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A SELF–ADAPTIVE PROJECTION METHOD FOR A CLASS OF VARIANT VARIATIONAL INEQUALITIES

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Abstract. In this paper, we consider the general variant variational inequality of the type: Find a vector $u^* \in \mathbb{R}^n$, such that

$$Q(u^*) \in \Omega, \qquad \langle v - Q(u^*), Tu^* \rangle \geqslant 0, \qquad \forall v \in \Omega,$$

where T, Q are operators. We suggest and analyze a very simple self-adaptive iterative method for solving this class of general variational inequalities. Under certain conditions, the global convergence of the proposed method is proved. An example is given to illustrate the efficiency and implementation of the proposed method. Preliminary numerical results show that the proposed method is applicable.

1. Introduction

The classical variational inequality, denoted by $\mathrm{VI}(\Omega, F)$, is to determine a vector u^* in a nonempty closed convex subset Ω of the n-dimensional Euclidean space $\mathbb{R}^n$, such that

$$\langle v - u^*, F(u^*) \rangle \geqslant 0, \quad \forall v \in \Omega,$$

where F is an operator from $\mathbb{R}^n$ into itself. Variational inequality problems are of fundamental importance for a wide range of problems in science and technology, such as mathematical programming, traffic engineering, economics and equilibrium problems, see [1–35].

It is well-known that $\mathrm{VI}(\Omega, F)$ is equivalent to the projection equation

$$u = P_\Omega[u - \beta_k F(u)],$$

where $P_\Omega(\cdot)$ denotes the orthogonal projection map on Ω and β_k is a judiciously chosen positive step size. Thus the solution of the $\mathrm{VI}(\Omega, F)$ is equivalent to finding a zer of the residue function

$$e(u, \beta) := u - P_\Omega[u - \beta_k F(u)].$$

We now have a variety of techniques to suggest and analyze various iterative algorithms for solving variational inequalities [1–30]. The simplest is the Goldstein's projection

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method [6] which, starting with any $u^0 \in \mathbb{R}^n$, iteratively updates u^{k+1} according to the formula

$$u^{k+1} = P_\Omega[u^k - \beta_k F(u^k)], \tag{1.1}$$

where β_k is a chosen positive step-size. In contrast with Douglas-Rachford operator splitting method [9, 17] for $\mathrm{VI}(\Omega, F)$, this projection method can be viewed as a simple explicit method.

In this paper, we consider a class of general variational inequalities of finding $u \in \mathbb{R}^n$ such that

$$Q(u) \in \Omega, \quad \langle v - Q(u), Tu \rangle \geqslant 0, \quad \forall v \in \Omega, \tag{1.2}$$

where $Q, T : \mathbb{R}^n \to \mathbb{R}^n$ are given nonlinear operators and $\Omega \subset \mathbb{R}^n$ is a nonempty closed convex set. Problem of the type (1.2) was introduced and considered by Noor [19]. For the formulation, applications, numerical methods and other aspects of the general variational inequalities (1.2), see [9–29] and the references therein. In this paper, our main aim is to develop a simple method for solving problem (1.2). In order to obtain the new iterate of the proposed method, we do not need to compute additionally the value of function Q. This is very important especially in practical problems in which the cost of computing or approximating the value of function Q is very expensive. Furthermore, numerical experiments show that the proposed method may be efficient for some large scale problems, which clearly illustrates its simplicity and its efficiency.

Throughout this paper, we assume that the operator Q is strongly monotone with respect to the operator T with a positive modulus $\alpha > 0$ such that

$$\langle Tx - Ty, Q(x) - Q(y) \rangle \geqslant \alpha \|x - y\|^2, \quad \forall x, y \in \mathbb{R}^{\ltimes}$$

and Lipschitz continuous with a positive constant $L > 0$ such that

$$\|Q(x) - Q(y)\| \leqslant L\|x - y\|, \quad \forall x, y \in \mathbb{R}^{\ltimes}.$$

Note that, for $L = 1$, the operator Q is called nonexpansive. Without loss of generality, we assume that the operator T is nonexpansive, unless otherwise specified. We would like to mention that we need neither the Lipschitz constant L nor the strong monotone α to design the algorithm. We assume that the solution set of (1.2), denoted by S^*, is nonempty.

