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ASYMPTOTIC BEHAVIOR OF HARDY OPERATORS

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Abstract. We consider the asymptotic behavior of certain Hardy operators.

1. Introduction

We consider operators of the form

$$T_\phi f(x) = \frac{1}{\Phi(x)} \int_0^x \phi(s) f(s)\, ds$$

and its dual

$$\tilde{T}_\phi f(x) = \phi(x) \int_x^a \frac{f(s)}{\Phi(s)}\, ds$$

where ϕ is a positive nondecreasing function on $(0,a)$, and

$$\Phi(x) = \int_0^x \phi(s)\, ds.$$

The L^p-continuity of such operators have been studied extensively in the literature, since the L^p-continuity in the cases $\phi_\gamma(s) = s^\gamma$, $\gamma > 0$, are Hardy's inequalities, two of the most useful inequalities in Analysis. For two comprehensive recent books on inequalitites of the Hardy type, see [3] and [4].

We have not been able to find in the literature any work on the interesting related direction: under what condition are families of these operators approximations of the identity. That is to say, under what conditions does one have

$$\lim_{\gamma \to \infty} T_{\phi_\gamma} f(x) = f(x)$$

both in the L^p sense and in the almost everywhere sense. It has been observed by A. Erdélyi in [1], and several times since then, that when ϕ is a power function, the operators T_ϕ and $\tilde{T}_\phi$ are convolution operators on the multiplicative group R_+, with its Haar measure. As observed in [1], it follows that various versions of Hardy's inequalities can be viewed as special cases of Young's inequalities for convolutions. In the same vein, the approximation of the identity results we prove, when ϕ is a power function, can be viewed as special cases of results for convolutions. The novelty in this paper is that it applies to a much broader class of kernels.

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2. The operator T_{ϕ_γ}

In the sequel, when we consider intervals of the form $(0,a)$, we shall assume that $0 < a \leqslant \infty$, unless otherwise specified. Also, $\mathfrak{W}$ will denote the family of sets of positive nondecreasing functions $\{\phi_\gamma\}$ indexed by $\gamma \in \mathbf{R}$ on $(0,a)$. Assume that for all sufficiently large γ, $\phi_\gamma \in L^1(I)$ for each bounded subinterval I of $(0,a)$, and denoting

$$\Phi_\gamma(x) = \int_0^x \phi_\gamma(s)\,ds$$

assume also that for $u \in (0,1)$, $x \in (0,a)$,

$$\lim_{\gamma\to\infty} \frac{\phi_\gamma(ux)}{\Phi_\gamma(x)} = 0. \tag{1}$$

THEOREM 1. *Assume that* $\{\phi_\gamma\} \in \mathfrak{W}$. *Assume also that there exists* $\gamma_0 \in \mathbf{R}$ *such that for all* $\gamma \geqslant \gamma_0$

$$\frac{\phi_\gamma(x)}{\phi_{\gamma_0}(x)}$$

is nondecreasing on $(0,a)$.

Let f *be a measurable function on* $(0,a)$ *such that* $\phi_{\gamma_0} f \in L^1(I)$ *for each bounded subinterval* I *of* $(0,a)$. *Then at any Lebesgue point* $x \in (0,a)$ *of* f,

$$\lim_{\gamma\to\infty} \frac{1}{\Phi_\gamma(x)} \int_0^x \phi_\gamma(s)|f(s)-f(x)|\,ds = 0.$$

Proof. We can assume that for each $\gamma \in \mathbf{R}$, ϕ_γ is right-continuous.

$$\begin{aligned}
&\frac{1}{\Phi_\gamma(x)} \int_0^x \phi_\gamma(s)|f(s)-f(x)|\,ds \\
&= \frac{1}{\Phi_\gamma(x)} \int_0^{x-\delta} \phi_\gamma(s)|f(s)-f(x)|\,ds + \frac{1}{\Phi_\gamma(x)} \int_{x-\delta}^x \phi_\gamma(s)|f(s)-f(x)|\,ds.
\end{aligned}$$

Let $\varepsilon > 0$ be given. There exists $\delta_0 > 0$ be such that for $0 < \delta < \delta_0$,

$$\frac{1}{\delta} \int_{x-\delta}^x |f(s)-f(x)|\,ds < \varepsilon.$$

For $0 < \delta < \delta_0$,

$$\begin{aligned}
&\frac{1}{\Phi_\gamma(x)}\int_{x-\delta}^{x}\phi_\gamma(s)|f(s)-f(x)|\,ds\\
&=\frac{1}{\Phi_\gamma(x)}\int_{x-\delta}^{x}\left(\phi_\gamma(x-\delta)+\int_{(x-\delta,s]}d\phi_\gamma(t)\right)|f(s)-f(x)|\,ds\\
&=\frac{1}{\Phi_\gamma(x)}\int_{(x-\delta,x]}\left(\int_t^x|f(s)-f(x)|\,ds\right)d\phi_\gamma(t)+\frac{\phi_\gamma(x-\delta)}{\Phi_\gamma(x)}\int_{x-\delta}^{x}|f(s)-f(x)|\,ds\\
&\leqslant\frac{\varepsilon}{\Phi_\gamma(x)}\left(\int_{(x-\delta,x]}(x-t)\,d\phi_\gamma(t)+\delta\phi_\gamma(x-\delta)\right)\\
&=\frac{\varepsilon}{\Phi_\gamma(x)}\int_{x-\delta}^{x}\phi_\gamma(t)\,dt\leqslant\varepsilon.
\end{aligned}$$

