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INTEGRAL, DISCRETE AND FUNCTIONAL VARIANTS OF JENSEN'S INEQUALITY

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Abstract. We deal with the convex functions on the bounded closed convex sets with common barycenter. More precisely, the integral arithmetic means of convex function f are compared on these two sets A and B with $A \subset B$. The paper shows that series of inequalities

$$\frac{1}{\mu(A)}\int_A f(x)\,d\mu(x) \leqslant \frac{1}{\mu(B)}\int_B f(x)\,d\mu(x) \leqslant \frac{1}{\mu(B\setminus A)}\int_{B\setminus A} f(x)\,d\mu(x)$$

hold for convex functions of one variable, but it doesn't generally hold for convex functions of several variables. The article also gives discrete and functional variants of the mentioned inequality, and their applications to quasi-arithmetic and power means.

1. The integral variants of Jensen's inequality

The notion of the arithmetic mean can be expanded on sets and functions with help of measure and integral theory. This concept is barycenter for sets and integral arithmetic mean for functions. Consequently, the barycenter of μ-measurable set $S \subset \mathbb{R}^n$ (barycenter with respect to measure μ) is the n-tuple

$$s_\mu = \frac{1}{\mu(S)}\int_S s\,d\mu(s)$$

with coordinates

$$(s_\mu)_i = \frac{1}{\mu(S)}\int_S s_i\,d\mu(s) \quad \text{for} \quad i = 1,\dots,n.$$

The point s_μ is called μ-barycenter. The integral arithmetic mean of μ-integrable function g on set S (integral arithmetic mean with respect to measure μ) is the real number

$$g_{s_\mu} = \frac{1}{\mu(S)}\int_S g(s)\,d\mu(s).$$

The number g_{s_μ} is called μ-integral arithmetic mean. A basic connection between the barycenter and the integral arithmetic mean is written in the integral form of Jensen's inequality.

Through this paper $I \subseteq \mathbb{R}$ will be an interval.

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The integral form of Jensen's inequality. *Let* $(S,\mathbb{S},\mu)$ *be a finite measure space. Let* $g : S \to I$ *be a* μ*-integrable function. Then inequality*

$$f\left(\frac{1}{\mu(S)}\int_S g(s)\,d\mu(s)\right) \leqslant \frac{1}{\mu(S)}\int_S (f\circ g)(s)\,d\mu(s)$$

holds for every convex function $f : I \to \mathbb{R}$ *provided that* $f\circ g$ *is* μ*-integrable.*

This integral variant can be found in [8], and more general integral variant can be found in [10].

This paper is animated with some ideas from [3]. We recall general known facts from [2] and [3].

Through this paper μ will be a finite measure.

LEMMA 1. *Let* A *and* B *be bounded closed sets from* $\mathbb{R}^n$ *such that* $A\subset B$ *and* $0<\mu(A)<\mu(B)$. *If*

$$\frac{1}{\mu(A)}\int_A x\,d\mu(x) = \frac{1}{\mu(B)}\int_B x\,d\mu(x),$$

then

$$\frac{1}{\mu(A)}\int_A x\,d\mu(x) = \frac{1}{\mu(B\setminus A)}\int_{B\setminus A} x\,d\mu(x).$$

Proof. Using the properties of integral and the assumption, it follows

$$\begin{aligned}\frac{1}{\mu(B\setminus A)}\int_{B\setminus A} x\,d\mu(x) &= \frac{1}{\mu(B\setminus A)}\left(\int_B x\,d\mu(x) - \int_A x\,d\mu(x)\right)\\ &= \frac{1}{\mu(B\setminus A)}\left(\frac{\mu(B)}{\mu(A)}\int_A x\,d\mu(x) - \int_A x\,d\mu(x)\right)\\ &= \frac{\mu(B)-\mu(A)}{\mu(B\setminus A)\,\mu(A)}\int_A x\,d\mu(x) = \frac{1}{\mu(A)}\int_A x\,d\mu(x). \qquad \square\end{aligned}$$

Geometrical and mechanical meaning of Lemma 1: If λ is μ-barycenter of the sets A and B with $A\subset B$, then λ is μ-barycenter of the set $B\setminus A$ also.

