Journal of
Mathematical
Inequalities
Volume 5, Number 2 (2011), 157–168

EXPONENTIAL CONVEXITY AND JENSEN'S INEQUALITY FOR DIVIDED DIFFERENCES

Z. PAVIĆ, J. PEČARIĆ AND A. VUKELIĆ

Abstract. In this paper we obtain means which involve divided differences for n-convex functions. We examine their monotonicity property using exponentially convex functions.

1. Introduction

Let f be a real-valued function defined on the segment $[a,b]$. The divided difference of order n of the function f at distinct points $x_0,...,x_n \in [a,b]$, is defined recursively (see [3], [10] and [12]) by

$$f[x_i] = f(x_i) \text{ for } i = 0,\ldots,n$$

and

$$f[x_0,\ldots,x_n] = \frac{f[x_1,\ldots,x_n] - f[x_0,\ldots,x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0,\ldots,x_n]$ is independent of the order of the points $x_0,\ldots,x_n$.
The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x,\ldots,x}_{j \text{ times}}] = \frac{f^{(j-1)}(x)}{(j-1)!}.$$

For divided difference, in the case of distinct points, the following holds:

$$f[x_0,\ldots,x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)}, \text{ where } \omega(x) = \prod_{j=0}^{n}(x - x_j),$$

so we have that

$$f[x_0,\ldots,x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{j=1, j\neq i}^{n}(x_i - x_j)}.$$

A function $f : [a,b] \to \mathbb{R}$ is said to be n-*convex* if the n-th order divided difference of f satisfies

$$f[x_0,\ldots,x_n] \geqslant 0 \quad \text{for all} \quad a \leqslant x_0 < \ldots < x_n \leqslant b.$$

The following theorem is proved (see [5]):

Mathematics subject classification (2010): 26D15, 26D20, 26D99.
Keywords and phrases: Jensen inequality, divided differences, n-convex function, Schur convex function, log-convexity, exponentially convex function, majorization, continuous extensions, Schur means.

THEOREM 1. *Let f be an $(n+2)$-convex function on (a,b) and $\mathbf{x} \in (a,b)^{n+1}$. Then*

$$G(\mathbf{x}) = f[x_0,\dots,x_n]$$

is a convex function of the vector $\mathbf{x} = (x_0,\dots,x_n)$. Consequently,

$$f\left[\sum_{i=0}^{m} a_i x_0^i,\dots,\sum_{i=0}^{m} a_i x_n^i\right] \leqslant \sum_{i=0}^{m} a_i f[x_0^i,\dots,x_n^i] \tag{1.1}$$

holds for all $a_i \geqslant 0$ such that $\sum_{i=0}^{m} a_i = 1$.

Schur polynomial in $n+1$ variables $x_0,\dots,x_n$ of degree $d = d_0+\dots+d_n$ ($d_0 \geqslant \dots \geqslant d_n$) is defined as

$$S_{(d_0,\dots,d_n)}(x_0,\dots,x_n) = \frac{\det\left[x_i^{d_{n-j}+j}\right]_{i,j=0}^{n}}{\det\left[\,x_i^j\,\right]_{i,j=0}^{n}}.$$

The numerator consists of alternating polynomials (they change the sign under any transposition of the variables) and so they are all divisible by the denominator which is Vandermonde determinant. Schur polynomial is also symmetric because the numerator and denominator are both alternating.

For $n \geqslant 1$ using Schur polinomial and Vandermonde determinant (extended with logarithmic function)

$$V(\mathbf{x};p,q) = \det\begin{bmatrix} 1 & x_0 & {x_0}^2 & \dots & {x_0}^{n-1} & {x_0}^p \ln^q x_0 \\ 1 & x_1 & {x_1}^2 & \dots & {x_1}^{n-1} & {x_1}^p \ln^q x_1 \\ 1 & x_2 & {x_2}^2 & \dots & {x_2}^{n-1} & {x_2}^p \ln^q x_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & {x_n}^2 & \dots & {x_n}^{n-1} & {x_n}^p \ln^q x_n \end{bmatrix}$$

we obtain:

PROPOSITION 1. *For monomial function $h(x) = x^{n+k}$, where $k \geqslant 1$ is an integer, holds*

$$h[x_0,\dots,x_n] = S_{(k,\underbrace{0,\dots,0}_{n\ times})}(x_0,\dots,x_n) = \frac{V(\mathbf{x};n+k,0)}{V(\mathbf{x};n,0)}.$$

For potential function $f(x) = x^p = x^{n+p-n}$, where p is a real number, holds

$$f[x_0,\dots,x_n] = \frac{V(\mathbf{x};p,0)}{V(\mathbf{x};n,0)}.$$

Further, for a partition π the Schur polynomial can be expressed by a sum

$$S_\pi(x_0,\dots,x_n) = \sum_T \mathbf{x}^T = \sum_T x_0^{t_0}\dots x_n^{t_n}.$$

The summation is over all semistandard Young tableaux T of shape π where the exponents $t_0,\dots t_n$ give the weight of T in which each t_j counts the occurences of the number j in T (see [2] and [7]). So, we have the next proposition:

PROPOSITION 2. *For monomial function $h(x)=x^{n+k}$, where $k\geqslant 1$ is an integer, holds*

$$h[x_0,\dots,x_n]=S_{(k,\underbrace{0,\dots,0}_{n\ times})}(x_0,\dots,x_n)=\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_k=0}^{i_{k-1}}x_{i_1}x_{i_2}\cdots x_{i_k}.$$

The goal of this paper is to define means using the inequality (1.1). We will examine their monotonicity property using exponential convex functions. Also, we will consider some special cases.