Also, for $\gamma \geqslant \gamma_0$,

$$\begin{aligned}
&\limsup_{\gamma\to\infty}\frac{1}{\Phi_\gamma(x)}\int_0^{x-\delta}\phi_\gamma(s)|f(s)-f(x)|\,ds\\
&=\limsup_{\gamma\to\infty}\frac{1}{\Phi_\gamma(x)}\int_0^{x-\delta}\frac{\phi_\gamma(s)}{\phi_{\gamma_0}(s)}\phi_{\gamma_0}(s)|f(s)-f(x)|\,ds\\
&\leqslant\limsup_{\gamma\to\infty}\frac{\phi_\gamma(x-\delta)}{\Phi_\gamma(x)}\frac{1}{\phi_{\gamma_0}(x-\delta)}\int_0^{x-\delta}\phi_{\gamma_0}(s)(|f(s)|+|f(x)|)\,ds.
\end{aligned}$$

By (1),

$$\lim_{\gamma\to\infty}\frac{\phi_\gamma(x-\delta)}{\Phi_\gamma(x)}=0$$

and so

$$\limsup_{\gamma\to\infty}\frac{1}{\Phi_\gamma(x)}\int_0^{x-\delta}\phi_\gamma(s)|f(s)-f(x)|\,ds=0.$$

Since $\varepsilon > 0$ was arbitrary,

$$\lim_{\gamma\to\infty}\frac{1}{\Phi_\gamma(x)}\int_0^{x}\phi_\gamma(s)|f(s)-f(x)|\,ds=0. \qquad \square$$

DEFINITION 1. For $\gamma \in \mathbf{R}$, define the operator A_γ on functions, f, such that $s^\gamma f(s) \in L^1(I)$ for each bounded subinterval I of $(0,a)$ by

$$A_\gamma f(x)=\frac{\gamma+1}{x^{\gamma+1}}\int_0^x s^\gamma f(s)\,ds.$$

Also, for $\gamma\in\mathbf{R}$, define the operator $\tilde{A}_\gamma$ on functions, f, such that $s^{-\gamma-1}f(s)\in L^1(x,a)$ for each $x\in(0,a)$ by

$$\tilde{A}_\gamma f(x)=\gamma x^\gamma\int_x^a s^{-\gamma-1}f(s)\,ds.$$

Applying Theorem 1 with $\phi_\gamma(x) = x^\gamma$ proves:

COROLLARY 1. *Let f be measurable on $(0,a)$. Suppose there exists $\gamma_0 \geqslant 0$ such that $x^{\gamma_0} f(x) \in L^1(I)$ for each bounded subinterval I of $(0,a)$. If $x \in (0,a)$ is a Lebesgue point of f, then*

$$\lim_{\gamma\to\infty} A_\gamma f(x) = f(x).$$

DEFINITION 2. Let $E \subseteq \mathbf{R}$ be measurable. For f measurable on E and $\omega(x) > 0$ a.e. and measurable on E, for $0 < p \leqslant \infty$, let

$$\|f\|_{L^p_\omega(E)} = \left(\int_E (\omega(x)|f(x)|)^p \, dx\right)^{\frac{1}{p}}$$

and let

$$L^p_\omega(E) = \{f : \|f\|_{L^p_\omega(E)} < \infty\}.$$

We write L^p_ω for $L^p_\omega(E)$ when E is clear from the context. We write L^p or $L^p(E)$ if $\omega = 1$ a.e.

Let us consider convergence in $L^p_{x^\alpha}(0,a)$.

Since for $0 < p < \infty$, $\alpha \in \mathbf{R}$, continuous functions with compact support are dense in $L^p_{x^\alpha}$,

THEOREM 2. *If $f \in L^p_{x^\alpha}$, $0 < p < \infty$, $\alpha \in \mathbf{R}$, then*

$$\lim_{t\to 1} \|f(tx) - f(x)\|_{L^p_{x^\alpha}} = 0.$$

THEOREM 3. *Assume that $\{\phi_\gamma\} \in \mathfrak{W}$. Assume also that there exist $\Psi_\gamma(u)$ so that for $u \in (0,1)$,*

$$\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)} \leqslant \Psi_\gamma(u)$$

for all $x \in (0,a)$, and so that

$$\limsup_{\gamma\to\infty} \|\Psi_\gamma\|_{L^1(0,1)} = C < \infty$$

and for all $\beta > 0$ and $0 < \theta < 1$,

$$\lim_{\gamma\to\infty} \|u^{-\beta}\Psi_\gamma(u)\|_{L^1(0,\theta)} = 0.$$

Then for $f \in L^p_{x^\alpha}(0,a)$, $1 \leqslant p < \infty$, $\alpha \in \mathbf{R}$,

$$\lim_{\gamma\to\infty} \left\| \frac{1}{\Phi_\gamma(x)} \int_0^x \phi_\gamma(s) f(s)\, ds - f(x) \right\|_{L^p_{x^\alpha}(0,a)} = 0.$$

Proof. Note that for γ sufficiently large, for $r \in (0,a)$,

$$\int_0^r \phi_\gamma(x)|f(x)|\,dx \leqslant \left(\int_0^r (x^\alpha|f(x)|)^p\,dx\right)^{\frac{1}{p}} \left(\int_0^r (x^{-\alpha}\phi_\gamma(x))^{p'}\,dx\right)^{\frac{1}{p'}}$$
$$\leqslant \|f\|_{L^p_{x^\alpha}(0,a)}(\phi_\gamma(r))^{\frac{1}{p}} \left(\int_0^r x^{-\alpha p'}\phi_\gamma(x)\,dx\right)^{\frac{1}{p'}}$$

and

$$\begin{aligned}\left(\int_0^r x^{-\alpha p'}\phi_\gamma(x)\,dx\right)^{\frac{1}{p'}} &= \left(\int_0^1 r(ur)^{-\alpha p'}\phi_\gamma(ur)\,du\right)^{\frac{1}{p'}}\\ &= r^{-\alpha}\left(\int_0^1 u^{-\alpha p'}[r\phi_\gamma(ur)]\,du\right)^{\frac{1}{p'}}\\ &= r^{-\alpha}(\Phi_\gamma(r))^{\frac{1}{p'}}\left(\int_0^1 u^{-\alpha p'}\frac{r\phi_\gamma(ur)}{\Phi_\gamma(r)}\,du\right)^{\frac{1}{p'}}\\ &\leqslant r^{-\alpha}(\Phi_\gamma(r))^{\frac{1}{p'}}\left(\int_0^1 u^{-\alpha p'}\Psi_\gamma(u)\,du\right)^{\frac{1}{p'}} < \infty.\end{aligned}$$

Thus, $\phi_\gamma f \in L^1(0,r)$ for all $r \in (0,a)$.