LEMMA 2. *Let* A *and* B *be bounded closed sets from* $\mathbb{R}^n$ *such that* $A\subset B$ *and* $0<\mu(A)<\mu(B)$. *If one of three equalities*

$$\frac{1}{\mu(A)}\int_A x\,d\mu(x) = \frac{1}{\mu(B)}\int_B x\,d\mu(x) = \frac{1}{\mu(B\setminus A)}\int_{B\setminus A} x\,d\mu(x)$$

is valid, then equalities

$$\frac{1}{\mu(A)}\int_A h(x)\,d\mu(x) = \frac{1}{\mu(B)}\int_B h(x)\,d\mu(x) = \frac{1}{\mu(B\setminus A)}\int_{B\setminus A} h(x)\,d\mu(x)$$

hold for any affine function $h(x)=h(x_1,\ldots,x_n)=\sum_{i=1}^n k_i x_i + l$.

Proof. From

$$\frac{1}{\mu(A)}\int_A x_i\, d\mu(x) = \lambda_i$$

follows

$$\frac{1}{\mu(A)}\int_A h(x)\, d\mu(x) = \sum_{i=1}^{n}\frac{k_i}{\mu(A)}\int_A x_i\, d\mu(x) + \frac{l}{\mu(A)}\int_A d\mu(x) = \sum_{i=1}^{n} k_i\lambda_i + l. \qquad \square$$

In order to get a practical description of Lemma 2 let define measure ν on $h(B)$ by the rule

$$\nu(h(A)) = |k|\mu(A)$$

for all μ-measurable subsets A of B, excluding the case $k = 0$. Geometrical and mechanical meaning of Lemma 2 is the following: If λ is μ-barycenter of the sets $A, B, B\setminus A$ with $A \subset B$; then $h(\lambda) = k\lambda + l$ is ν-barycenter of the sets $h(A), h(B), h(B\setminus A)$.

PROPOSITION 1. *Let A and B be bounded closed intervals from $\mathbb{R}$ such that $A \subset B$ and $0 < \mu(A) < \mu(B)$. If one of three equalities*

$$\frac{1}{\mu(A)}\int_A x\, d\mu(x) = \frac{1}{\mu(B)}\int_B x\, d\mu(x) = \frac{1}{\mu(B\setminus A)}\int_{B\setminus A} x\, d\mu(x)$$

is valid, then series of inequalities

$$\frac{1}{\mu(A)}\int_A f(x)\, d\mu(x) \leqslant \frac{1}{\mu(B)}\int_B f(x)\, d\mu(x) \leqslant \frac{1}{\mu(B\setminus A)}\int_{B\setminus A} f(x)\, d\mu(x)$$

hold for every convex function $f : B \to \mathbb{R}$.

Proof. If $A = [a_1, a_2]$, let $y = h_A^{cho}(x)$ be a chord line through points $T_1(a_1, f(a_1))$ and $T_2(a_2, f(a_2))$. From the convexity of f and Lemma 2, it follows

$$\begin{aligned}\frac{1}{\mu(A)}\int_A f(x)\, d\mu(x) &\leqslant \frac{1}{\mu(A)}\int_A h_A^{cho}(x)\, d\mu(x) = \frac{1}{\mu(B\setminus A)}\int_{B\setminus A} h_A^{cho}(x)\, d\mu(x)\\ &\leqslant \frac{1}{\mu(B\setminus A)}\int_{B\setminus A} f(x)\, d\mu(x).\end{aligned}$$

Using this result we will prove the first inequality:

$$\begin{aligned}\frac{1}{\mu(A)}\int_A f(x)\, d\mu(x) &= \frac{\mu(B)}{\mu(B)\,\mu(A)}\int_A f(x)\, d\mu(x) = \frac{\mu(A)+\mu(B\setminus A)}{\mu(B)\,\mu(A)}\int_A f(x)\, d\mu(x)\\ &= \frac{1}{\mu(B)}\left(\int_A f(x)\, d\mu(x) + \frac{\mu(B\setminus A)}{\mu(A)}\int_A f(x)\, d\mu(x)\right)\\ &\leqslant \frac{1}{\mu(B)}\left(\int_A f(x)\, d\mu(x) + \int_{B\setminus A} f(x)\, d\mu(x)\right)\\ &= \frac{1}{\mu(B)}\int_B f(x)\, d\mu(x)\end{aligned}$$