2. Preliminaries

In this section, we summarize some preliminary results which are useful in the following analysis. At first, we give some basic properties of the projection mapping. In the next step, we present an useful equivalent expression of the variational inequality problem.

LEMMA 2.1. *For a given $u \in \Omega$ and $z \in \mathbb{R}^n$, the inequality*

$$\langle u - z, v - u \rangle \geqslant 0, \quad \forall v \in \Omega$$

holds if and only if $u = P_\Omega(z)$, where P_Ω is the projection operator.

It follows from Lemma 2.1 that

$$\langle z - P_\Omega(z), P_\Omega(z) - v\rangle \geqslant 0, \qquad \forall z \in \mathbb{R}^n,\ v \in \Omega. \tag{2.1}$$

LEMMA 2.2. *Let Ω be a closed convex set in $\mathbb{R}^n$, then we have*

$$\|(v - P_\Omega(v)) - (w - P_\Omega(w))\| \leqslant \|v - w\| \qquad \forall v, w \in \mathbb{R}^n. \tag{2.2}$$

Proof. Using (2.1), we can prove that

$$\langle v - w, P_\Omega(v) - P_\Omega(w)\rangle \geqslant \|P_\Omega(v) - P_\Omega(w)\|^2 \qquad \forall v, w \in \mathbb{R}^n,$$

by using the above inequality, we obtain

$$\begin{aligned} &\|(v - P_\Omega(v)) - (w - P_\Omega(w))\|^2 \\ &\quad = \|v - w\|^2 - 2\langle v - w, P_\Omega(w) - P_\Omega(w)\rangle + \|P_\Omega(v) - P_\Omega(w)\|^2 \\ &\quad \leqslant \|v - w\|^2 - \|P_\Omega(v) - P_\Omega(w)\|^2. \end{aligned}$$

This implies that

$$\|(v - P_\Omega(v)) - (w - P_\Omega(w))\| \leqslant \|v - w\|.$$

We obtain the desired result. □

LEMMA 2.3. [32] *Denote $R_\gamma(u) = Q(u) - P_\Omega[Q(u) - \frac{1}{\gamma}Tu]$, then for all $u \in \mathbb{R}^n$ and $\gamma' \geqslant \gamma > 0$, it holds that*

$$\|R_{\gamma'}(u)\| \leqslant \|R_\gamma(u)\|. \tag{2.3}$$

LEMMA 2.4. [19] *$u^* \in \mathbb{R}^n$ is solution of problem (1.2) if and only if $u^* \in \mathbb{R}^n$ satisfies the relation:*

$$Q(u^*) = P_K[Q(u^*) - \beta T u^*], \tag{2.4}$$

From Lemma 2.4, it is clear that u is solution of (1.2) if and only if u is a zero point of the function

$$r(u,\beta) := \frac{1}{\beta}\{Q(u) - P_\Omega[Q(u) - \beta T u]\}. \tag{2.5}$$

It is clear that

$$u \in S^* \Leftrightarrow r(u,\beta) = 0.$$

For a given $\gamma > 0$, the equation (2.5) can be written as

$$u = u - \gamma r(u,\beta) \tag{2.6}$$

which plays an important part in the whole analysis of the proposed method.

3. Self-adaptive Method

In this section, we describe the proposed method and we prove some lemmas, which are used in the next section. Note that the Goldstein's projection scheme (1.1) can be viewed as

$$u^{k+1} = u^k - e(u^k, \beta_k). \tag{3.1}$$

By analogy, based on $r(u,\beta)$ we can construct a similar method for solving problem (1.2). Then, we consider the following iterative scheme:

ALGORITHM 3.1.

Step 0. Given a non negative sequence $\eta_k > 0$ with $\sum_{k=1}^{+\infty} \eta_k < +\infty, \beta_0 > 0, \varepsilon > 0,$ $\mu \in [0.5,1), \delta \in (0,1), \delta_0 \in (0,1), \varepsilon > 0,$ and $u^0 \in \mathbb{R}^n$, set $\gamma_0 := \beta_0$ and $k = 0$.