For $0<\theta<1$,

$$\begin{aligned}&\left\|\frac{1}{\Phi_\gamma(x)}\int_0^x \phi_\gamma(s)f(s)\,ds - f(x)\right\|_{L^p_{x^\alpha}}\\ &\leqslant \left(\int_0^a\left(\frac{x^\alpha}{\Phi_\gamma(x)}\int_0^x \phi_\gamma(s)|f(s)-f(x)|\,ds\right)^p dx\right)^{\frac{1}{p}}\\ &= \left(\int_0^a\left(\frac{x^{\alpha+1}}{\Phi_\gamma(x)}\int_0^1 \phi_\gamma(xu)|f(xu)-f(x)|\,du\right)^p dx\right)^{\frac{1}{p}}\\ &\leqslant \int_0^1\left(\int_0^a\left(\frac{x^{\alpha+1}}{\Phi_\gamma(x)}\phi_\gamma(ux)|f(ux)-f(x)|\right)^p dx\right)^{\frac{1}{p}} du\\ &= \int_0^\theta\left(\int_0^a\left(\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)}x^\alpha|f(ux)-f(x)|\right)^p dx\right)^{\frac{1}{p}} du\\ &\quad + \int_\theta^1\left(\int_0^a\left(\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)}x^\alpha|f(ux)-f(x)|\right)^p dx\right)^{\frac{1}{p}} du.\end{aligned}$$

By Theorem 2, for every $\varepsilon>0$, there exists $0<\theta_\varepsilon<1$ such that for $\theta_\varepsilon<u<1$,

$$\left(\int_0^a (x^\alpha|f(ux)-f(x)|)^p\,dx\right)^{\frac{1}{p}} < \frac{\varepsilon}{C}$$

and so

$$\begin{aligned}
&\int_{\theta_\varepsilon}^{1}\left(\int_0^a\left(\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)}x^\alpha|f(ux)-f(x)|\right)^p dx\right)^{\frac{1}{p}} du \\
&\leqslant \int_{\theta_\varepsilon}^{1}\Psi_\gamma(u)\left(\int_0^a (x^\alpha|f(ux)-f(x)|)^p\,dx\right)^{\frac{1}{p}} du \\
&< \frac{\varepsilon}{C}\int_{\theta_\varepsilon}^{1}\Psi_\gamma(u)\,du \leqslant \frac{\varepsilon}{C}\int_0^1\Psi_\gamma(u)\,du
\end{aligned}$$

and so

$$\limsup_{\gamma\to\infty}\int_{\theta_\varepsilon}^{1}\left(\int_0^a\left(\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)}x^\alpha|f(ux)-f(x)|\right)^p dx\right)^{\frac{1}{p}} du \leqslant \varepsilon.$$

Also,

$$\begin{aligned}
&\limsup_{\gamma\to\infty}\int_0^{\theta_\varepsilon}\left(\int_0^a\left(\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)}x^\alpha|f(ux)-f(x)|\right)^p dx\right)^{\frac{1}{p}} du \\
&\leqslant \limsup_{\gamma\to\infty}\int_0^{\theta_\varepsilon}\Psi_\gamma(u)\left(\int_0^a (x^\alpha|f(ux)-f(x)|)^p\,dx\right)^{\frac{1}{p}} du \\
&\leqslant \limsup_{\gamma\to\infty}\int_0^{\theta_\varepsilon}\Psi_\gamma(u)\left(\left(\int_0^a (x^\alpha|f(ux)|)^p\,dx\right)^{\frac{1}{p}}+\left(\int_0^a (x^\alpha|f(x)|)^p\,dx\right)^{\frac{1}{p}}\right)du \\
&\leqslant \|f\|_{L^p_{x^\alpha}}\limsup_{\gamma\to\infty}\int_0^{\theta_\varepsilon}\Psi_\gamma(u)\left(\frac{1}{u^{\alpha+\frac{1}{p}}}+1\right)du = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{\gamma\to\infty}\left\|\frac{1}{\Phi_\gamma(x)}\int_0^x\phi_\gamma(s)f(s)\,ds - f(x)\right\|_{L^p_{x^\alpha}} \\
&\leqslant \limsup_{\gamma\to\infty}\int_0^{\theta_\varepsilon}\left(\int_0^a\left(\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)}x^\alpha|f(ux)-f(x)|\right)^p dx\right)^{\frac{1}{p}} du \\
&\quad+\limsup_{\gamma\to\infty}\int_{\theta_\varepsilon}^{1}\left(\int_0^a\left(\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)}x^\alpha|f(ux)-f(x)|\right)^p dx\right)^{\frac{1}{p}} du \\
&\leqslant \varepsilon
\end{aligned}$$

and so

$$\lim_{\gamma\to\infty}\left\|\frac{1}{\Phi_\gamma(x)}\int_0^x\phi_\gamma(s)f(s)\,ds - f(x)\right\|_{L^p_{x^\alpha}} = 0. \qquad \square$$