2. Jensen means for divided differences

THEOREM 2. *Let $f\in C^{n+2}([a,b])$ and $\mathbf{x}^i\in(a,b)^{n+1}$ for $i=0,\dots,m$. If $a_i\geqslant 0$ such that $\sum_{i=0}^{m}a_i=1$ and $\sum_{j=0}^{n}\sum_{k=0}^{j}\left(\sum_{i=0}^{m}a_ix_j^ix_k^i-\sum_{i=0}^{m}a_ix_j^i\sum_{l=0}^{m}a_lx_k^l\right)\neq 0$, then there exists $\xi\in(a,b)$ such that*

$$\sum_{i=0}^{m}a_if[x_0^i,\dots,x_n^i]-f\left[\sum_{i=0}^{m}a_ix_0^i,\dots,\sum_{i=0}^{m}a_ix_n^i\right]$$
$$=\frac{f^{(n+2)}(\xi)}{(n+2)!}\sum_{j=0}^{n}\sum_{k=0}^{j}\left(\sum_{i=0}^{m}a_ix_j^ix_k^i-\sum_{i=0}^{m}a_ix_j^i\sum_{l=0}^{m}a_lx_k^l\right). \tag{2.1}$$

Proof. Let us denote $\alpha=\min f^{(n+2)}$ and $\beta=\max f^{(n+2)}$. We first consider the following function $\phi_1(x)=\frac{\beta x^{n+2}}{(n+2)!}-f(x)$. Then $\phi_1^{(n+2)}(x)=\beta-f^{(n+2)}(x)\geqslant 0, x\in[a,b]$, so ϕ_1 is an $(n+2)$-convex function. Applying Theorem 1 on an $(n+2)$-convex function ϕ_1 with $\phi(x)=\frac{x^{n+2}}{(n+2)!}$ we have

$$\beta\cdot\sum_{i=0}^{m}a_i\phi[x_0^i,\dots,x_n^i]-\sum_{i=0}^{m}a_if[x_0^i,\dots,x_n^i]$$
$$\geqslant\beta\cdot\phi\left[\sum_{i=0}^{m}a_ix_0^i,\dots,\sum_{i=0}^{m}a_ix_n^i\right]-f\left[\sum_{i=0}^{m}a_ix_0^i,\dots,\sum_{i=0}^{m}a_ix_n^i\right],$$

i.e.

$$\sum_{i=0}^{m}a_if[x_0^i,\dots,x_n^i]-f\left[\sum_{i=0}^{m}a_ix_0^i,\dots,\sum_{i=0}^{m}a_ix_n^i\right]$$
$$\leqslant\beta\left(\sum_{i=0}^{m}a_i\phi[x_0^i,\dots,x_n^i]-\phi\left[\sum_{i=0}^{m}a_ix_0^i,\dots,\sum_{i=0}^{m}a_ix_n^i\right]\right).$$

Similarly, a function $\phi_2(x) = f(x) - \alpha \cdot \phi(x)$ is an $(n+2)$-convex function. Inequality from the Theorem 1 with $(n+2)$-convex function ϕ_2 becomes

$$\sum_{i=0}^{m} a_i f[x_0^i, \dots, x_n^i] - \alpha \cdot \sum_{i=0}^{m} a_i \phi[x_0^i, \dots, x_n^i]$$
$$\geqslant f\left[\sum_{i=0}^{m} a_i x_0^i, \dots, \sum_{i=0}^{m} a_i x_n^i\right] - \alpha \cdot \phi\left[\sum_{i=0}^{m} a_i x_0^i, \dots, \sum_{i=0}^{m} a_i x_n^i\right],$$

i.e.

$$\sum_{i=0}^{m} a_i f[x_0^i, \dots, x_n^i] - f\left[\sum_{i=0}^{m} a_i x_0^i, \dots, \sum_{i=0}^{m} a_i x_n^i\right]$$
$$\geqslant \alpha\left(\sum_{i=0}^{m} a_i \phi[x_0^i, \dots, x_n^i] - \phi\left[\sum_{i=0}^{m} a_i x_0^i, \dots, \sum_{i=0}^{m} a_i x_n^i\right]\right).$$

From Proposition 2 with $k = 2$ and the fact that the function $\phi(x)$ is $(n+2)$-convex we have

$$\sum_{i=0}^{m} a_i \phi[x_0^i, \dots, x_n^i] - \phi\left[\sum_{i=0}^{m} a_i x_0^i, \dots, \sum_{i=0}^{m} a_i x_n^i\right]$$
$$= \frac{1}{(n+2)!}\left[\sum_{j=0}^{n}\sum_{k=0}^{j}\left(\sum_{i=0}^{m} a_i x_j^i x_k^i - \sum_{i=0}^{m} a_i x_j^i \sum_{l=0}^{m} a_l x_k^l\right)\right] > 0.$$

We can now conclude that there exists $\xi \in (a,b)$ that we are looking for in (2.1). □

REMARK 1. Let us note that the left-hand side in equation (2.1) is greater than or equal to zero if $f^{(n+2)} \geqslant 0$ which is the statement of Theorem 1.

