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TWO SHARP INEQUALITIES FOR LEHMER MEAN, IDENTRIC MEAN AND LOGARITHMIC MEAN

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Abstract. For $r \in \mathbb{R}$, the Lehmer mean of two positive numbers a and b is defined by

$$L_r(a,b) = \frac{a^{r+1} + b^{r+1}}{a^r + b^r}.$$

In this paper, we establish two sharp inequalities as follows: $I(a,b) > L_{-\frac{1}{6}}(a,b)$ and $L(a,b) > L_{-\frac{1}{3}}(a,b)$ for all $a,b > 0$ with $a \neq b$. Here $I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ and $L(a,b) = \frac{b-a}{\log b - \log a}$ denote the identric mean and logarithmic mean of two positive numbers a and b with $a \neq b$, respectively.

1. Introduction

For $r \in \mathbb{R}$, the Lehmer mean of two positive numbers a and b is defined by

$$L_r(a,b) = \frac{a^{r+1} + b^{r+1}}{a^r + b^r}.$$

It is well-known that $L_r(a,b)$ is continuous and strictly increasing with respect to $r \in \mathbb{R}$ for fixed a and b with $a \neq b$, and its properties can be found in [1–5]. If we denote by $A(a,b) = \frac{a+b}{2}$, $I(a,b) = \frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}$, $L(a,b) = \frac{b-a}{\ln b - \ln a}$, $G(a,b) = \sqrt{ab}$ and $H(a,b) = \frac{2ab}{a+b}$ the arithmetic mean, identric mean, logarithmic mean, geometric mean and harmonic mean of two positive numbers a and b with $a \neq b$, respectively, then

$$\min\{a,b\} < H(a,b) = L_{-1}(a,b) < G(a,b) = L_{-\frac{1}{2}}(a,b) < L(a,b)$$
$$< I(a,b) < A(a,b) = L_0(a,b) < \max\{a,b\}.$$

In the recent past, the logarithmic mean and identric mean have been the subject intensive research. In particular, many remarkable inequalities for L and I can be found in the literature [6–12], the following Theorem A and its proof can be found in [13].

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THEOREM A. *For all positive numbers a and b with $a \neq b$, we have*

$$\sqrt{G(a,b)A(a,b)} < \sqrt{L(a,b)I(a,b)} < \frac{1}{2}(L(a,b)+I(a,b)) < \frac{1}{2}(G(a,b)+A(a,b)).$$

For $p \in \mathbb{R}$, the p-th power mean $M_p(a,b)$ of two positive numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}$$

The properties of these means are given in [14]. Several authors discussed the relationship of certain means to M_r. The following sharp bounds for L, I, $(IL)^{\frac{1}{2}}$ and $\frac{I+L}{2}$ in terms of power means are proved in [13, 15–19].

THEOREM B. *For all positive real numbers a and b with $a \neq b$ we have*

$$M_0(a,b) < L(a,b) < M_{\frac{1}{3}}(a,b), \quad M_{\frac{2}{3}}(a,b) < I(a,b) < M_{\log 2}(a,b),$$

$$M_0(a,b) < I^{\frac{1}{2}}(a,b)L^{\frac{1}{2}}(a,b) < M_{\frac{1}{2}}(a,b)$$

and

$$\frac{1}{2}[I(a,b)+L(a,b)] < M_{\frac{1}{2}}(a,b).$$

The main purpose of this paper is to answer the questions: What are the greatest values α and β, such that $I(a,b) > L_\alpha(a,b)$ and $L(a,b) > L_\beta(a,b)$ for all $a,b > 0$ with $a \neq b$.

2. Main Results

THEOREM 2.1. *$I(a,b) > L_{-\frac{1}{6}}(a,b)$ for all $a,b > 0$ with $a \neq b$, and $L_{-\frac{1}{6}}(a,b)$ is the best possible lower Lehmer mean bound for the identric mean $I(a,b)$.*

Proof. Without loss of generality, we assume that $a > b$. Let $t = \sqrt[6]{\frac{a}{b}} > 1$, then simple computation leads to

$$\begin{aligned} &\log I(a,b) - \log L_{-\frac{1}{6}}(a,b) \\ &= \frac{5t^6+1}{t^6-1}\log t - \log(t^5+1) + \log(t+1) - 1. \end{aligned} \tag{2.1}$$

Let $f(t) = \frac{5t^6+1}{t^6-1}\log t - \log(t^5+1) + \log(t+1) - 1$, then

$$\lim_{t \to 1} f(t) = 0, \tag{2.2}$$

$$f'(t) = \frac{t^5}{(t^6-1)^2}g(t), \tag{2.3}$$

where

$$g(t) = -36\log t + \frac{g_1(t)}{t^6(t^4 - t^3 + t^2 - t + 1)} \tag{2.4}$$

and

$$g_1(t) = t^{16} - 2t^{15} + 3t^{14} - 4t^{13} + 5t^{12} + 4t^{10} - 2t^9 + 2t^7, \\ -4t^6 - 5t^4 + 4t^3 - 3t^2 + 2t - 1.$$