Taking $\phi_\gamma(x) = x^\gamma$, we obtain:

COROLLARY 2. *For* $f \in L^p_{x^\alpha}(0,a)$, $1 \leqslant p < \infty$, $\alpha \in \mathbf{R}$,

$$\lim_{\gamma\to\infty}\|(A_\gamma - I)f\|_{L^p_{x^\alpha}(0,a)} = 0. \tag{2}$$

DEFINITION 3. Let $f \geqslant 0$. Define for $\beta \geqslant 0$

$$QMD(\beta) = \{f : x^{-\beta} f(x) \text{ is nonincreasing}\}$$

and

$$QMD = \bigcup_{\beta \geqslant 0} QMD(\beta).$$

Theorem 4 in [2] states that for $f \in QMD(\beta)$ on $(0,a)$, $\beta > 0$, for $0 < p \leqslant \infty$, $\tau > 0$,

$$\left(\int_0^a \left(x^{-\tau} \int_0^x f(s) \frac{ds}{s}\right)^p \frac{dx}{x}\right)^{\frac{1}{p}} \sim \left(\int_0^a (x^{-\tau} f(x))^p \frac{dx}{x}\right)^{\frac{1}{p}}.$$

and

$$\left(\int_0^a \left(x^{\tau} \int_x^a f(s) \frac{ds}{s}\right)^p \frac{dx}{x}\right)^{\frac{1}{p}} \sim \left(\int_0^a (x^{\tau} f(x))^p \frac{dx}{x}\right)^{\frac{1}{p}} \tag{3}$$

where $g_1 \sim g_2$ means that there exists a constant, C, so that

$$\frac{1}{C} g_1 \leqslant g_2 \leqslant C g_1.$$

We will need an explicit constant for one of these inequalities:

$$\left(\int_0^a \left(x^{-\tau} \int_0^x f(s) \frac{ds}{s}\right)^p \frac{dx}{x}\right)^{\frac{1}{p}} \leqslant \frac{1}{p} \tau^{-\frac{1}{p}} \beta^{\frac{1}{p}-1} \left(\int_0^a (x^{-\tau} f(x))^p \frac{dx}{x}\right)^{\frac{1}{p}}. \tag{4}$$

The following lemma is essentially proved in [5]. See also Theorem 5 in [2].

LEMMA 1. *Assume that $f \in QMD(\beta)$ on $(0,a)$, and*

$$\int_x^a f(t) \frac{dt}{t} < \infty \tag{5}$$

for all $x \in (0,a)$. Then $f = g - \beta h$, where $g, h \geqslant 0$ are nonincreasing functions.

If $f \in (QMD \cap L^p_{x^\alpha})(0,a)$, $\alpha > -\frac{1}{p}$, then (5) holds, and

$$\|g\|_{L^p_{x^\alpha}} \sim \|f\|_{L^p_{x^\alpha}}$$

and

$$\|h\|_{L^p_{x^\alpha}} \sim \|f\|_{L^p_{x^\alpha}}$$

Proof. Let

$$h(x) = \int_x^a f(s) \frac{ds}{s}$$

and

$$g(x) = f(x) + \beta h(x).$$

We shall see that $g(x)$ is a nondecreasing function. For $0<u<1$, $x^{-\beta}f(x)\leqslant (ux)^{-\beta}f(ux)$, and so $f(x)\leqslant u^{-\beta}f(ux)$. Thus, for $0<x_0<x_1$,

$$\begin{aligned}
g(x_1)-g(x_0) &= f(x_1)-f(x_0)-\beta\int_{x_0}^{x_1} f(s)\frac{ds}{s}\\
&= f(x_1)-f(x_0)-\beta\int_{\frac{x_0}{x_1}}^{1} f(ux_1)\frac{du}{u}\\
&\leqslant f(x_1)-f(x_0)-\beta\int_{\frac{x_0}{x_1}}^{1} u^{\beta} f(x_1)\frac{du}{u}\\
&= f(x_1)-f(x_0)-f(x_1)\left(1-\left(\frac{x_0}{x_1}\right)^{\beta}\right)\\
&= -f(x_0)+f(x_1)\left(\frac{x_0}{x_1}\right)^{\beta}\leqslant 0.
\end{aligned}$$

Assume that $f\in L^p_{x^\alpha}(0,a)$. From Hardy's inequality in the case $p\geqslant 1$, and from its generalization to $0<p<1$ for $f\in QMD$, (3), it follows that

$$\int_x^a f(s)\frac{ds}{s}<\infty$$

for all $x\in(0,a)$, and furthermore

$$\|h\|_{L^p_{x^\alpha}}\sim C\|f\|_{L^p_{x^\alpha}}$$

so that

$$\|f\|_{L^p_{x^\alpha}}\leqslant\|g\|_{L^p_{x^\alpha}}\leqslant C(\|f\|_{L^p_{x^\alpha}}+\|h\|_{L^p_{x^\alpha}})\leqslant C\|f\|_{L^p_{x^\alpha}}$$

and thus

$$\|g\|_{L^p_{x^\alpha}}\sim\|f\|_{L^p_{x^\alpha}} \qquad \square$$

THEOREM 4. *Assume that* $f\in(QMD\cap L^p_{x^\alpha})(0,a)$, *with* $\alpha>-\frac{1}{p}$. *Then (2) also holds for* $0<p<1$.