Let us prove the second inequality:

$$\begin{aligned}\frac{1}{\mu(B)}\int_B f(x)\,d\mu(x) &= \frac{1}{\mu(B)}\left(\int_A f(x)\,d\mu(x)+\int_{B\setminus A} f(x)\,d\mu(x)\right)\\ &\leqslant \frac{1}{\mu(B)}\left(\frac{\mu(A)}{\mu(B\setminus A)}\int_{B\setminus A} f(x)\,d\mu(x)+\int_{B\setminus A} f(x)\,d\mu(x)\right)\\ &= \frac{1}{\mu(B\setminus A)}\int_{B\setminus A} f(x)\,d\mu(x) \qquad \square\end{aligned}$$

Proposition 1 can be expressed in more general situation with measure space $(S,\mathbb{S},\mu)$ and μ-integrable function $g: S\to\mathbb{R}$. However, it is assumed that μ-integral arithmetic means of a function g are the same on a pair of measurable sets $S_A, S_B\in\mathbb{S}$ with $S_A\subseteq S_B$.

THEOREM 1. *Let $(S,\mathbb{S},\mu)$ be a finite measure space. Let $g: S\to I$ be a μ-integrable function. Let A and B be bounded closed intervals from $\mathbb{R}$ so that $A\subset B\subseteq I$ and $0<\mu(S_A)<\mu(S_B)$, where $S_A=g^{-1}(A)$ and $S_B=g^{-1}(B)$. If one of three equalities*

$$\frac{1}{\mu(S_A)}\int_{S_A} g(s)\,d\mu(s)=\frac{1}{\mu(S_B)}\int_{S_B} g(s)\,d\mu(s)=\frac{1}{\mu(S_{B\setminus A})}\int_{S_{B\setminus A}} g(s)\,d\mu(s)$$

is valid, then series of inequalities

$$\begin{aligned}\frac{1}{\mu(S_A)}\int_{S_A}(f\circ g)(s)\,d\mu(s) &\leqslant \frac{1}{\mu(S_B)}\int_{S_B}(f\circ g)(s)\,d\mu(s)\\ &\leqslant \frac{1}{\mu(S_{B\setminus A})}\int_{S_{B\setminus A}}(f\circ g)(s)\,d\mu(s)\end{aligned}$$

hold for every convex function $f: I\to\mathbb{R}$.

Proof. Convex function f is continuous on Int$B$ and bounded on B. Therefore, the composition $f\circ g$ is μ-integrable on S_B. Series of inequalities follow from the proof of Proposition 1 by successively inserting

$$S_A\ ,\ S_B\ ,\ S_{B\setminus A}\ ,\ g(s)\ ,\ d\mu(s)$$

instead of

$$A\ ,\ B\ ,\ B\setminus A\ ,\ x\ ,\ d\mu(x).$$

At deriving the inequalities, an obvious fact $S_{B\setminus A}=S_B\setminus S_A$ also can be useful. $\square$

These series of inequalities can be extended from the left side if we use the integral Jensen inequality

$$f\left(\frac{1}{\mu(S_A)}\int_{S_A} g(s)\,d\mu(s)\right)\leqslant\frac{1}{\mu(S_A)}\int_{S_A}(f\circ g)(s)\,d\mu(s).$$

So, in the short form, we have

$$f\left(\frac{1}{\mu(S_A)}\int_{S_A} g\,d\mu\right) \leqslant \frac{1}{\mu(S_A)}\int_{S_A} f\circ g\,d\mu$$
$$\leqslant \frac{1}{\mu(S_B)}\int_{S_B} f\circ g\,d\mu \leqslant \frac{1}{\mu(S_{B\setminus A})}\int_{S_{B\setminus A}} f\circ g\,d\mu.$$

Stochastic meaning of Theorem 1 for probability space $(S,\mathbb{S},\mu)$ i.e. for $\mu(S)=1$ is the following: If the expectations of random variable g are the same on sets A and B, and therefore on $A\setminus B$ also,

$$E[g|A]=E[g|B]=E[g|B\setminus A],$$

then for the expectations of random variable $f\circ g$ on these sets hold

$$E[f\circ g|A]\leqslant E[f\circ g|B]\leqslant E[f\circ g|B\setminus A].$$

If we allow contraction of closed interval A towards point

$$a=E[g|B],$$

that is if $A=\{a\}$, then from the left side of the above double inequality follows Jensen's inequality for expectation on set B:

$$f(E[g|B])=f(a)=E[f\circ g|\{a\}]\leqslant E[f\circ g|B].$$

REMARK 1. If a function f is concave, then the reverse inequalities hold in Integral form of Jensen's inequality, Proposition 1 and Theorem 1.