Differentiating $g(t)$ yields

$$g'(t) = \frac{(t+1)^2(t-1)^4(t^2+t+1)(t^2-t+1)g_2(t)}{t^7(t^4 - t^3 + t^2 - t + 1)^2}, \tag{2.5}$$

where

$$\begin{aligned} g_2(t) &= 6t^{10} - 5t^9 + 22t^8 - 18t^7 + 50t^6 - 23t^5 \\ &\quad +50t^4 - 18t^3 + 22t^2 - 5t + 6 \\ &= t(t-1)(5t^8 + 18t^6 + 23t^4 + 18t^2 + 5) \\ &\quad +t^{10} + 4t^8 + 27t^6 + 32t^4 + 17t^2 + 6. \end{aligned} \tag{2.6}$$

From (2.6) we clearly see that $g_2(t) > 0$ for $t > 1$, then (2.4) and (2.5) imply that $g(t) > g(1) = 0$ for $t > 1$. Therefore, $I(a,b) > L_{-\frac{1}{6}}(a,b)$ follows from (2.1)–(2.3) and $g(t) > 0$ for $t > 1$.

Next, we prove that $L_{-\frac{1}{6}}(a,b)$ is the best possible lower Lehmer mean bound for the identric mean $I(a,b)$.

For any $\varepsilon \in (0, \frac{1}{6})$ and $x > 0$, one has

$$\begin{aligned} &L_{-\frac{1}{6}+\varepsilon}(1,1+x) - I(1,1+x) \\ &= \frac{(x+1)^{\frac{5}{6}+\varepsilon} + 1}{(x+1)^{-\frac{1}{6}+\varepsilon} + 1} - e^{\frac{x+1}{x}\log(1+x) - 1} \\ &= \frac{h(x)}{(x+1)^{-\frac{1}{6}+\varepsilon} + 1}, \end{aligned} \tag{2.7}$$

where $h(x) = (x+1)^{\frac{5}{6}+\varepsilon} + 1 - [(x+1)^{-\frac{1}{6}+\varepsilon} + 1]e^{\frac{x+1}{x}\log(1+x) - 1}$.

Letting $x \to 0$ and making use of Taylor expansion, we get

$$\begin{aligned} h(x) &= 2 + (\tfrac{5}{6} + \varepsilon)x + (\tfrac{5}{6} + \varepsilon)(\tfrac{\varepsilon}{2} - \tfrac{1}{12})x^2 + o(x^2) - [2 + (\varepsilon - \tfrac{1}{6})x \\ &\quad + (\varepsilon - \tfrac{1}{6})(\tfrac{\varepsilon}{2} - \tfrac{7}{12})x^2 + o(x^2)][1 + \tfrac{1}{2}x - \tfrac{1}{24}x^2 + o(x^2)] \\ &= 2 + (\tfrac{5}{6} + \varepsilon)x + (\tfrac{5}{6} + \varepsilon)(\tfrac{\varepsilon}{2} - \tfrac{1}{12})x^2 - 2 - x - (\varepsilon - \tfrac{1}{6})x \\ &\quad - (\tfrac{\varepsilon}{2} - \tfrac{1}{12})x^2 - (\varepsilon - \tfrac{1}{6})(\tfrac{\varepsilon}{2} - \tfrac{7}{12})x^2 + \tfrac{1}{12}x^2 + o(x^2) \\ &= \tfrac{1}{2}\varepsilon x^2 + o(x^2). \end{aligned} \tag{2.8}$$

Equations (2.7) and (2.8) imply that for any $0 < \varepsilon < \frac{1}{6}$, there exists $0 < \delta_1 = \delta_1(\varepsilon) < 1$, such that $L_{-\frac{1}{6}+\varepsilon}(1,1+x) > I(1,1+x)$ for $x \in (0, \delta_1)$. $\square$

REMARK 2.1. For $\varepsilon > 0$, one has

$$\lim_{t\to+\infty} \frac{I(1,t)}{L_{-\varepsilon}(1,t)} = \lim_{t\to+\infty} \frac{\frac{1}{e}t^{\frac{t}{t-1}}}{\frac{1+t^{-\varepsilon+1}}{1+t^{-\varepsilon}}} = +\infty. \tag{2.9}$$

Therefore $L_0(a,b) = A(a,b)$ is the best possible upper Lehmer mean bound for the identric mean $I(a,b)$. It follows from (2.9) and the well-known result $L_0(a,b) = A(a,b) > I(a,b)$ for all $a,b > 0$ with $a \neq b$.