Proof. Let $f\geqslant 0$ be nonincreasing on $(0,a)$. For $0<\theta<1$,

$$\begin{aligned}
\|(A_\gamma-I)f\|^p_{L^p_{x^\alpha}} &= \int_0^a\left(x^\alpha\left|\frac{\gamma+1}{x^{\gamma+1}}\int_0^x s^\gamma f(s)\,ds-f(x)\right|\right)^p dx\\
&=(\gamma+1)^p\int_0^a\left(x^\alpha\int_0^1 t^\gamma(f(tx)-f(x))\,dt\right)^p dx\\
&\leqslant(\gamma+1)^p\int_0^a\left(x^\alpha\int_0^\theta t^\gamma(f(tx)-f(x))\,dt\right)^p dx\\
&\quad+(\gamma+1)^p\int_0^a\left(x^\alpha\int_\theta^1 t^\gamma(f(tx)-f(x))\,dt\right)^p dx.
\end{aligned}$$

Let $\varepsilon > 0$ be given. By Theorem 2, there exists $0 < \theta_\varepsilon < 1$ such that

$$\|f(\theta_\varepsilon x) - f(x)\|^p_{L^p_{x^\alpha}} < \varepsilon.$$

Thus,

$$\begin{aligned}
&(\gamma+1)^p \int_0^a \left(x^\alpha \int_{\theta_\varepsilon}^1 t^\gamma (f(tx) - f(x))\,dt \right)^p dx \\
&\leqslant (\gamma+1)^p \int_0^a \left(x^\alpha \int_{\theta_\varepsilon}^1 t^\gamma (f(\theta_\varepsilon x) - f(x))\,dt \right)^p dx \\
&\leqslant \int_0^a (x^\alpha (f(\theta_\varepsilon x) - f(x)))^p\,dx < \varepsilon.
\end{aligned}$$

Also,

$$\begin{aligned}
&(\gamma+1)^p \int_0^a \left(x^\alpha \int_0^{\theta_\varepsilon} t^\gamma (f(tx) - f(x))\,dt \right)^p dx \\
&\leqslant (\gamma+1)^p \int_0^a \left(x^\alpha \int_0^{\theta_\varepsilon} t^\gamma f(tx)\,dt \right)^p dx + (\gamma+1)^p \int_0^a \left(x^\alpha \int_0^{\theta_\varepsilon} t^\gamma f(x)\,dt \right)^p dx.
\end{aligned}$$

But

$$(\gamma+1)^p \int_0^a \left(x^\alpha \int_0^{\theta_\varepsilon} t^\gamma f(x)\,dt \right)^p dx = \theta_\varepsilon^{(\gamma+1)p} \|f\|^p_{L^p_{x^\alpha}}$$

and since $\theta_\varepsilon < 1$,

$$\lim_{\gamma\to\infty} (\gamma+1)^p \int_0^a \left(x^\alpha \int_0^{\theta_\varepsilon} t^\gamma f(x)\,dt \right)^p dx = 0.$$

It remains to show that

$$\lim_{\gamma\to\infty} (\gamma+1) \left(\int_0^a \left(x^\alpha \int_0^{\theta_\varepsilon} t^\gamma f(tx)\,dt \right)^p dx \right)^{\frac{1}{p}} = 0.$$

To estimate the integral, we use (4).

$$\begin{aligned}
&\limsup_{\gamma\to\infty} (\gamma+1) \left(\int_0^a \left(x^\alpha \int_0^{\theta_\varepsilon} t^\gamma f(tx)\,dt \right)^p dx \right)^{\frac{1}{p}} \\
&= \limsup_{\gamma\to\infty} (\gamma+1) \left(\int_0^a \left(x^{\alpha+\frac{1}{p}-\gamma-1} \int_0^{\theta_\varepsilon} (xt)^{\gamma+1} f(tx) \frac{dt}{t} \right)^p \frac{dx}{x} \right)^{\frac{1}{p}} \\
&= \limsup_{\gamma\to\infty} (\gamma+1) \left(\int_0^a \left(x^{\alpha+\frac{1}{p}-\gamma-1} \int_0^{\theta_\varepsilon x} u^{\gamma+1} f(u) \frac{du}{u} \right)^p \frac{dx}{x} \right)^{\frac{1}{p}} \\
&= \limsup_{\gamma\to\infty} (\gamma+1) \theta_\varepsilon^{\gamma+1-\alpha-\frac{1}{p}} \left(\int_0^a \left((\theta_\varepsilon x)^{\alpha+\frac{1}{p}-\gamma-1} \int_0^{\theta_\varepsilon x} u^{\gamma+1} f(u) \frac{du}{u} \right)^p \frac{dx}{x} \right)^{\frac{1}{p}}
\end{aligned}$$

$$= \limsup_{\gamma\to\infty}(\gamma+1)\theta_\varepsilon^{\gamma+1-\alpha-\frac{1}{p}}\left(\int_0^{\theta_\varepsilon a}\left(z^{\alpha+\frac{1}{p}-\gamma-1}\int_0^z u^{\gamma+1}f(u)\frac{du}{u}\right)^p\frac{dz}{z}\right)^{\frac{1}{p}}$$
$$\leqslant \limsup_{\gamma\to\infty}(\gamma+1)\theta_\varepsilon^{\gamma+1-\alpha-\frac{1}{p}}\left(\int_0^{a}\left(z^{\alpha+\frac{1}{p}-\gamma-1}\int_0^z u^{\gamma+1}f(u)\frac{du}{u}\right)^p\frac{dz}{z}\right)^{\frac{1}{p}}$$
$$\leqslant \limsup_{\gamma\to\infty}\frac{\gamma+1}{p}\left(\gamma+1-\frac{1}{p}-\alpha\right)^{-\frac{1}{p}}(\gamma+1)^{\frac{1}{p}-1}\theta_\varepsilon^{\gamma+1-\alpha-\frac{1}{p}}\left(\int_0^a (x^\alpha f(x))^p\,dx\right)^{\frac{1}{p}}$$
$$= 0.$$

