbers

$$\begin{aligned} &x_1 := n+2, \qquad x_2 := x_3 := n+1, \qquad x_4 := n; \\ &y_1 := y_2 := n+\frac{3}{2}, \qquad\qquad y_3 := y_4 := n+\frac{1}{2}. \end{aligned}$$

A simple computation shows that

- $x_1 \geqslant \cdots \geqslant x_4$ and $y_1 \geqslant \cdots \geqslant y_4$;
- $x_1 + \cdots + x_k \geqslant y_1 + \cdots + y_k \quad$ for all $k \in \{1,2,3\}$;
- $x_1 + \cdots + x_4 = y_1 + \cdots + y_4$.

Thus (18) ensures the validity of (17). Therefore (15) holds, as claimed.

By (14) and (15) it follows that

$$\left(\omega_n - \frac{1}{2}\right) f(n+1) \leqslant \frac{f(n) - f(n+1)}{4},$$

and this inequality implies the upper estimate in (13). □

REMARK 1. Let $0 < \alpha < 1$, and let $f(x) := \frac{1}{x^\alpha}$ for all $x \in [1,\infty)$. Then we have

$$b_n = 1 + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} - \frac{n^{1-\alpha}}{1-\alpha} + \frac{1}{1-\alpha}.$$

Let $\gamma_f$ be the limit of $(b_n)$, and let

$$C_\alpha := \gamma_f - \frac{1}{1-\alpha} = \lim_{n\to\infty} \left(1 + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} - \frac{n^{1-\alpha}}{1-\alpha}\right).$$

Further, let $\omega_n \in [0,1]$ be the unique real number satisfying

$$1 + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} - \frac{(n+\omega_n)^{1-\alpha}}{1-\alpha} + \frac{1}{1-\alpha} = \gamma_f,$$

i.e.,

$$1 + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} = C_\alpha + \frac{(n+\omega_n)^{1-\alpha}}{1-\alpha}.$$

By Theorem 2 it follows that $\frac{1}{2} \leqslant \omega_n \leqslant \frac{1}{4}\left[1 + \left(1 + \frac{1}{n}\right)^\alpha\right]$, for every $n \in \mathbb{N}$, hence $\lim_{n\to\infty} \omega_n = \frac{1}{2}$.

REMARK 2. In the special case when $f(x) := \frac{1}{x}$, the number $\omega_n$ satisfying (7) is given by

$$\omega_n := \exp\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma\right) - n. \tag{19}$$

Theorem 2 provides the estimate

$$\frac{1}{2} \leqslant \exp\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma\right) - n \leqslant \frac{1}{2} + \frac{1}{4n},$$

i.e.,

$$0 \leqslant c_n - \gamma \leqslant \ln\left(n + \frac{1}{2} + \frac{1}{4n}\right) - \ln\left(n + \frac{1}{2}\right)$$
$$= \ln\left(1 + \frac{1}{4n\left(n+\frac{1}{2}\right)}\right) < \frac{1}{4n\left(n+\frac{1}{2}\right)} < \frac{1}{4n^2}.$$

Although this estimate is not so accurate as (9), it has the advantage that it does not appear as an isolated fact, but was derived as a special case of a more general result.

REMARK 3. By (15) it follows that under the assumptions of Theorem 2 one has

$$0 \leqslant c_n - \gamma_f \leqslant F_n\left(\frac{1}{2}\right) \leqslant \frac{f(n) - f(n+1)}{4},$$

whence

$$0 \leqslant \frac{1}{f(n)}\left(c_n - \gamma_f\right) \leqslant \frac{1}{4}\left[1 - \frac{f(n+1)}{f(n)}\right].$$

If, in addition, $f$ satisfies (10), then

$$\lim_{n\to\infty} \frac{1}{f(n)}\left(c_n - \gamma_f\right) = 0,$$

i.e., $(c_n)$ converges faster to $\gamma_f$ than $(b_n)$.

If $f$ does not satisfy (10), then the limit of the sequence $(\omega_n)$ is no longer $\frac{1}{2}$. In this case we have the following result concerning the convergence of $(\omega_n)$.

THEOREM 3. *Let* $f : [1,\infty) \to (0,\infty)$ *be a continuous decreasing convex function satisfying* (5)*, and let* $\omega_n \in \left[\frac{1}{2}, 1\right]$ *be the unique real number defined by* (7)*. If* $\lim_{x\to\infty} \frac{f(x+t)}{f(x)}$ *exists in* $\mathbb{R}$ *for every* $t \in [0,1]$*, then*

$$\lim_{n\to\infty} \omega_n = L(a), \tag{20}$$

*where* $a := \lim_{x\to\infty} \frac{f(x+1)}{f(x)}$ *and* $L : [0,1] \to \left[\frac{1}{2}, 1\right]$ *is the function defined by*

$$L(x) := \begin{cases} 1 & \text{if} \quad x = 0 \\ \ln\left(\frac{x\ln x}{x-1}\right)/\ln x & \text{if} \quad 0 < x < 1 \\ 1/2 & \text{if} \quad x = 1. \end{cases}$$

*Proof.* If $a = 1$, then the conclusion follows by Theorem 2.