COROLLARY 1. *Let $f, g \in C^{n+2}([a,b])$ and $\mathbf{x}^i \in (a,b)^{n+1}$. If $a_i \geqslant 0$ and $\sum_{i=0}^{m} a_i = 1$, then there exists $\xi \in (a,b)$ such that*

$$\frac{\sum_{i=0}^{m} a_i f[x_0^i, \dots, x_n^i] - f\left[\sum_{i=0}^{m} a_i x_0^i, \dots, \sum_{i=0}^{m} a_i x_n^i\right]}{\sum_{i=0}^{m} a_i g[x_0^i, \dots, x_n^i] - g\left[\sum_{i=0}^{m} a_i x_0^i, \dots, \sum_{i=0}^{m} a_i x_n^i\right]} = \frac{f^{(n+2)}(\xi)}{g^{(n+2)}(\xi)} \tag{2.2}$$

provided that both denominators not equal zero.

Proof. We use the following standard technique: Let us define the linear functional $L(h) = \sum_{i=0}^{m} a_i h[x_0^i, \dots, x_n^i] - h\left[\sum_{i=0}^{m} a_i x_0^i, \dots, \sum_{i=0}^{m} a_i x_n^i\right]$. Next, we define $\psi(t) = f(t)L(g) - g(t)L(f)$. According to Theorem 2, applied on ψ, there exists $\xi \in (a,b)$ so that

$$L(\psi) = \frac{\psi^{(n+2)}(\xi)}{(n+2)!}\sum_{j=0}^{n}\sum_{k=0}^{j}\left(\sum_{i=0}^{m} a_i x_j^i x_k^i - \sum_{i=0}^{m} a_i x_j^i \sum_{l=0}^{m} a_l x_k^l\right).$$

From $L(\psi)=0$, it follows $f^{(n+2)}(\xi)L(g)-g^{(n+2)}(\xi)L(f)=0$ and (2.2) is proved. $\square$

The above corollary enables us to define various types of means, because if the function $f^{(n+2)}/g^{(n+2)}$ has inverse, from (2.2) we can get

$$\xi=\left(\frac{f^{(n+2)}}{g^{(n+2)}}\right)^{-1}\left(\frac{\sum_{i=0}^{m}a_if[x_0^i,\dots,x_n^i]-f\left[\sum_{i=0}^{m}a_ix_0^i,\dots,\sum_{i=0}^{m}a_ix_n^i\right]}{\sum_{i=0}^{m}a_ig[x_0^i,\dots,x_n^i]-g\left[\sum_{i=0}^{m}a_ix_0^i,\dots,\sum_{i=0}^{m}a_ix_n^i\right]}\right).$$

Specially, if we take substitutions $f(t)=t^p,\ g(t)=t^q$ with $t>0$ in the above identity and use the method of continuous extensions, we get **Jensen means for divided differences**. Let us use the following notation:

$$\mathbf{X}=\begin{bmatrix} x_0^0 & x_1^0 & x_2^0 & \dots & x_{n-1}^0 & x_n^0\\ x_0^1 & x_1^1 & x_2^1 & \dots & x_{n-1}^1 & x_n^1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ x_0^m & x_1^m & x_2^m & \dots & x_{n-1}^m & x_n^m\end{bmatrix},\quad \mathbf{a}=(a_0,\dots,a_m),$$