Thus, (2) holds if f is monotone nonincreasing. Now assume that $f\in QMD(\beta)$ on $(0,a)$, and $0<p<1$. By Lemma 1,

$$\int_x^a f(t)\frac{dt}{t}<\infty$$

for all $x\in(0,a)$, so that there exist $f_1,f_2\geqslant 0$ nonincreasing such that $f=f_1-f_2$, with

$$\|f_j\|_{L^p_{x^\alpha}}\leqslant C\|f\|_{L^p_{x^\alpha}}.$$

Then

$$\limsup_{\gamma\to\infty}\|(A_\gamma-I)f\|^p_{L^p_{x^\alpha}} = \limsup_{\gamma\to\infty}\|(A_\gamma-I)(f_1-f_2)\|^p_{L^p_{x^\alpha}}$$
$$\leqslant \limsup_{\gamma\to\infty}(\|(A_\gamma-I)f_1\|^p_{L^p_{x^\alpha}}+\|(A_\gamma-I)f_2\|^p_{L^p_{x^\alpha}})=0. \qquad \square$$

THEOREM 5. *Assume that* $\{\phi_\gamma\}\in\mathfrak{W}$. *Assume also that there exist* $\Psi_\gamma(u)$ *so that for* $u\in(0,1)$,

$$\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)}\leqslant\Psi_\gamma(u) \tag{6}$$

for all $x\in(0,a)$, *and*

$$\lim_{\gamma\to\infty}\Psi_\gamma(u)=0. \tag{7}$$

Then for f *uniformly continuous on* $(0,a)$, $0<a<\infty$,

$$\lim_{\gamma\to\infty}\sup_{0<x<a}\left|\frac{1}{\Phi_\gamma(x)}\int_0^x\phi_\gamma(s)f(s)\,ds-f(x)\right|=0.$$

Proof. Let $\varepsilon>0$. Since f is uniformly continuous on $(0,a)$, there exists $0<\delta<a$ such that $|f(u)-f(v)|<\varepsilon$ for $u,v\in(0,a)$ and $|u-v|<\delta$. By (7), there exists γ_0 such that for $\gamma\geqslant\gamma_0$,

$$\Psi_\gamma\left(\frac{a-\delta}{a}\right)<\frac{\varepsilon}{2\sup_{0<t<a}|f(t)|}.$$
$$\left|\frac{1}{\Phi_\gamma(x)}\int_0^x\phi_\gamma(s)f(s)\,ds-f(x)\right|\leqslant\frac{1}{\Phi_\gamma(x)}\int_0^x\phi_\gamma(s)|f(s)-f(x)|\,ds.$$

For $0 < x \leqslant \delta$,

$$\frac{1}{\Phi_\gamma(x)}\int_0^x \phi_\gamma(s)|f(s)-f(x)|\,ds \leqslant \frac{\varepsilon}{\Phi_\gamma(x)}\int_0^x \phi_\gamma(s)\,ds = \varepsilon$$

while for $\delta < x < a$,

$$\begin{aligned}
&\frac{1}{\Phi_\gamma(x)}\int_0^x \phi_\gamma(s)|f(s)-f(x)|\,ds \\
&= \frac{1}{\Phi_\gamma(x)}\left(\int_0^{x-\delta} \phi_\gamma(s)|f(s)-f(x)|\,ds + \int_{x-\delta}^x \phi_\gamma(s)|f(s)-f(x)|\,ds\right) \\
&\leqslant \frac{1}{\Phi_\gamma(x)}\left((2\sup_{0<t<a}|f(t)|)(x-\delta)\phi_\gamma(x-\delta) + \varepsilon\int_0^x \phi_\gamma(s)\,ds\right) \\
&\leqslant (2\sup_{0<t<a}|f(t)|)\frac{x\phi_\gamma(x-\delta)}{\Phi_\gamma(x)} + \varepsilon \\
&\leqslant (2\sup_{0<t<a}|f(t)|)\frac{x\phi_\gamma\frac{(a-\delta)x}{a}}{\Phi_\gamma(x)} + \varepsilon \\
&\leqslant (2\sup_{0<t<a}|f(t)|)\Psi_\gamma\left(\frac{a-\delta}{a}\right) + \varepsilon.
\end{aligned}$$

Thus,

$$\sup_{0<x<a}\left|\frac{1}{\Phi_\gamma(x)}\int_0^x \phi_\gamma(s)f(s)\,ds - f(x)\right| \leqslant (2\sup_{0<t<a}|f(t)|)\Psi_\gamma\left(\frac{a-\delta}{a}\right) + \varepsilon.$$

Therefore,

$$\limsup_{\gamma\to\infty}\sup_{0<x<a}\left|\frac{1}{\Phi_\gamma(x)}\int_0^x \phi_\gamma(s)f(s)\,ds - f(x)\right| \leqslant 2\varepsilon$$

and so

$$\lim_{\gamma\to\infty}\sup_{0<x<a}\left|\frac{1}{\Phi_\gamma(x)}\int_0^x \phi_\gamma(s)f(s)\,ds - f(x)\right| = 0. \qquad \square$$