2. The counterexamples for convex functions of two and three variables on polytopes with common barycenter

It is important to determine whether Proposition 1 is valid in the case when convex sets A and B belong to Euclidean space $\mathbb{R}^n$. Unfortunately, already for $n=2$ the nothing in Proposition 1 generally doesn't hold. In the next two examples the reverse inequalities hold for bounded closed convex sets, Lebesgue measure μ $(d\mu(x)=dx)$ and convex functions of two or three variables.

EXAMPLE 1. For polytopes A(four vertices) and B(six vertices) with common barycenter in origin,

$$A=\operatorname{co}\left\{\left(\pm 1,\pm\frac{1}{2}\right)\right\} \quad \textit{and} \quad B=\operatorname{co}\left\{\left(\pm 1,\pm\frac{1}{2}\right),\left(0,\pm\frac{3}{2}\right)\right\}$$

where co denotes convex hull, and convex function

$$f(x)=f(x_1,x_2)=\begin{cases} x_1 \ \textit{for}\ x_1\geqslant 0\\ 0 \ \ \textit{for}\ x_1\leqslant 0\end{cases}$$

the following holds:

$$\frac{1}{\mu(A)}\int_A f(x)\,d\mu(x) > \frac{1}{\mu(B)}\int_B f(x)\,d\mu(x) > \frac{1}{\mu(B\setminus A)}\int_{B\setminus A} f(x)\,d\mu(x) \tag{1}$$

Proof. Evidently, it follows $\mu(A)=2$, $\mu(B)=4$ and $\mu(B\setminus A)=2$. We have:

$$\frac{1}{\mu(A)}\int_A f(x)\,dx = \frac{1}{2}\int_0^1 x_1\,dx_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_2 = \frac{1}{4}$$

$$\frac{1}{\mu(B)}\int_B f(x)\,dx = \frac{5}{24}$$

$$\frac{1}{\mu(B\setminus A)}\int_{B\setminus A} f(x)\,dx = \frac{1}{6}$$

and the serious of inequalities in (1) hold. □

EXAMPLE 2. For polytopes A (eight vertices) and B (ten vertices) with common barycenter in origin,

$$A = co\left\{\left(\pm 1, \pm\frac{1}{2}, \pm 1\right)\right\} \quad and \quad B = co\left\{\left(\pm 1, \pm\frac{1}{2}, \pm 1\right), \left(0, \pm\frac{3}{2}, 0\right)\right\}$$

and convex function

$$f(x) = f(x_1,x_2,x_3) = \begin{cases} x_1 & for\ \ x_1 \geqslant 0 \\ 0 & for\ \ x_1 \leqslant 0 \end{cases}$$

the following holds:

$$\frac{1}{\mu(A)}\int_A f(x)\,d\mu(x) > \frac{1}{\mu(B)}\int_B f(x)\,d\mu(x) > \frac{1}{\mu(B\setminus A)}\int_{B\setminus A} f(x)\,d\mu(x) \tag{2}$$

Proof. Evidently, it follows $\mu(A)=4$, $\mu(B)=\frac{20}{3}$ and $\mu(B\setminus A)=\frac{8}{3}$. We have:

$$\frac{1}{\mu(A)}\int_A f(x)\,dx = \frac{1}{4}\int_0^1 x_1\,dx_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_2 \int_{-1}^{1} dx_3 = \frac{1}{4}$$

$$\frac{1}{\mu(B)}\int_B f(x)\,dx = \frac{9}{40}$$

$$\frac{1}{\mu(B\setminus A)}\int_{B\setminus A} f(x)\,dx = \frac{3}{16}$$

and so the serious of inequalities in (2) hold. □

3. The discrete variants of Jensen's inequality and applications to quasi-arithmetic and power means

The theory of convex functions can be subordinated to the theory of convex sets, since a convex function is the one whose epigraph is a convex set. However, the subject of convexity gets the full meaning when we consider a convex function on a convex set. The basic result on convex sets and convex functions is the well-known Jensen's inequality.