$$V\mathbf{x}pq=V(\mathbf{x};p,q)\ \text{ and }\ N=\{0,1,\dots,n+1\}$$

The Jensen means for divided differences is defined as

$$E_{Jen}(\mathbf{X},\mathbf{a};p,q)=\begin{cases}\left(\prod_{i=0}^{n+1}\frac{q-i}{p-i}\frac{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jp0}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}p0}{V\mathbf{aX}n0}}{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jq0}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}q0}{V\mathbf{aX}n0}}\right)^{\frac{1}{p-q}} & \text{for } p\neq q;\ p\notin N, q\notin N\\ \left(\prod_{\substack{i=0\\ i\neq k}}^{n+1}\frac{(q-k)(q-i)}{k-i}\frac{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jk1}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}k1}{V\mathbf{aX}n0}}{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jq0}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}q0}{V\mathbf{aX}n0}}\right)^{\frac{1}{k-q}} & \text{for } p\neq q;\ p=k\in N, q\notin N\\ \left(\prod_{\substack{i=0\\ i\neq l}}^{n+1}\frac{l-i}{(p-l)(p-i)}\frac{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jp0}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}p0}{V\mathbf{aX}n0}}{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jl1}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}l1}{V\mathbf{aX}n0}}\right)^{\frac{1}{p-l}} & \text{for } p\neq q;\ p\notin N, q=l\in N\\ \left(\prod_{\substack{i=0\\ i\neq k,l}}^{n+1}\frac{l-i}{i-k}\frac{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jk1}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}k1}{V\mathbf{aX}n0}}{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jl1}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}l1}{V\mathbf{aX}n0}}\right)^{\frac{1}{k-l}} & \text{for } p\neq q;\ p=k\in N, q=l\in N\\ \exp\left(\sum_{i=0}^{n+1}\frac{1}{i-p}+\frac{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jp1}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}p1}{V\mathbf{aX}n0}}{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jp0}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}p0}{V\mathbf{aX}n0}}\right) & \text{for } p=q;\ p\notin N, q\notin N\\ \exp\left(\sum_{\substack{i=0\\ i\neq k}}^{n+1}\frac{1}{i-k}+\frac{1}{2}\frac{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jk2}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}k2}{V\mathbf{aX}n0}}{\sum_{j=0}^{m}a_j\frac{V\mathbf{x}^jk1}{V\mathbf{x}^jn0}-\frac{V\mathbf{aX}k1}{V\mathbf{aX}n0}}\right) & \text{for } p=q;\ p=q=k\in N\end{cases}\tag{2.3}$$

A mapping $E_{Jen} : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function in two variables p and q. The skeleton of all expressions for this means is a fraction

$$\frac{\sum_{i=0}^{m} a_i f[x_0^i,\ldots,x_n^i] - f\left[\sum_{i=0}^{m} a_i x_0^i,\ldots,\sum_{i=0}^{m} a_i x_n^i\right]}{\sum_{i=0}^{m} a_i g[x_0^i,\ldots,x_n^i] - g\left[\sum_{i=0}^{m} a_i x_0^i,\ldots,\sum_{i=0}^{m} a_i x_n^i\right]} = \frac{\sum_{i=0}^{m} a_i \frac{V\mathbf{x}^i p0}{V\mathbf{x}^i n0} - \frac{V\mathbf{aX}p0}{V\mathbf{aX}n0}}{\sum_{i=0}^{m} a_i \frac{V\mathbf{x}^i q0}{V\mathbf{x}^i n0} - \frac{V\mathbf{aX}q0}{V\mathbf{aX}n0}}$$

which follows from Proposition 1. The most parts of expressions proceed from

$$\lim_{p\to k} \frac{\sum_{i=0}^{m} a_i \frac{V\mathbf{x}^i p0}{V\mathbf{x}^i n0} - \frac{V\mathbf{aX}p0}{V\mathbf{aX}n0}}{p-k} = \sum_{i=0}^{m} a_i \frac{V\mathbf{x}^i k1}{V\mathbf{x}^i n0} - \frac{V\mathbf{aX}k1}{V\mathbf{aX}n0} \qquad \text{for } k\in N. \tag{2.4}$$

The quotient under limit becomes an indeterminate form when $p\to k\in N$ because $V\mathbf{x}k0 = 0$ for $k = 0,1,...,n-1$ and $\sum_{i=0}^{m} a_i \frac{V\mathbf{x}^i k0}{V\mathbf{x}^i n0} - \frac{V\mathbf{aX}k0}{V\mathbf{aX}n0} = 1-1 = 0$ for $k=n$. When $p\to k = n+1$:

$$\sum_{i=0}^{m} a_i \frac{V\mathbf{x}^i k0}{V\mathbf{x}^i n0} - \frac{V\mathbf{aX}k0}{V\mathbf{aX}n0} = \sum_{i=0}^{m} a_i \sum_{j=0}^{n} x_j^i - \sum_{j=0}^{n}\sum_{i=0}^{m} a_i x_j^i = 0.$$

So, we can apply L'Hospital's rule and also the formula $\frac{d}{dp}V\mathbf{x}pq = V\mathbf{x}\ p\ q{+}1$ to get result (2.4). If $n=0$ the expressions for means $E_{Jen}(\mathbf{X},\mathbf{a};p,q)$ proceed from the continuous extensions of function

$$(p,q)\to\left(\frac{q}{p}\frac{\sum_{i=0}^{m} a_i (x_0^i)^p - (\sum_{i=0}^{m} a_i x_0^i)^p}{\sum_{i=0}^{m} a_i (x_0^i)^q - (\sum_{i=0}^{m} a_i x_0^i)^q}\right)^{\frac{1}{p-q}}.$$

3. Monotonicity of Jensen means for divided differences

Now we can introduce exponentially convex functions that will play an important role in this section. First, we give here a definition of exponentially convex function as it was done originally by Bernstein in [4] (see also [1],[8], [9]).