THEOREM 2.2. *$L(a,b) > L_{-\frac{1}{3}}(a,b)$ for all $a,b > 0$ with $a \neq b$, and $L_{-\frac{1}{3}}(a,b)$ is the best possible lower Lehmer mean bound for the logarithmic mean $L(a,b)$.*

Proof. Without loss of generality, we assume that $a > b$. Let $t = \sqrt[3]{\frac{a}{b}} > 1$, then

$$\begin{aligned} &L(a,b) - L_{-\frac{1}{3}}(a,b) \\ &= b\left[\frac{t^3-1}{3\log t} - \frac{t(t^2+1)}{t+1}\right] \\ &= \frac{b}{3(t+1)\log t}[-3t(t^2+1)\log t + t^4 + t^3 - t - 1]. \end{aligned} \tag{2.10}$$

Let $f(t) = -3t(t^2+1)\log t + t^4 + t^3 - t - 1$, then simple computations lead to

$$f(1) = 0, \tag{2.11}$$

$$f'(t) = -3(1+3t^2)\log t + 4t^3 - 4, \tag{2.12}$$

$$f'(1) = 0, \tag{2.13}$$

$$f''(t) = \frac{3}{t} f_1(t), \tag{2.14}$$

where

$$f_1(t) = -6t^2\log t + 4t^3 - 3t^2 - 1, \tag{2.15}$$

$$f_1(1) = 0, \tag{2.16}$$

$$f_1'(t) = 12t(t - 1 - \log t). \tag{2.17}$$

Let $f_2(t) = t - 1 - \log t$, then

$$f_2(1) = 0, \tag{2.18}$$

$$f_2'(t) = \frac{t-1}{t} > 0. \tag{2.19}$$

Therefore the inequality $L(a,b) > L_{-\frac{1}{3}}(a,b)$ follows from (2.10)–(2.19).

Now we prove that $L_{-\frac{1}{3}}(a,b)$ is the best possible lower Lehmer mean bound for the logarithmic mean $L(a,b)$.

For any $\varepsilon \in (0, \frac{1}{3})$ and $x > 0$, one has

$$\begin{aligned} &L_{-\frac{1}{3}+\varepsilon}(1,1+x) - L(1,1+x) \\ &= \frac{(1+x)^{\frac{2}{3}+\varepsilon}+1}{(1+x)^{-\frac{1}{3}+\varepsilon}+1} - \frac{x}{\log(1+x)} \\ &= \frac{g(x)}{[(1+x)^{-\frac{1}{3}+\varepsilon}+1]\log(1+x)}, \end{aligned} \tag{2.20}$$

where $g(x) = [1+(1+x)^{\frac{2}{3}+\varepsilon]}\log(1+x) - x[1+(1+x)^{-\frac{1}{3}+\varepsilon}]$.

Letting $x \to 0$ and making use of Taylor expansion, we get

$$\begin{aligned} g(x) = &\left[2+\left(\frac{2}{3}+\varepsilon\right)x+\left(\frac{2}{3}+\varepsilon\right)\left(\frac{\varepsilon}{2}-\frac{1}{6}\right)x^2+o(x^2)\right]\left[x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)\right] \\ &-x\left[2+\left(\varepsilon-\frac{1}{3}\right)x+\left(\varepsilon-\frac{1}{3}\right)\left(\frac{\varepsilon}{2}-\frac{2}{3}\right)x^2+o(x^2)\right] \\ = x&\left[2+\left(\varepsilon-\frac{1}{3}\right)x-\left(\frac{1}{3}+\frac{\varepsilon}{2}\right)x^2+\frac{2}{3}x^2+\left(\frac{2}{3}+\varepsilon\right)\left(\frac{\varepsilon}{2}-\frac{1}{6}\right)x^2\right. \\ &\left.-2-\left(\varepsilon-\frac{1}{3}\right)x-\left(\varepsilon-\frac{1}{3}\right)\left(\frac{\varepsilon}{2}-\frac{2}{3}\right)x^2+o(x^2)\right] \\ = &\frac{1}{2}\varepsilon x^3+o(x^3). \end{aligned} \tag{2.21}$$

Equations (2.20) and (2.21) imply that for any $0 < \varepsilon < \frac{1}{3}$, there exists $0 < \delta_2 = \delta_2(\varepsilon) < 1$, such that $L_{-\frac{1}{3}+\varepsilon}(1,1+x) > L(1,1+x)$ for $x \in (0, \delta_2)$. $\square$

REMARK 2.2. For $\varepsilon > 0$, one has

$$\lim_{t\to+\infty}\frac{L(1,t)}{L_{-\varepsilon}(1,t)} = \lim_{t\to+\infty}\frac{t+t^{\varepsilon+1}-t^{\varepsilon}-1}{(t^{\varepsilon}+t)\log t} = +\infty. \tag{2.22}$$

Therefore $L_0(a,b) = A(a,b)$ is the best possible upper Lehmer mean bound for the logarithmic mean $L(a,b)$. It follows from (2.22) and the well-known result $L(a,b) < A(a,b)$ for all $a,b > 0$ with $a \neq b$.

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