The discrete form of Jensen's inequality. *Let vectors $x_1,\dots,x_n$ belong to a convex set C in a real vector space. Let scalars $p_1,\dots,p_n$ be non-negative and $\mathbf{p}_n=\sum_{i=1}^n p_i$ be positive. Then inequality*

$$f\left(\frac{1}{\mathbf{p}_n}\sum_{i=1}^n p_i x_i\right)\leqslant \frac{1}{\mathbf{p}_n}\sum_{i=1}^n p_i f(x_i)$$

holds for every convex function $f:C\to\mathbb{R}$.

In other words, the f-value of the convex combinations of vectors is less than or equal to the convex combination of the f-values of vectors. This inequality can be used as an alternative definition of convexity.

The mechanical interpretation of Jensen's inequality can be proposed when we consider a set of n point-particles T_i with masses q_i, and so with total mass $\mathbf{q}_n=\sum_{i=1}^n q_i$. If the particles T_i are located over a convex curve $y=f(x)$ in plane at $T_i(x_i,y_i)$, then the following holds:

$$f\left(\frac{1}{\mathbf{q}_n}\sum_{i=1}^n q_i x_i\right)\leqslant \frac{1}{\mathbf{q}_n}\sum_{i=1}^n q_i y_i\,.$$

Now we suppose that a measure μ from Section 1 is discrete. Further, with respect to our integral case where $A\subset B$ and $0<\mu(A)<\mu(B)$, let:

$$A=\{x_i : i=1,\dots,m<n\}\quad,\quad B=\{x_i : i=1,\dots,n\}$$

$$\mu(x_i)=p_i\quad\text{for}\quad i=1,\dots,n$$

$$0<\mu(A)=\sum_{i=1}^m p_i=\mathbf{p}_m<\mathbf{p}_n=\sum_{i=1}^n p_i=\mu(B)$$

Consequently, it gives an idea for a discrete variant of Proposition 1:

PROPOSITION 2. *Let numbers $x_1,\dots,x_n$ belong to an interval $I\subseteq\mathbb{R}$ so that $x_i\notin$ $\operatorname{co}\{x_1,\dots,x_m\}$ for $i=m+1,\dots,n$. Let numbers $p_1,\dots,p_n$ be non-negative so that $0<\sum_{i=1}^m p_i=\mathbf{p}_m<\mathbf{p}_n=\sum_{i=1}^n p_i$. If one of three equalities*

$$\frac{1}{\mathbf{p}_m}\sum_{i=1}^m p_i x_i=\frac{1}{\mathbf{p}_n}\sum_{i=1}^n p_i x_i=\frac{1}{\mathbf{p}_n-\mathbf{p}_m}\sum_{i=m+1}^n p_i x_i$$

is valid, then series of inequalities

$$f\left(\frac{1}{\mathbf{p}_m}\sum_{i=1}^{m} p_i x_i\right) \leqslant \frac{1}{\mathbf{p}_m}\sum_{i=1}^{m} p_i f(x_i) \leqslant \frac{1}{\mathbf{p}_n}\sum_{i=1}^{n} p_i f(x_i) \leqslant \frac{1}{\mathbf{p}_n-\mathbf{p}_m}\sum_{i=m+1}^{n} p_i f(x_i)$$

hold for every convex function $f: I \to \mathbb{R}$.

REMARK 2. If a function f is concave, then the reverse inequalities hold in Discrete form of Jensen's inequality and Proposition 2.

Let $x_1, \ldots, x_n$ be real numbers from an interval I and $\alpha_1, \ldots, \alpha_n$ be non-negative real numbers such that $\sum_{i=1}^{n} \alpha_i = 1$. Let $\varphi: I \to \mathbb{R}$ be a strictly monotone function. The weighted quasi-arithmetic mean of $\{x_i\}$ with respect to a function φ is defined as

$$\mathscr{M}_n(\varphi\,,\,\alpha_i\,,\,x_i) = \varphi^{-1}\left(\sum_{i=1}^{n} \alpha_i\,\varphi(x_i)\right).$$

If all x_i are positive, and by putting strictly monotone functions

$$\varphi_r(x) = \begin{cases} x^r & \text{for } r \neq 0 \\ \ln x & \text{for } r = 0 \end{cases}$$

in an expression $\mathscr{M}_n(\varphi_r, \alpha_i, x_i)$, we obtain the weighted r-power means

$$\mathscr{M}_n^{[r]}(\alpha_i\,,\,x_i) = \begin{cases} \left(\sum_{i=1}^{n} \alpha_i x_i^r\right)^{\frac{1}{r}} & \text{for } r \neq 0 \\ \exp\left(\sum_{i=1}^{n} \alpha_i \ln x_i\right) & \text{for } r = 0 \end{cases}.$$