Step 1. If $\|r(u^k,1)\| \leqslant \varepsilon$ then stop. Otherwise go to Step 2.

Step 2. If $u^k \notin S^*$, then find the smallest non negative integer l_k, such that

$$\beta_{k+1} := \frac{\gamma_k}{\mu^{l_k}} \qquad \text{and} \qquad u^{k+1} := u^k - r(u^k, \beta_{k+1}),$$

satisfies

$$\frac{1}{\beta_{k+1}} \|Q(u^k) - Q(u^{k+1})\|^2 \leqslant (2-\delta)\langle Tu^k - Tu^{k+1}, Q(u^k) - Q(u^{k+1})\rangle.$$

Step 3. Selection of γ_{k+1}, if

$$\frac{1}{\beta_{k+1}} \|Q(u^k) - Q(u^{k+1})\| \leqslant \delta_0(\langle Tu^k - Tu^{k+1}, Q(u^k) - Q(u^{k+1})\rangle,$$

then

$$\gamma_{k+1} := \frac{\beta_{k+1}}{1+\eta_{k+1}},$$

otherwise

$$\gamma_{k+1} := \beta_{k+1}.$$

Set $k := k+1$, and go to Step 1.

REMARK 3.1. For the given u^k, the new iterate of Li and Yuan's algorithm in [15] is generated by

$$u^{k+1} = u^k - \alpha_k d(u^k, \tilde{u}^k, \beta_k)$$

where

$$\tilde{u}^k = u^k - r(u^k, \beta_k)$$

$$d(u^k, \tilde{u}^k, \beta_k) = \frac{1}{\beta_k}\{Q(\tilde{u}^k) - P_\Omega[Q(u^k) - \beta_k Tu^k]\}$$

and

$$\alpha_k = \frac{\gamma_k \langle r(u^k,\beta_k), d(u^k,\tilde{u}^k,\beta_k)\rangle}{\|d(u^k,\tilde{u}^k,\beta_k)\|^2}, \quad \gamma_k \in (0,2).$$

From Step 2, we have

$$u^{k+1} = -\frac{1}{\beta_{k+1}}\{(Q(u^k) - \beta_{k+1}Tu^k) - P_\Omega[Q(u^k) - \beta_{k+1}Tu^k]\}. \tag{3.2}$$

The main amount of computational effort when computing the new iterate is to obtain or approximate the value of the function Q at a given vector. The algorithm of Li and Yuan from [15] needs the value of Q both at $\tilde{u}^k$ as well as at u^k. In contrast, our algorithm computes (or approximates) $Q(u^k)$ and computes the next iterate by an additional post processing step which solely involves the application of P_Ω. The cost of this post processing step is relatively minor in comparison with the cost of evaluating $Q(\tilde{u}^k)$. This is the main advantage of our method.

REMARK 3.2. The numerical performance of the method depends significantly on the initial step-size parameter β_0. The method converges quite quickly when a proper fixed step-size parameter is chosen. However, this proper step-size parameter is unknown beforehand. If the step-size parameter is too large or too small, the number of iterations can increase significantly. To overcome this difficulty, in the case of our present work, we have used a self-adaptive technique to adjust the step-size parameter at each iteration. The main advantage of this technique is that it can adjust the step-size parameter automatically.

In the following lemma, we show that the sequence β_k is bounded both from above as well as from below.