DEFINITION 1. A function $\varphi : (a,b)\to\mathbb{R}$ is exponentially convex if it is continuous and $\sum_{i,j=1}^{n} \xi_i\xi_j\,\varphi(x_i+x_j) \geqslant 0$ for every $n\in\mathbb{N}$ and all choices $\xi_1,\ldots,\xi_n\in\mathbb{R}$ and all choices $x_1,\ldots,x_n\in\ (a,b)$ so that $x_i+x_j\in(a,b)$.

In the rest of the paper we will rely on the following proposition and its corollaries.

PROPOSITION 3. *Let $\varphi : (a,b)\to\mathbb{R}$. Then the following statements are equivalent:*

(i) φ is exponentially convex.

(ii) φ is continuous and

$$\sum_{i,j=1}^{n} \xi_i\xi_j\,\varphi\left(\frac{x_i+x_j}{2}\right) \geqslant 0 \tag{3.1}$$

for every $n\in\mathbb{N}$ and all choices $\xi_1,\ldots,\xi_n\in\mathbb{R}$ and all choices $x_1,\ldots,x_n\in(a,b)$.

Using basic calculus, we have these two corollaries.

COROLLARY 2. *If φ is exponentially convex, then*

$$\det\left[\varphi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n} \geqslant 0$$

for every $n \in \mathbb{N}$ and all choices $x_1,\ldots,x_n \in (a,b)$.

COROLLARY 3. *Exponentially convex function $\varphi : (a,b) \to (0,\infty)$ is log-convex:*

$$\varphi\left(\frac{x+y}{2}\right) \leqslant \sqrt{\varphi(x)\varphi(y)} \quad \text{for all } x,y \in (a,b).$$

Using exponential convexity we will prove monotonicity of Jensen means $E(\mathbf{X},\mathbf{a};p,q)$ in both parameters p and q.

THEOREM 3. *Let $x_0^i, x_1^i, \ldots, x_n^i$ for $i = 0,\ldots,m$ be mutually different positive real numbers so that $\sum_{j=0}^{n}\sum_{k=0}^{j}\left(\sum_{i=0}^{m} a_i x_j^i x_k^i - \sum_{i=0}^{m} a_i x_j^i \sum_{l=0}^{m} a_l x_k^l\right) \neq 0$ for all $a_i \geqslant 0$ and $\sum_{i=0}^{m} a_i = 1$. Also let*

$$\varphi(\lambda) = \sum_{i=0}^{m} a_i f_\lambda[x_0^i,\ldots,x_n^i] - f_\lambda\left[\sum_{i=0}^{m} a_i x_0^i,\ldots,\sum_{i=0}^{m} a_i x_n^i\right],$$

where f_λ is defined with

$$f_\lambda(t) = \begin{cases} \frac{t^\lambda}{\lambda(\lambda-1)\cdots(\lambda-n-1)} & \text{if } \lambda \notin \{0,1,\ldots,n+1\}, \\ \frac{t^k \ln t}{(-1)^{n+1-k}k!(n+1-k)!} & \text{if } \lambda = k \in \{0,1,\ldots,n+1\}. \end{cases} \tag{3.2}$$

(i) The function φ is exponentially convex.

(ii) For every $n \in \mathbb{N}$ and all choices $t_1,\ldots,t_n \in \mathbb{R}$ the matrix $\left[\varphi\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^{n}$ is a positive semi-definite matrix. Particularly

$$\det\left[\varphi\left(\tfrac{t_i+t_j}{2}\right)\right]_{i,j=1}^{n} \geqslant 0. \tag{3.3}$$

Proof. (i) First, the expression

$$\varphi(\lambda) = \begin{cases} \frac{1}{\lambda(\lambda-1)\cdots(\lambda-n-1)}\left[\sum_{i=0}^{m} a_i \frac{V(\mathbf{x}^i;\lambda,0)}{V(\mathbf{x}^i;n,0)} - \frac{V(\mathbf{aX};\lambda,0)}{V(\mathbf{aX};n,0)}\right] & \text{if } \lambda \notin \{0,1,\ldots,n+1\}, \\ \frac{1}{(-1)^{n+1-k}k!(n+1-k)!}\left[\sum_{i=0}^{m} a_i \frac{V(\mathbf{x}^i;k,1)}{V(\mathbf{x}^i;n,0)} - \frac{V(\mathbf{aX};k,1)}{V(\mathbf{aX};n,0)}\right] & \text{if } \lambda = k \in \{0,1,\ldots,n+1\}, \end{cases}$$

shows that φ is a continuous function. Further, let us consider the function $f(t) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{t_i+t_j}{2}}(t)$, where $\xi_i \in \mathbb{R}$ and $t_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Since $f_\lambda^{(n+2)}(t) = t^{\lambda-n-2}$, it follows

$$f^{(n+2)}(t) = \sum_{i,j=1}^{n} \xi_i \xi_j t^{\frac{t_i+t_j}{2}-n-2} = \left(\sum_{i=1}^{n} \xi_i t^{\frac{t_i-n-2}{2}} \right)^2 \geqslant 0.$$

This is equivalent to inequality $\sum_{i=0}^{m} a_i f[x_0^i, \ldots, x_n^i] - f\left[\sum_{i=0}^{m} a_i x_0^i, \ldots, \sum_{i=0}^{m} a_i x_n^i\right] \geqslant 0$, i.e.

$$\sum_{i,j=1}^{n} \xi_i \xi_j \varphi\left(\frac{t_i+t_j}{2}\right) \geqslant 0. \tag{3.4}$$

We now conclude that φ is an exponentially convex function.