COROLLARY 1. *Let numbers* $x_1, \ldots, x_n$ *belong to an interval* $I \subseteq \mathbb{R}$ *so that* $x_i \notin \operatorname{co}\{x_1, \ldots, x_m\}$ *for* $i = m+1, \ldots, n$. *Let numbers* $p_1, \ldots, p_n$ *be non-negative so that* $0 < \sum_{i=1}^{m} p_i = \mathbf{p}_m < \mathbf{p}_n = \sum_{i=1}^{n} p_i$. *Let function* $\varphi: I \to \mathbb{R}$ *be strictly monotone. If one of three equalities*

$$\mathscr{M}_m\left(\frac{p_i}{\mathbf{p}_m}\,,\,\varphi(x_i)\right) = \mathscr{M}_n\left(\frac{p_i}{\mathbf{p}_n}\,,\,\varphi(x_i)\right) = \mathscr{M}_{n-m}\left(\frac{p_{m+i}}{\mathbf{p}_n-\mathbf{p}_m}\,,\,\varphi(x_{m+i})\right)$$

is valid, then series of inequalities

$$\mathscr{M}_m\left(f\,,\,\frac{p_i}{\mathbf{p}_m}\,,\,x_i\right) \leqslant \mathscr{M}_n\left(f\,,\,\frac{p_i}{\mathbf{p}_n}\,,\,x_i\right) \leqslant \mathscr{M}_{n-m}\left(f\,,\,\frac{p_{m+i}}{\mathbf{p}_n-\mathbf{p}_m}\,,\,x_{m+i}\right)$$

hold for every strictly increasing function $f: I \to \mathbb{R}$ *which is convex with respect to* φ, *i.e.* $f \circ \varphi^{-1}$ *is convex.*

Proof. If we use Proposition 2 with convex function $f \circ \varphi^{-1}$ on assumption

$$\frac{1}{\mathbf{p}_m}\sum_{i=1}^{m} p_i\,\varphi(x_i) = \frac{1}{\mathbf{p}_n}\sum_{i=1}^{n} p_i\,\varphi(x_i) = \frac{1}{\mathbf{p}_n-\mathbf{p}_m}\sum_{i=m+1}^{n} p_i\,\varphi(x_i),$$

we get

$$\frac{1}{\mathbf{p}_m}\sum_{i=1}^{m} p_i f(x_i) \leqslant \frac{1}{\mathbf{p}_n}\sum_{i=1}^{n} p_i f(x_i) \leqslant \frac{1}{\mathbf{p}_n-\mathbf{p}_m}\sum_{i=m+1}^{n} p_i f(x_i).$$

The wanted result follows after applying of an increasing function f^{-1}. □

REMARK 3. If f is strictly decreasing and concave with respect to φ, then series of inequalities in Corollary 1 are also valid. If either f is strictly increasing and concave with respect to φ or f is strictly decreasing and convex with respect to φ, then series of inequalities in Corollary 1 are reversed.

Especially, when quasi-arithmetic mean becomes r-power mean we have the following result.

COROLLARY 2. *Let numbers $x_1,\dots,x_n$ be positive so that $x_i \notin \operatorname{co}\{x_1,\dots,x_m\}$ for $i=m+1,\dots,n$. Let numbers $p_1,\dots,p_n$ be non-negative so that $0<\sum_{i=1}^{m} p_i = \mathbf{p}_m < \mathbf{p}_n = \sum_{i=1}^{n} p_i$. If one of three equalities*

$$\frac{1}{\mathbf{p}_m}\sum_{i=1}^{m} p_i x_i = \frac{1}{\mathbf{p}_n}\sum_{i=1}^{n} p_i x_i = \frac{1}{\mathbf{p}_n-\mathbf{p}_m}\sum_{i=m+1}^{n} p_i x_i$$

is valid, then series of inequalities

$$\mathscr{M}_m^{[r]}\left(\frac{p_i}{\mathbf{p}_m}, x_i\right) \geqslant \mathscr{M}_n^{[r]}\left(\frac{p_i}{\mathbf{p}_n}, x_i\right) \geqslant \mathscr{M}_{n-m}^{[r]}\left(\frac{p_{m+i}}{\mathbf{p}_n-\mathbf{p}_m}, x_{m+i}\right) \quad for\ r \leqslant 1$$

$$\mathscr{M}_m^{[r]}\left(\frac{p_i}{\mathbf{p}_m}, x_i\right) \leqslant \mathscr{M}_n^{[r]}\left(\frac{p_i}{\mathbf{p}_n}, x_i\right) \leqslant \mathscr{M}_{n-m}^{[r]}\left(\frac{p_{m+i}}{\mathbf{p}_n-\mathbf{p}_m}, x_{m+i}\right) \quad for\ r \geqslant 1$$

hold.