LEMMA 3.1. *In each iteration of Algorithm 3.1, the procedure of searching β_{k+1} will terminate in finite steps. Furthermore, there is a constant $M > 0$ and there are two positive real numbers denoted by $\beta_{\min} := \beta_0 M$ and $\beta_{\max} := \frac{L^2}{(2-\delta)\alpha\mu}$ such that:*

$$\beta_{\max} \geqslant \beta_{k+1} \geqslant \beta_{\min} > 0, \quad \forall k > 0.$$

Proof. By Step 2 in the proposed method, we derive that

$$\beta_{k+1} \geqslant \frac{\beta_k}{1+\eta_k}, \qquad \forall k > 0.$$

Since

$$\sum_{k=1}^{+\infty} \eta_k < +\infty,$$

it follows that there is a constant $M > 0$ such that

$$\frac{1}{\prod_{k=1}^{+\infty}(1+\eta_k)} \geqslant M.$$

Then

$$\beta_{k+1} \geqslant \beta_{\min} := \beta_0 M, \qquad \forall k > 0.$$

Since the mapping Q is Lipschitz continuous with a constant L on the feasible set Ω, we have

$$\frac{1}{\beta_{k+1}}\|Q(u^k)-Q(u^{k+1})\|^2 \leqslant \frac{1}{\beta_{k+1}}L^2\|u^k-u^{k+1}\|^2. \tag{3.3}$$

Because Q is strongly monotone with respect to the operator T with a constant modulus $\alpha>0$, it yields

$$(2-\delta)\alpha\|u^k-u^{k+1}\|^2 \leqslant (2-\delta)\langle Tu^k-Tu^{k+1}, Q(u^k)-Q(u^{k+1})\rangle. \tag{3.4}$$

From (3.3) and (3.4), it follows that the inequality of Step 2 is satisfied if

$$\beta_{k+1} \geqslant \frac{L^2}{(2-\delta)\alpha}.$$

The parameter l_k in the algorithm is the minimum non negative integer fulfilling the condition of Step 2, this means that

$$\beta_{k+1} \leqslant \beta_{\max} := \frac{L^2}{(2-\delta)\alpha\mu}, \qquad \forall k>0.$$

The proof is completed. □

In the next lemma we prove that $\|r(u,\beta)\|$ is a non increasing function for $\beta>0$. Note that the proof of this Lemma is different from that in [15, Lemma 2].

LEMMA 3.2. *For a given $u\in\mathbb{R}^n$, let $0<\beta<\beta'$. Then it holds*

$$\|r(u,\beta')\| \leqslant \|r(u,\beta)\| \leqslant \frac{\beta'}{\beta}\|r(u,\beta')\|.$$

Proof. First, for all $v,w\in\mathbb{R}^n$ and for any nonempty closed convex set Ω, we have

$$\langle P_\Omega(v)-P_\Omega(w), v-P_\Omega(v)\rangle \geqslant 0.$$

Setting $v=Q(u)-\beta Tu$, $w=Q(u)-\beta' Tu$ and respectively , we obtain

$$\langle P_\Omega[Q(u)-\beta Tu]-P_\Omega[Q(u)-\beta' Tu], \beta'\{Q(u)-\beta Tu-P_\Omega[Q(u)-\beta Tu]\rangle\} \geqslant 0, \tag{3.5}$$

and

$$\langle P_\Omega[Q(u)-\beta' Tu]-P_\Omega[Q(u)-\beta Tu], \beta\{Q(u)-\beta' Tu-P_\Omega[Q(u)-\beta' Tu]\rangle\} \geqslant 0. \tag{3.6}$$

Using (2.5), adding (3.5) and (3.6), we get

$$\langle \beta' r(u,\beta')-\beta r(u,\beta), \beta'\beta r(u,\beta)-\beta'\beta r(u,\beta')\rangle \geqslant 0.$$

Thus, we have

$$\langle(\beta'+\beta) r(u,\beta'), r(u,\beta)\rangle \geqslant \beta'\|r(u,\beta')\|^2+\beta\|r(u,\beta)\|^2.$$

It's easily to show that

$$(\beta'+\beta)\|r(u,\beta')\|^2+(\beta'+\beta)\|r(u,\beta)\|^2 \geqslant 2\beta'\|r(u,\beta')\|^2+2\beta\|r(u,\beta)\|^2.$$

And

$$\|r(u,\beta)\|^2 \geqslant \|r(u,\beta')\|^2,$$

i.e.