(ii) This follows from nonnegativity of quadratic form (3.4). □

LEMMA 1. *If $\varphi : (a,b) \to (0,\infty)$ is log-convex and $s,t,u,v \in (a,b)$ so that $s \neq t, u \neq v, s \leqslant u, t \leqslant u$ (or $s \leqslant v, t \leqslant v$); then*

$$\left(\frac{\varphi(t)}{\varphi(s)}\right)^{\frac{1}{t-s}} \leqslant \left(\frac{\varphi(v)}{\varphi(u)}\right)^{\frac{1}{v-u}}.$$

COROLLARY 4. *Let $p \leqslant u$ and $q \leqslant v$. Then*

$$E_{Jen}(\mathbf{X}, \mathbf{a}; p, q) \leqslant E_{Jen}(\mathbf{X}, \mathbf{a}; u, v) \tag{3.5}$$

for all $x_0^i, x_1^i, \ldots, x_n^i$ mutually different positive real numbers so that $\sum_{j=0}^{n} \sum_{k=0}^{j} \left(\sum_{i=0}^{m} a_i x_j^i x_k^i - \sum_{i=0}^{m} a_i x_j^i \sum_{l=0}^{m} a_l x_k^l \right) \neq 0$ *for all $a_i \geqslant 0$ and $\sum_{i=0}^{m} a_i = 1$.*

Proof. Let us first observe that for $p \neq q$ holds

$$E_{Jen}(\mathbf{X}, \mathbf{a}; p, q) = \left(\frac{\varphi(p)}{\varphi(q)}\right)^{\frac{1}{p-q}}. \tag{3.6}$$

According to Theorem 3 φ is a positive exponentially convex function and therefore, according to Corollary 3, a log-convex function. So, we can apply Lemma 1 on φ and get

$$\left(\frac{\varphi(t)}{\varphi(s)}\right)^{\frac{1}{t-s}} \leqslant \left(\frac{\varphi(v)}{\varphi(u)}\right)^{\frac{1}{v-u}} \tag{3.7}$$

for $s,t,u,v \in \mathbb{R}$ so that $s \neq t, u \neq v, s \leqslant u, t \leqslant u$ (or $s \leqslant v, t \leqslant v$). Using continuous extensions of (3.6) and (3.7) we finally conclude that, $p \leqslant u$ and $q \leqslant v$ implies

$$E_{Jen}(\mathbf{X}, \mathbf{a}; p, q) \leqslant E_{Jen}(\mathbf{X}, \mathbf{a}; u, v). \qquad \square$$

4. Special cases

If in (2.3) we put $x_0^i = x^i$, $x_j^i = x^i + h_j$ and $a_i = \frac{p_i}{P_m}$, where $i = 0, \ldots, m$, $P_m = \sum_{i=0}^m p_i$, $j = 1, \ldots, n$ and $x^i + h_j \in (a,b)$, we get means related to inequality (see [16]):

$$\frac{1}{P_m}\sum_{i=0}^m p_i f[x^i, x^i + h_1, \ldots, x^i + h_n] \geqslant f[\bar{x}, \bar{x} + h_1, \ldots, \bar{x} + h_n],$$

where $\bar{x} = \frac{1}{P_m}\sum_{i=0}^m p_i x^i$.

If we put $n = 1$, $h = x_1^i - x_0^i$ and $x_i = x_0^i$, $y_i = x_1^i$ for $i = 0, \ldots, m$ in above case we get means related to inequality (see [13]):

$$\frac{1}{P_m}\sum_{i=0}^m p_i f(x_i) - f\left(\frac{1}{P_m}\sum_{i=0}^m p_i x_i\right) \leqslant \frac{1}{P_m}\sum_{i=0}^m p_i f(y_i) - f\left(\frac{1}{P_m}\sum_{i=0}^m p_i y_i\right).$$

If in (2.3) we put $n = 1$, $x_0^i = 2a - x_i$, $x_1^i = x_i$, where $i = 0, \ldots, m$ and $x_i \in [0, 2a]$, we get means related to inequality (see [15] and [12]):

$$\sum_{i=0}^m a_i \frac{f(x_i) - f(2a - x_i)}{x_i - a} \leqslant \frac{f\left(\sum_{i=0}^m a_i x_i\right) - f\left(2a - \sum_{i=0}^m a_i x_i\right)}{\sum_{i=0}^m a_i x_i - a}.$$