4. The functional approach

The results constructed for integrals in the first section can be generalized on linear functionals. In this section, we will mainly use Jessen's generalization of Jensen's inequality for unital positive linear functionals. Let S be a non-empty set and $\mathscr{S}$ be a vector space of real-valued functions $g:\mathscr{S}\to\mathbb{R}$. The space $\mathscr{S}$ which contains a unit function $\mathbf{1}$, by definition $\mathbf{1}(s)=1$ for every $s\in S$, we will denote with $\mathscr{S}_{\mathbf{1}}$. Linear functional $P:\mathscr{S}_{\mathbf{1}}\to\mathbb{R}$ is unital or normalized if $P(\mathbf{1})=1$.

We will also use the basic property of a convex function which says that the graph of a convex curve $y=f(x)$ on the bounded closed interval $[a,b]$ is located between a support line $y=h_{x_0}^{sup}(x)$ taken in $x_0\in(a,b)$ and a chord line $y=h_{[a,b]}^{cho}(x)$:

$$h_{x_0}^{sup}(x) \leqslant f(x) \leqslant h_{[a,b]}^{cho}(x) \quad \text{for every } x\in[a,b].$$

The weighted functional form of Jensen's inequality. *Let $w \in \mathscr{S}_{\mathbf{1}}$ be a positive function, $g : S \to I$ be a function so that $w \cdot g \in \mathscr{S}_{\mathbf{1}}$, and $P : \mathscr{S}_{\mathbf{1}} \to \mathbb{R}$ be a positive linear functional so that $P(w) > 0$. Then inequality*

$$f\left(\frac{1}{P(w)}P(w \cdot g)\right) \leqslant \frac{1}{P(w)}P(w \cdot (f \circ g))$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that $w \cdot (f \circ g) \in \mathscr{S}_{\mathbf{1}}$.

COROLLARY 3. *Let $w \in \mathscr{S}_{\mathbf{1}}$ be a positive function, $g : S \to [a,b]$ be a function so that $w \cdot g \in \mathscr{S}_{\mathbf{1}}$, and $P : \mathscr{S}_{\mathbf{1}} \to \mathbb{R}$ be a positive linear functional so that $P(w) > 0$. Then series of inequalities*

$$h_{x_0}^{sup}\left(\frac{P(w \cdot g)}{P(w)}\right) \leqslant f\left(\frac{P(w \cdot g)}{P(w)}\right) \leqslant \frac{P(w \cdot (f \circ g))}{P(w)} \leqslant h_{[a,b]}^{cho}\left(\frac{P(w \cdot g)}{P(w)}\right) \tag{3}$$

hold for every continuous convex function $f : [a,b] \to \mathbb{R}$ provided that $w \cdot (f \circ g) \in \mathscr{S}_{\mathbf{1}}$.

Now, we expose the functional variant of Theorem 1. Let χ_{S_A} denotes the characteristic function of the set $S_A = g^{-1}(A)$ for function $g : S \to I$ and subset $A \subseteq I$.

THEOREM 2. *Let $g : S \to I$ be a function and $P : \mathscr{S} \to \mathbb{R}$ be a positive linear functional. Let A and B be bounded closed intervals so that $A \subset B \subseteq I$, χ_{S_A}, χ_{S_B}, $\chi_{S_A} \cdot g$, $\chi_{S_B} \cdot g \in \mathscr{S}$ and $0 < P(\chi_{S_A}) < P(\chi_{S_B})$. If one of three equalities*

$$\frac{1}{P(\chi_{S_A})}P(\chi_{S_A} \cdot g) = \frac{1}{P(\chi_{S_B})}P(\chi_{S_B} \cdot g) = \frac{1}{P(\chi_{S_{B\setminus A}})}P(\chi_{S_{B\setminus A}} \cdot g)$$