REMARK 1. If f is not uniformly continuous on $(0,a)$, the conclusion of Theorem 5 may fail: take $f(x) = |\sin\frac{1}{x}|$ and $\phi_\gamma(x) = x^\gamma$ on $(0,1)$. Then for $k \geqslant 1$,

$$\begin{aligned}
A_\gamma f\left(\frac{1}{k\pi}\right) - f\left(\frac{1}{k\pi}\right) &= A_\gamma f\left(\frac{1}{k\pi}\right) = (\gamma+1)(k\pi)^{\gamma+1}\int_0^{\frac{1}{k\pi}} s^\gamma\left|\sin\frac{1}{s}\right|\,ds \\
&= (\gamma+1)(k\pi)^{\gamma+1}\int_{k\pi}^\infty u^{-\gamma-2}|\sin u|\,du \\
&\geqslant \frac{\sqrt{2}}{2}(\gamma+1)(k\pi)^{\gamma+1}\sum_{l=k}^\infty \int_{(l+\frac{1}{4})\pi}^{(l+\frac{3}{4})\pi} u^{-\gamma-2}\,du \\
&\geqslant \frac{\sqrt{2}}{2}(\gamma+1)(k\pi)^{\gamma+1}\sum_{l=k}^\infty \frac{\pi}{2}\left(\left(l+\frac{3}{4}\right)\pi\right)^{-\gamma-2}
\end{aligned}$$

$$\geqslant \frac{\sqrt{2}}{4}(\gamma+1)k^{\gamma+1}\sum_{l=k+1}^{\infty} l^{-\gamma-2}$$
$$\geqslant \frac{\sqrt{2}}{4}(\gamma+1)k^{\gamma+1}\int_{k+1}^{\infty} s^{-\gamma-2}\,ds = \frac{\sqrt{2}}{4}\left(\frac{k}{k+1}\right)^{\gamma+1}$$

and so for all γ,

$$\sup_{x\in(0,1)} (A_\gamma f(x) - f(x)) \geqslant \frac{\sqrt{2}}{4}.$$

3. The operator $\tilde{T}_{\phi_\gamma}$

THEOREM 6. *Assume that* $\{\phi_\gamma\} \in \mathfrak{W}$. *Assume also that for* $x \in (0,a)$,

$$\lim_{\gamma\to\infty} \phi_\gamma(x)\int_x^a \frac{1}{\Phi_\gamma(s)}\,ds = 1, \tag{8}$$

and for all $0 < \delta < a - x$,

$$\lim_{\gamma\to\infty} \phi_\gamma(x)\int_{x+\delta}^a \frac{1}{\Phi_\gamma(s)}\,ds = 0. \tag{9}$$

Let $f \in L^1(0,a)$. *Then at any Lebesgue point* $x \in (0,a)$ *of* f,

$$\lim_{\gamma\to\infty} \phi_\gamma(x)\int_x^a \frac{f(s)}{\Phi_\gamma(s)}\,ds = f(x).$$

Proof.

$$\left|\phi_\gamma(x)\int_x^a \frac{f(s)}{\Phi_\gamma(s)}\,ds - f(x)\right|$$
$$\leqslant \left|\phi_\gamma(x)\int_x^a \frac{f(s)}{\Phi_\gamma(s)}\,ds - \phi_\gamma(x)f(x)\int_x^a \frac{1}{\Phi_\gamma(s)}\,ds\right|$$
$$+\left|\phi_\gamma(x)f(x)\int_x^a \frac{1}{\Phi_\gamma(s)}\,ds - f(x)\right|$$
$$\leqslant \phi_\gamma(x)\int_x^a \frac{|f(s)-f(x)|}{\Phi_\gamma(s)}\,ds + |f(x)|\left|\phi_\gamma(x)\int_x^a \frac{1}{\Phi_\gamma(s)}\,ds - 1\right|.$$

Of course,

$$\lim_{\gamma\to\infty}\left|\phi_\gamma(x)\int_x^a \frac{1}{\Phi_\gamma(s)}\,ds - 1\right| = 0$$

and so we consider the first term.

Let $\varepsilon > 0$ be given. There exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$,

$$\frac{1}{\delta}\int_x^{x+\delta} |f(s)-f(x)|\,ds < \varepsilon.$$

For $0 < \delta < \delta_0$,

$$\begin{aligned}
&\phi_\gamma(x)\int_x^a \frac{|f(s)-f(x)|}{\Phi_\gamma(s)}\,ds\\
&= \phi_\gamma(x)\int_x^{x+\delta} \frac{|f(s)-f(x)|}{\Phi_\gamma(s)}\,ds + \phi_\gamma(x)\int_{x+\delta}^a \frac{|f(s)-f(x)|}{\Phi_\gamma(s)}\,ds.
\end{aligned}$$

For the first term,

$$\begin{aligned}
&\phi_\gamma(x)\int_x^{x+\delta} \frac{|f(s)-f(x)|}{\Phi_\gamma(s)}\,ds\\
&= \phi_\gamma(x)\int_x^{x+\delta}\left(\frac{1}{\Phi_\gamma(x+\delta)} - \int_{(s,x+\delta]} d\left(\frac{1}{\Phi_\gamma}\right)(t)\right)|f(s)-f(x)|\,ds\\
&= \frac{\phi_\gamma(x)}{\Phi_\gamma(x+\delta)}\int_x^{x+\delta}|f(s)-f(x)|\,ds\\
&\quad -\phi_\gamma(x)\int_{(x,x+\delta]}\left(\int_x^t |f(s)-f(x)|\,ds\right) d\left(\frac{1}{\Phi_\gamma}\right)(t).
\end{aligned}$$