$$\|r(u,\beta)\| \geqslant \|r(u,\beta')\|.$$

For the proof of the right-hand inequalities. Note that

$$\alpha\|r(u,\alpha)\| = \|R_{\frac{1}{\alpha}}(u)\|,$$

and from Lemma 2.3, we have

$$\beta\|r(u,\beta)\| = \|R_{\frac{1}{\beta}}(u)\| \leqslant \|R_{\frac{1}{\beta'}}(u)\| = \beta'\|r(u,\beta')\|, \tag{3.7}$$

from which, we have

$$\|r(u,\beta)\| \leqslant \frac{\beta'}{\beta}\|r(u,\beta')\|.$$

Therefore, the assertion of this lemma is proved. □

4. Convergence Analysis

In this section, we consider the global convergence of the proposed method. For this purpose, we need the following result.

THEOREM 4.1. *The sequence $\{u^k\}$ generated by the proposed method satisfies*

$$\|r(u^{k+1},\beta_{k+1})\| \leqslant \left(1-\frac{\delta\alpha}{\beta_{\max}}\right)^{\frac{1}{2}}(1+\eta_k)\|r(u^k,\beta_k)\|. \tag{4.1}$$

Proof. It follows from (2.5) and (3.2) that

$$\begin{aligned} r(u^{k+1},\beta_{k+1}) = &\frac{1}{\beta_{k+1}}\{(Q(u^{k+1})-\beta_{k+1}Tu^{k+1})-P_\Omega[Q(u^{k+1})-\beta_{k+1}Tu^{k+1}]\} \\ &-\frac{1}{\beta_{k+1}}\{(Q(u^k)-\beta_{k+1}Tu^k)-P_\Omega[Q(u^k)-\beta_{k+1}Tu^k]\}, \end{aligned} \tag{4.2}$$

substituting $v=Q(u^{k+1})-\beta_{k+1}Tu^{k+1}$ and $w=Q(u^k)-\beta_{k+1}Tu^k$ in (2.2), we obtain

$$\begin{aligned} \|r(u^{k+1},\beta_{k+1})\|^2 &\leqslant \|(Tu^{k+1}-Tu^k)-\frac{1}{\beta_{k+1}}(Q(u^{k+1})-Q(u^k))\|^2 \\ &\leqslant \|Tu^{k+1}-Tu^k\|^2-\frac{2}{\beta_{k+1}}\langle Tu^{k+1}-Tu^k,Q(u^{k+1})-Q(u^k)\rangle \\ &\quad+\frac{1}{\beta_{k+1}^2}\|Q(u^{k+1})-Q(u^k)\|^2. \end{aligned} \tag{4.3}$$

From Step 2, we have

$$\frac{1}{\beta_{k+1}^2}\|Q(u^k)-Q(u^{k+1})\|^2 \leqslant \frac{2-\delta}{\beta_{k+1}}\langle Tu^k-Tu^{k+1},Q(u^k)-Q(u^{k+1})\rangle. \tag{4.4}$$

Substituting (4.4) in (4.3) and using the strongly monotone of Q with respect to the operator $T,$ we obtain

$$\begin{aligned}\|r(u^{k+1},\beta_{k+1})\|^2 &\leqslant \left(1-\frac{\delta\alpha}{\beta_{\max}}\right)\|u^{k+1}-u^k\|^2\\ &= \left(1-\frac{\delta\alpha}{\beta_{\max}}\right)\|r(u^k,\beta_{k+1})\|^2. \end{aligned}\tag{4.5}$$

Because $\beta_{k+1} \geqslant \frac{\beta_k}{1+\eta_k}$ and $\beta_k \geqslant \frac{\beta_k}{1+\eta_k}$, based on Lemma 3.2, we get

$$\|r(u^k,\beta_{k+1})\| \leqslant \left\|r\left(u^k,\frac{\beta_k}{1+\eta_k}\right)\right\| \leqslant \frac{\beta_k}{\frac{\beta_k}{1+\eta_k}}\|r(u^k,\beta_k)\|,$$

then

$$\|r(u^k,\beta_{k+1})\| \leqslant (1+\eta_k)\|r(u^k,\beta_k)\|. \tag{4.6}$$

Combining (4.5) and (4.6), we have

$$\|r(u^{k+1},\beta_{k+1})\| \leqslant \left(1-\frac{\delta\alpha}{\beta_{\max}}\right)^{\frac{1}{2}}(1+\eta_k)\|r(u^k,\beta_k)\|.$$

Then (4.1) holds and the theorem is proved. □

From the above theorem, we get the convergence of the proposed method as follows.