Next consider the case $0 < a < 1$. It is easily seen (see also [7, proof of Theorem 3]) that

$$\lim_{x\to\infty} \frac{f(x+t)}{f(x)} = a^t \quad \text{for all } t \in [0,\infty). \tag{21}$$

Following V. Timofte [7] in his proof of Theorem 3, let $\omega := L(a)$, and let $(u_n)$ be the sequence defined by

$$u_n := f(1) + f(2) + \cdots + f(n) - \int_1^{n+\omega} f(t)dt - \gamma_f.$$

Then we have $\omega \in \left[\frac{1}{2}, 1\right]$. Taking into account (7) we deduce that

$$u_n = \int_{n+\omega}^{n+\omega_n} f(t)dt,$$

whence $u_n = (\omega_n - \omega) f(n + \lambda_n)$ with $\lambda_n \in \left[\frac{1}{2}, 1\right]$, by virtue of the mean value theorem for integrals. Therefore we have

$$|\omega_n - \omega| = \frac{|u_n|}{f(n+\lambda_n)} \leqslant \frac{|u_n|}{f(n+1)}.$$

By using (21) and the Cesáro-Stolz theorem it can be proved that

$$\lim_{n\to\infty} \frac{u_n}{f(n+1)} = 0,$$

whence $\lim_{n\to\infty} \omega_n = \omega = L(a)$ (we omit the details because they are the same as in [7, pp. 94–95]).

Finally, suppose that $a = 0$. In order to prove that $\lim_{n\to\infty} \omega_n = 1$, let $\varepsilon \in (0,1)$ be arbitrarily chosen, and let $(d_n)$ be the sequence defined by

$$d_n := f(1) + f(2) + \cdots + f(n) - \int_1^{n+1-\varepsilon} f(t)dt.$$

By virtue of (21) we have

$$\lim_{x\to\infty} \frac{f(x+t)}{f(x)} = 0 \quad \text{for all } t \in [0,\infty),$$

whence

$$\lim_{n\to\infty} \frac{f(n+1)}{f\left(n+1-\frac{\varepsilon}{2}\right)} = 0.$$

Choose $n_0 \in \mathbb{N}$ such that

$$\frac{f(n+1)}{f\left(n+1-\frac{\varepsilon}{2}\right)} < \frac{\varepsilon}{2} \quad \text{for all } n \geqslant n_0.$$

Then for all $n \geqslant n_0$ we have

$$\begin{aligned} d_{n+1} - d_n &= f(n+1) - \int_{n+1-\varepsilon}^{n+2-\varepsilon} f(t)dt \\ &\leqslant f(n+1) - \int_{n+1-\varepsilon}^{n+1-\frac{\varepsilon}{2}} f(t)dt \\ &\leqslant f(n+1) - \frac{\varepsilon}{2} f\left(n+1-\frac{\varepsilon}{2}\right) < 0. \end{aligned}$$

Consequently, the sequence $(d_n)_{n\geqslant n_0}$ is strictly decreasing. On the other hand, this sequence converges to $\gamma_f$ because of (5), whence $d_n > \gamma_f$ for all $n \geqslant n_0$. This means that $F_n(1-\varepsilon) > 0$ for all $n \geqslant n_0$, $F_n$ being the function defined by (6). Taking into account that $F_n$ is strictly decreasing on $[0,\infty)$ and that $F_n(\omega_n) = 0$, it follows that

$$1-\varepsilon < \omega_n \leqslant 1 \quad \text{for all } n \geqslant n_0.$$

Since $\varepsilon \in (0,1)$ was arbitrarily chosen, we conclude that $\lim_{n\to\infty} \omega_n = 1$. □

## 3. Monotonicity of the sequence $(\omega_n)$

Surprisingly, also the monotonicity of the sequence $(\omega_n)$ defined by (7) is the same as that of the sequence $(\theta_n)$ defined by (2).