is valid, then series of inequalities

$$\frac{1}{P(\chi_{S_A})}P(\chi_{S_A} \cdot (f \circ g)) \leqslant \frac{1}{P(\chi_{S_B})}P(\chi_{S_B} \cdot (f \circ g)) \leqslant \frac{1}{P(\chi_{S_{B\setminus A}})}P(\chi_{S_{B\setminus A}} \cdot (f \circ g))$$

hold for every convex function $f : I \to \mathbb{R}$ provided that $\chi_{S_A} \cdot (f \circ g)$, $\chi_{S_B} \cdot (f \circ g) \in \mathscr{S}$.

Proof. We can prove the inequalities similarly as in Proposition 1 using a chord line $y = h_A^{cho}(x)$ and the convexity of a curve $y = f(x)$. □

If a function f is continuous and convex, then the series of inequalities in Theorem 2 can be extended from the left side by using the weighted functional form of Jensen's inequality, and so we can get

$$f\left(\frac{P(\chi_{S_A} \cdot g)}{P(\chi_{S_A})}\right) \leqslant \frac{P(\chi_{S_A} \cdot (f \circ g))}{P(\chi_{S_A})} \leqslant \frac{P(\chi_{S_B} \cdot (f \circ g))}{P(\chi_{S_B})} \leqslant \frac{P(\chi_{S_{B\setminus A}} \cdot (f \circ g))}{P(\chi_{S_{B\setminus A}})}.$$

We can also express Theorem 2 in terms of the generalized means. The generalized mean of a function $g \in \mathscr{S}$ with respect to the positive linear functional $P : \mathscr{S} \to \mathbb{R}$ is the number

$$\mathscr{M}(P,g) = P(g).$$

COROLLARY 4. *Let $g : S \to I$ be a function and $P : \mathscr{S} \to \mathbb{R}$ be a positive linear functional. Let A and B be bounded closed intervals so that $A \subset B \subseteq I$, $\chi_{S_A}, \chi_{S_B} \in \mathscr{S}$ and $0 < P(\chi_{S_A}) < P(\chi_{S_B})$. Let $\varphi : I \to \mathbb{R}$ be a strictly monotone function so that $\chi_{S_A} \cdot (\varphi \circ g), \chi_{S_B} \cdot (\varphi \circ g) \in \mathscr{S}$. If one of three equalities*

$$\mathscr{M}\left(P, \frac{\chi_{S_A} \cdot (\varphi \circ g)}{P(\chi_{S_A})}\right) = \mathscr{M}\left(P, \frac{\chi_{S_B} \cdot (\varphi \circ g)}{P(\chi_{S_B})}\right) = \mathscr{M}\left(P, \frac{\chi_{S_{B\setminus A}} \cdot (\varphi \circ g)}{P(\chi_{S_{B\setminus A}})}\right)$$

is valid, then series of inequalities

$$\mathscr{M}\left(P, \frac{\chi_{S_A} \cdot (f \circ g)}{P(\chi_{S_A})}\right) \leqslant \mathscr{M}\left(P, \frac{\chi_{S_B} \cdot (f \circ g)}{P(\chi_{S_B})}\right) \leqslant \mathscr{M}\left(P, \frac{\chi_{S_{B\setminus A}} \cdot (f \circ g)}{P(\chi_{S_{B\setminus A}})}\right)$$

hold for every function $f : I \to \mathbb{R}$ which is convex with respect to φ provided that $\chi_{S_A} \cdot (f \circ g), \chi_{S_B} \cdot (f \circ g) \in \mathscr{S}$.

REMARK 4. Theorem 1 follows from Theorem 2 if we put

$$P(g) = \frac{1}{\int_S d\mu(s)} \int_S g(s)\, d\mu(s) = \frac{1}{\mu(S)} \int_S g(s)\, d\mu(s).$$

Then, these special cases hold:

$$P(\chi_{S_A}) = \frac{1}{\int_S d\mu(s)} \int_{S_A} d\mu(s) = \frac{1}{\mu(S)}\, \mu(S_A)$$

$$P(\chi_{S_A} \cdot g) = \frac{1}{\int_S d\mu(s)} \int_{S_A} g(s)\, d\mu(s) = \frac{1}{\mu(S)} \int_{S_A} g(s)\, d\mu(s)$$

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