THEOREM 4.2. *The proposed method for solving* (1.2) *is globally convergent.*

Proof. From (4.1), we obtain

$$\|r(u^{k+1},\beta_{k+1})\| \leqslant \left(1-\frac{\delta\alpha}{\beta_{\max}}\right)^{\frac{k+1}{2}}\prod_{i=0}^{k}(1+\eta_i)\|r(u^0,\beta_0)\|.$$

Since $0 \leqslant 1-\frac{\delta\alpha}{\beta_{\max}} < 1$ and $\prod_{i=0}^{k}(1+\eta_i) < +\infty$, it follows that

$$\|r(u^k,\beta_k)\| \to 0,$$

and by using Lemma 3.2, we have

$$\|r(u^k,\beta_{\max})\| \to 0.$$

From $\lim_{k\to\infty}\eta_k = 0$, we know that there exist $k_0 > 0$ and $c_0 < 1$ such that

$$\left(1-\frac{\delta\alpha}{\beta_{\max}}\right)^{\frac{1}{2}}(1+\eta_k) \leqslant c_0, \qquad \forall k \geqslant k_0, \tag{4.7}$$

which combined with (4.1) gives

$$\begin{aligned}\|r(u^{k+1},\beta_{k+2})\| &\leqslant (1+\eta_{k+1})\left(1-\frac{\delta\alpha}{\beta_{\max}}\right)^{\frac{1}{2}}\|r(u^k,\beta_{k+1})\| \\ &\leqslant c_0\|r(u^k,\beta_{k+1})\| \qquad \forall k \geqslant k_0.\end{aligned} \tag{4.8}$$

Note that $r(u^k,\beta_{k+1}) = u^k - u^{k+1}$, and from (4.8) we get

$$\|u^{k+2}-u^{k+1}\| \leqslant c_0\|u^{k+1}-u^k\|, \qquad \forall k \geqslant k_0.$$

Therefore, $\{u^k\}$ is a Cauchy sequence and converges to its cluster point, say u'. Because $\lim_{k\to\infty} u^k = u'$ and $r(u,\beta_{\max})$ is continuous on Ω, it follows that

$$r(u',\beta_{\max}) = \lim_{k\to\infty} r(u^k,\beta_{\max}) = 0,$$

and u' is a solution of (1.2). Since the problem has unique solution, then we have $u' = u^*$. Therefore, the generated sequence converges to the unique solution u^*. □

5. Numerical Results

To verify the theoretical assertions, we consider the following least distance problem:

$$\begin{aligned}&\min\ \frac{1}{2}\|x-c\|^2 \\ &\text{s.t.}\quad Ax \in \Omega\end{aligned}$$

where $A \in \mathbb{R}^{n\times n}, c \in \mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is a closed convex set. This problem can be written as

$$\begin{aligned}&\min \frac{1}{2}\|x-c\|^2 \\ &\text{s.t.}\quad Ax-\xi = 0, \quad \xi \in \Omega.\end{aligned} \tag{5.1}$$

The Lagrangian function of problem (5.1) is

$$L(x,\xi,y) = \frac{1}{2}\langle x,x\rangle - \langle c,x\rangle - \langle y, Ax-\xi\rangle.$$

Thus, we have

$$L(x^*,\xi^*,y) \leqslant L(x^*,\xi^*,y^*) \leqslant L(x,\xi,y^*),$$

where $(x^*, \xi^*, y^*) \in \mathbb{R}^n \times \Omega \times \mathbb{R}^n$ is saddle point of the Lagrangian function. From the above inequalities for all $\xi \in \Omega$ we can obtain

$$\begin{cases} x^* = \langle A, y^* \rangle + c \\ \langle \xi - \xi^*, y^* \rangle \geqslant 0 \\ Ax^* = \xi^* \end{cases} \tag{5.2}$$