THEOREM 4. *Let $f:[1,\infty)\to(0,\infty)$ be a twice differentiable decreasing convex function satisfying* (5)*, and let $\omega_n \in \left[\frac{1}{2},1\right]$ be the unique real number defined by* (7)*. If the function $f''/f'$ is monotone, then the sequence $(\omega_n)$ has the opposite monotonicity. Moreover, $\lim_{x\to\infty}\frac{f(x+t)}{f(x)}$ exists in $\mathbb{R}$ for every $t\in[0,1]$, whence* (20) *holds.*

*Proof.* Since the proof is similar to that of Theorem 6 in [7] we only sketch it by pointing out the differences. Assume, for instance, that $f''/f'$ is increasing on $[1,\infty)$. Let $F:[1,\infty)\to[0,\infty)$ be the strictly increasing function defined by

$$F(x) := \int_1^x f(t)dt.$$

By (7) it follows that

$$F(n+\omega_n) - F(n-1+\omega_{n-1}) = f(n) \quad \text{for all } n \geqslant 2.$$

Further, let $\varphi:[1,\infty)\times\left[\frac{1}{2},1\right]\to\mathbb{R}$ be the function defined by

$$\varphi(x,y) := F(x+y) - F(x+y-1) - f(x).$$

For any fixed $x\in[1,\infty)$ the partial function $\varphi(x,\cdot)$ is decreasing and satisfies the inequality $\varphi\left(x,\frac{1}{2}\right) > 0 \geqslant \varphi(x,1)$, whence there is a unique $y\in\left[\frac{1}{2},1\right]$ such that $\varphi(x,y)=0$. In other words, there is a unique function $\Theta:[1,\infty)\to\left[\frac{1}{2},1\right]$ such that $\varphi(x,\Theta(x))=0$, i.e.,

$$F(x+\Theta(x)) - F(x+\Theta(x)-1) = f(x) \quad \text{for all } x\in[1,\infty). \tag{22}$$

Furthermore, $\Theta$ is decreasing on $[1,\infty)$ (see [7, pp. 97–98] for details).

Next set

$$e_n := f(1)+f(2)+\cdots+f(n) - F(n+\Theta(n))$$

for every $n\in\mathbb{N}$. Taking into account that $F$ is strictly increasing, by (22) it follows that

$$\begin{aligned} e_n - e_{n+1} &= F(n+1+\Theta(n+1)) - f(n+1) - F(n+\Theta(n)) \\ &= F(n+\Theta(n+1)) - F(n+\Theta(n)) \leqslant 0 \end{aligned}$$

because $\Theta(n+1) \leqslant \Theta(n)$. Therefore, the sequence $(e_n)$ is increasing. On the other hand, $(e_n)$ converges to $\gamma_f$ because of (5), whence $e_n \leqslant \gamma_f$ for all $n \in \mathbb{N}$. Taking into account (7) and the definition of $e_n$, this inequality is equivalent to

$$f(1)+\cdots+f(n)-F(n+\Theta(n)) \leqslant f(1)+\cdots+f(n-1)-F(n-1+\omega_{n-1}),$$

i.e., to

$$F(n-1+\omega_{n-1}) \leqslant F(n+\Theta(n))-f(n)=F(n-1+\Theta(n)),$$

by virtue of (22). Since $F$ is increasing and $\varphi(n,\cdot)$ is decreasing, we deduce that $\omega_{n-1} \leqslant \Theta(n)$, whence

$$\varphi(n,\omega_{n-1}) \geqslant \varphi(n,\Theta(n))=0.$$

This inequality implies that

$$\begin{aligned} 0 &\leqslant F(n+\omega_{n-1})-F(n-1+\omega_{n-1})-f(n) \\ &= F(n+\omega_{n-1})-F(n+\omega_n), \end{aligned}$$

whence $\omega_{n-1} \geqslant \omega_n$. Thus the sequence $(\omega_n)$ is decreasing.

The last statement of the theorem follows easily by l'Hôpital's rule (see [7]). □

REMARK 4. If $f(x)=\frac{1}{x}$ for all $x \in [1,\infty)$, then by Theorem 4 it follows that the sequence $(\omega_n)$, defined by (19), is decreasing (this monotonicity of $(\omega_n)$ seems to be new).

## REFERENCES


[1] W. W. BRECKNER AND T. TRIF, *Convex Functions and Related Functional Equations. Selected Topics*, Cluj University Press, Cluj-Napoca, 2008.

[2] D. W. DETEMPLE, *A quicker convergence to Euler's constant*, Amer. Math. Monthly, **100** (1993), 468–470.

[3] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, 1934.

[4] C. P. NICULESCU AND L.-E. PERSSON, *Convex Functions and Their Applications. A Contemporary Approach*, Springer-Verlag, New York, 2006.

[5] A. W. ROBERTS AND D. E. VARBERG, *Convex Functions*, Academic Press, New York, London, 1973.

[6] J. SÁNDOR, *On generalized Euler constants and Schlömilch–Lemonnier type inequalities*, J. Math. Anal. Appl., **328** (2007), 1336–1342.

[7] V. TIMOFTE, *Integral estimates for convergent positive series*, J. Math. Anal. Appl., **303** (2005), 90–102.

[8] L. TOTH, *Problem E3432*, Amer. Math. Monthly, **98** (1991), 264.

[9] L. TOTH, *Problem E3432 (Solution)*, Amer. Math. Monthly, **99** (1992), 684–685.


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