4.1. Shur means

Let $\mathbf{x} = (x_0, \ldots, x_n)$ and $\mathbf{y} = (y_0, \ldots, y_n)$ denote two real $(n+1)$-tuples. We say that $\mathbf{x}$ *majorizes* $\mathbf{y}$ and put $\mathbf{x} \succ \mathbf{y}$ if

$$\sum_{i=0}^k x_{[i]} \geqslant \sum_{i=0}^k y_{[i]} \quad \text{for} \quad k = 0, 1, \ldots, n-1$$

and

$$\sum_{i=0}^n x_i = \sum_{i=0}^n y_i,$$

where

$$x_{[0]} \geqslant x_{[1]} \geqslant \cdots \geqslant x_{[n]} \quad \text{and} \quad y_{[0]} \geqslant y_{[1]} \geqslant \cdots \geqslant y_{[n]}$$

are the nonincreasing ordered components of $\mathbf{x}$ and $\mathbf{y}$.

An important tool in the study of majorization is a theorem (see [6]) which says that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} \succ \mathbf{y}$ if and only if $\mathbf{y} = \mathbf{x}\mathbf{P}$ for some double stohastic matrix $\mathbf{P}$. A square matrix $\mathbf{P}$ is said to be *stochastic* if its elements are all nonnegative and all row sums are one. If in addition to being stochastic, all column sums are one, the matrix is said to be *double stochastic*.

If we put $m = n$,

$$\mathbf{P} = \begin{bmatrix} a_0 & a_n & a_{n-1} & \ldots & a_2 & a_1 \\ a_1 & a_0 & a_n & \ldots & a_3 & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_0 & a_n \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_1 & a_0 \end{bmatrix},$$

and

$$\begin{aligned}
\mathbf{x}^0 &= (x_0, x_1, \ldots, x_{n-1}, x_n),\\
\mathbf{x}^1 &= (x_1, x_2, \ldots, x_n, x_0),\\
&\ldots,\\
\mathbf{x}^{n-1} &= (x_{n-1}, x_n, \ldots, x_{n-3}, x_{n-2}),\\
\mathbf{x}^n &= (x_n, x_0, \ldots, x_{n-2}, x_{n-1})
\end{aligned}$$

we have

$$\begin{aligned}
\mathbf{y} &= (y_0, y_1, \ldots, y_n)\\
&= (x_0, x_1, \ldots, x_n) \cdot \begin{bmatrix} a_0 & a_n & a_{n-1} & \ldots & a_2 & a_1\\ a_1 & a_0 & a_n & \ldots & a_3 & a_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_0 & a_n\\ a_n & a_{n-1} & a_{n-2} & \ldots & a_1 & a_0 \end{bmatrix}\\
&= \left(\sum_{i=0}^{m} a_i x_0^i, \ldots, \sum_{i=0}^{m} a_i x_n^i \right) = \mathbf{aX}.
\end{aligned}$$

Now, using Theorem 1 we get inequality (see [14]):

$$f[x_0, \ldots, x_n] \geqslant f[y_0, \ldots, y_n],$$

i.e. function G defined in Theorem 1 is Schur-convex. So, (2.3) in this case are Schur means (see [11]).

The following result (see [5]) is a consequence of the previous result:

THEOREM 4. *Let $f^{(n)}$ be convex in (a,b), $a \leqslant x_0 \leqslant \cdots \leqslant x_n \leqslant b$, and define $\bar{x} = \frac{1}{n+1} \sum_{i=0}^{n} x_i$. Then*

$$\frac{f^{(n)}(\bar{x})}{n!} = f\,[\underbrace{\bar{x}, \ldots, \bar{x}}_{n+1 \text{ times}}] \leqslant f[x_0, \ldots, x_n]. \tag{4.1}$$

If $x_0 \neq x_n$, then equality in (4.1) holds iff $f \in \Pi_{n+1}$ where Π_{n+1} denotes the class of polynomials of degree at most $n+1$.

For the Schur means related with above inequality we have the following:

$$E_{Sch}(\mathbf{x};p,q)=$$

$$\begin{cases}
\left(\prod_{i=0}^{n+1}\frac{q-i}{p-i}\,\frac{\frac{V\mathbf{x}p0}{V\mathbf{x}n0}-\frac{\prod_{i=0}^{n-1}(p-i)\bar{x}^{p}}{n!\cdot\bar{x}^{n}}}{\frac{V\mathbf{x}q0}{V\mathbf{x}n0}-\frac{\prod_{i=0}^{n-1}(q-i)\bar{x}^{q}}{n!\cdot\bar{x}^{n}}}\right)^{\frac{1}{p-q}} & \text{for } p\neq q;\ p\notin N, q\notin N\\
\left(\prod_{\substack{i=0\\ i\neq k}}^{n+1}\frac{(q-k)(q-i)}{k-i}\,\frac{\frac{V\mathbf{x}k1}{V\mathbf{x}n0}-\frac{\sum_{j=0}^{n-1}\prod_{\substack{i=0\\ i\neq j}}^{n-1}(k-i)\bar{x}^{k}+\prod_{i=0}^{n-1}(k-i)\bar{x}^{k}\ln\bar{x}}{n!\cdot\bar{x}^{n}}}{\frac{V\mathbf{x}q0}{V\mathbf{x}n0}-\frac{\prod_{i=0}^{n-1}(q-i)\bar{x}^{q}}{n!\cdot\bar{x}^{n}}}\right)^{\frac{1}{k-q}} & \text{for } p\neq q;\ p=k\in N, q\notin N\\
\left(\prod_{\substack{i=0\\ i\neq l}}^{n+1}\frac{l-i}{(p-l)(p-i)}\,\frac{\frac{V\mathbf{x}p0}{V\mathbf{x}n0}-\frac{\prod_{i=0}^{n-1}(p-i)\bar{x}^{p}}{n!\cdot\bar{x}^{n}}}{\frac{V\mathbf{x}l1}{V\mathbf{x}n0}-\frac{\sum_{j=0}^{n-1}\prod_{\substack{i=0\\ i\neq j}}^{n-1}(l-i)\bar{x}^{l}+\prod_{i=0}^{n-1}(l-i)\bar{x}^{l}\ln\bar{x}}{n!\cdot\bar{x}^{n}}}\right)^{\frac{1}{p-l}} & \text{for } p\neq q;\ p\notin N, q=l\in N\\
\left(\prod_{\substack{i=0\\ i\neq k,l}}^{n+1}\frac{l-i}{i-k}\,\frac{\frac{V\mathbf{x}k1}{V\mathbf{x}n0}-\frac{\sum_{j=0}^{n-1}\prod_{\substack{i=0\\ i\neq j}}^{n-1}(k-i)\bar{x}^{k}+\prod_{i=0}^{n-1}(k-i)\bar{x}^{k}\ln\bar{x}}{n!\cdot\bar{x}^{n}}}{\frac{V\mathbf{x}l1}{V\mathbf{x}n0}-\frac{\sum_{j=0}^{n-1}\prod_{\substack{i=0\\ i\neq j}}^{n-1}(l-i)\bar{x}^{l}+\prod_{i=0}^{n-1}(l-i)\bar{x}^{l}\ln\bar{x}}{n!\cdot\bar{x}^{n}}}\right)^{\frac{1}{k-l}} & \text{for } p\neq q;\ p=k\in N, q=l\in N\\
\exp\left(\sum_{i=0}^{n+1}\frac{1}{i-p}+\frac{\frac{V\mathbf{x}p1}{V\mathbf{x}n0}-\frac{\sum_{j=0}^{n-1}\prod_{\substack{i=0\\ i\neq j}}^{n-1}(p-i)\bar{x}^{p}+\prod_{i=0}^{n-1}(p-i)\bar{x}^{p}\ln\bar{x}}{n!\cdot\bar{x}^{n}}}{\frac{V\mathbf{x}p0}{V\mathbf{x}n0}-\frac{\prod_{i=0}^{n-1}(p-i)\bar{x}^{p}}{n!\cdot\bar{x}^{n}}}\right) & \text{for } p=q;\ p\notin N, q\notin N\\
\exp\left(\sum_{\substack{i=0\\ i\neq k}}^{n+1}\frac{1}{i-k}+\frac{1}{2}\,\frac{\frac{V\mathbf{x}k2}{V\mathbf{x}n0}-\frac{\sum_{j_1=0}^{n-1}\sum_{\substack{j_2=0\\ j_2\neq j_1}}^{n-1}\prod_{\substack{i=0\\ i\neq j_1\\ i\neq j_2}}^{n-1}(k-i)\bar{x}^{k}+2\sum_{j=0}^{n-1}\prod_{\substack{i=0\\ i\neq j}}^{n-1}(k-i)\bar{x}^{k}\ln\bar{x}+\prod_{i=0}^{n-1}(k-i)\bar{x}^{k}\ln^{2}\bar{x}}{n!\cdot\bar{x}^{n}}}{\frac{V\mathbf{x}k1}{V\mathbf{x}n0}-\frac{\sum_{j=0}^{n-1}\prod_{\substack{i=0\\ i\neq j}}^{n-1}(k-i)\bar{x}^{k}+\prod_{i=0}^{n-1}(k-i)\bar{x}^{k}\ln\bar{x}}{n!\cdot\bar{x}^{n}}}\right) & \\
\text{for } p=q;\ p=q=k\in N &
\end{cases}$$

Acknowledgement. The research of the authors was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grant $117-1170889-0888$.

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(Received February 24, 2011)

Z. Pavić
Mechanical Engineering Faculty
University of Osijek
Trg Ivane Brlić Mažuranić 2
35000 Slavonski Brod
Croatia
e-mail: Zlatko.Pavic@sfsb.hr

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6
10000 Zagreb, Croatia
and
Abdus Salam School of Mathematical Sciences
GC University
Lahore Gulberg 54660
Pakistan
e-mail: pecaric@element.hr

A. Vukelić
Faculty of Food Technology and Biotechnology
Mathematics department
University of Zagreb
Pierottijeva 6
10000 Zagreb
Croatia
e-mail: avukelic@pbf.hr