Substituting the first and the third equation in the second of system (5.2), we get

$$(AA^T y^* + Ac) \in \Omega, \qquad \langle \xi - (AA^T y^* + Ac)), y^* \rangle \geqslant 0 \qquad \forall \xi \in \Omega, \tag{5.3}$$

which is exactly the general variational inequality (1.2) with

$$Q(u) = AA^T u + Ac \quad \text{and} \quad Ty = y.$$

We form the test problem as follows: The matrix A is $n \times n$ matrix whose entries are randomly generated in the interval (-5,+5), the vector c is generated from a uniform distribution in the interval (-500,500) and the closed convex set Ω is defined as

$$\Omega := \{u \in \mathbb{R}^n |\ \|u\| \leqslant a\}.$$

Then the projection on Ω in the sense of Euclidean-norm is very easy to carry out, for example,

$$\forall u \in \mathbb{R}^n, P_\Omega[u] = \begin{cases} u, & \text{if} \quad \|u\| \leqslant a; \\ \frac{a}{\|u\|} u, & \text{if} \quad \|u\| > a. \end{cases}$$

Note that in the case $\|Ac\| > a, \|AA^T u^* + Ac\| = a$ (otherwise $u^* = 0$ is the trivial solution). Therefore, we test the problem with $a = \rho \|Ac\|$ and $\rho \in (0,1)$.

In all tests we take $\delta_0 = 0.5; \delta = 0.2$ and $\mu = 0.5$. All iterations start with $u^0 = (1, \ldots, 1)^T$, $\beta_0 = 1$ and stopped whenever $\frac{\|r(u,1)\|}{a} \leqslant 10^{-5}$. The adjustment factor η_k for these test problems is adopted based in the criterion below,

$$\eta_{k+1} = \begin{cases} 1 & \text{if} \quad u^{k+1} \text{ satisfies condition of Step 2;} \\ 0 & \text{otherwise,} \end{cases}$$

All codes were written in Matlab. The test results for problems (5.3) are reported in tables 1-2, k is the number of iterations and l denotes the number of evaluations of mapping Q.

Table 1. Numerical results for problem (5.3) with $n = 100$

	Method [15]		Algorithm 3.1	
ρ	k	l	k	l
0.1	58	185	51	79
0.3	13	53	13	26
0.5	8	45	7	19
0.7	5	33	5	16
0.9	3	27	4	15

Table 2. Numerical results for problem (5.3) with $n = 500$

	Method [15]		Algorithm 3.1	
ρ	k	l	k	l
0.1	39	103	39	52
0.3	9	43	9	22
0.5	5	35	5	18
0.7	4	33	4	17
0.9	3	31	3	16

6. Remarks

It is interesting to observe the difference of the convergent between Goldstein's projection method [6] and our method. We remark that the Goldestein's projection method updates u^{k+1} according to the formula

$$u^{k+1} = P_\Omega[u^k - \beta F(u^k)].$$

We note that if F is Lipschitz continuous (with a Lipschitz $L > 0$) and uniformly strong monotone (with a constant modulus α)

$$(u-v)^T(F(u) - F(v)) \geqslant \alpha \|u - v\|^2,$$

and β satisfies

$$0 < \beta < \frac{2\alpha}{L^2},$$

then this method is convergent. However, the estimation of α and L may lead to a slow convergence. The performance of our method for $T = I$, the identity operator, needs neither the Lipshchitz L nor the strong monotone modulus α to design the algorithm, but only depends on the choice of scaling parameter β. Our preliminary numerical results show that the method may be efficient for some large scale problems. Moreover, it demonstrates computationally that the new method needs less evaluations of mapping Q. It is evident that the smaller the ratio $a/\|AC\|$ is, the more iterations it takes to meet the stopping criterion. On the other hand, it seems that the number of evaluations of mapping Q as well as the overall number of iterations is not very sensitive to the size of the problem, that is the dimension of the space $\mathbb{R}^n$.

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