Of course,

$$\lim_{\gamma\to\infty}\frac{\phi_\gamma(x)}{\Phi_\gamma(x+\delta)}\int_x^{x+\delta}|f(s)-f(x)|\,ds = 0$$

while

$$\begin{aligned}
&\limsup_{\gamma\to\infty}\left(-\phi_\gamma(x)\int_{(x,x+\delta]}\left(\int_x^t |f(s)-f(x)|\,ds\right) d\left(\frac{1}{\Phi_\gamma}\right)(t)\right)\\
&\leqslant \limsup_{\gamma\to\infty}\left(-\varepsilon\phi_\gamma(x)\int_{(x,x+\delta]}(t-x)\,d\left(\frac{1}{\Phi_\gamma}\right)(t)\right)\\
&= \limsup_{\gamma\to\infty}\left(-\varepsilon\delta\frac{\phi_\gamma(x)}{\Phi_\gamma(x+\delta)} + \varepsilon\phi_\gamma(x)\int_x^{x+\delta}\frac{1}{\Phi_\gamma(t)}\,dt\right)\\
&\leqslant \limsup_{\gamma\to\infty}\left(-\varepsilon\delta\frac{\phi_\gamma(x)}{\Phi_\gamma(x+\delta)} + \varepsilon\phi_\gamma(x)\int_x^{a}\frac{1}{\Phi_\gamma(t)}\,dt\right) = \varepsilon
\end{aligned}$$

and so

$$\limsup_{\gamma\to\infty}\phi_\gamma(x)\int_x^{x+\delta}\frac{|f(s)-f(x)|}{\Phi_\gamma(s)}\,ds \leqslant \varepsilon.$$

Also, using (9),

$$\begin{aligned}
\lim_{\gamma\to\infty}\phi_\gamma(x)\int_{x+\delta}^a \frac{|f(s)-f(x)|}{\Phi_\gamma(s)}\,ds &\leqslant \lim_{\gamma\to\infty}\frac{\phi_\gamma(x)}{\Phi_\gamma(x+\delta)}\int_{x+\delta}^a |f(s)|\,ds\\
&\quad + |f(x)|\lim_{\gamma\to\infty}\phi_\gamma(x)\int_{x+\delta}^a \frac{1}{\Phi_\gamma(s)}\,ds = 0.
\end{aligned}$$

Thus,

$$\limsup_{\gamma\to\infty} |\phi_\gamma(x)\int_x^a \frac{f(s)}{\Phi_\gamma(s)}\,ds - f(x)| \leqslant \varepsilon$$

and so

$$\lim_{\gamma\to\infty} \phi_\gamma(x)\int_x^a \frac{f(s)}{\Phi_\gamma(s)}\,ds = f(x). \quad \Box$$

Applying Theorem 6 with $\phi_\gamma(x) = x^\gamma$ proves:

COROLLARY 3. *Let $f \in L^1(0,a)$. If $x \in (0,a)$ is a Lebesgue point of f, then*

$$\lim_{\gamma\to\infty} \tilde{A}_\gamma f(x) = f(x).$$

THEOREM 7. *Assume that $\{\phi_\gamma\} \in \mathfrak{W}$. Assume also:*
1. There exist $\Psi_\gamma(u)$ so that for $u \in (0,1)$,

$$\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)} \leqslant \Psi_\gamma(u)$$

for all $x \in (0,a)$, and so that

$$\limsup_{\gamma\to\infty} \|\Psi_\gamma(u)\|_{L^1(0,1)} = C < \infty$$

and for all $\beta > 0$ and $0 < \theta < 1$,

$$\lim_{\gamma\to\infty} \|u^{-\beta}\Psi_\gamma(u)\|_{L^1(0,\theta)} = 0.$$

2. There exists $B(\gamma)$ such that for all $x \in (0,a)$,

$$\left|\phi_\gamma(x)\int_x^a \frac{1}{\Phi_\gamma(s)}\,ds - 1\right| \leqslant B(\gamma)$$

and

$$\lim_{\gamma\to\infty} B(\gamma) = 0.$$

Then for $f \in L^p_{x^\alpha}(0,a)$, $1 \leqslant p < \infty$, $\alpha \in \mathbf{R}$,

$$\lim_{\gamma\to\infty} \left\|\phi_\gamma(x)\int_x^a \frac{f(s)}{\Phi_\gamma(s)}\,ds - f(x)\right\|_{L^p_{x^\alpha}(0,a)} = 0.$$

Proof.

$$\limsup_{\gamma\to\infty} \left\|\phi_\gamma(x)\int_x^a \frac{f(s)}{\Phi_\gamma(s)}\,ds - f(x)\right\|_{L^p_{x^\alpha}}$$

$$\leqslant \limsup_{\gamma\to\infty}\left\| f(x)\left(\phi_\gamma(x)\int_x^a \frac{1}{\Phi_\gamma(s)}\,ds-1\right)\right\|_{L^p_{x^\alpha}} + \limsup_{\gamma\to\infty}\left\|\phi_\gamma(x)\int_x^a \frac{f(s)-f(x)}{\Phi_\gamma(s)}\,ds\right\|_{L^p_{x^\alpha}}$$

$$\leqslant \limsup_{\gamma\to\infty} B(\gamma)\|f\|_{L^p_{x^\alpha}} + \limsup_{\gamma\to\infty}\left\|\phi_\gamma(x)\int_x^a \frac{f(s)-f(x)}{\Phi_\gamma(s)}\,ds\right\|_{L^p_{x^\alpha}}$$

By hypothesis, we need consider only the last term above. For $0<\theta<1$,

$$\begin{aligned}
&\limsup_{\gamma\to\infty}\left\|\phi_\gamma(x)\int_x^a \frac{f(s)-f(x)}{\Phi_\gamma(s)}\,ds\right\|_{L^p_{x^\alpha}}\\
&= \limsup_{\gamma\to\infty}\left(\int_0^a \left| x^\alpha\phi_\gamma(x)\int_x^a \frac{f(s)-f(x)}{\Phi_\gamma(s)}\,ds\right|^p dx\right)^{\frac1p}\\
&\leqslant \limsup_{\gamma\to\infty}\left(\int_0^a \left(x^\alpha\phi_\gamma(x)\int_x^a \frac{|f(s)-f(x)|}{\Phi_\gamma(s)}\,ds\right)^p dx\right)^{\frac1p}\\
&\leqslant \limsup_{\gamma\to\infty}\left(\int_0^a \left(x^\alpha\int_x^a \Psi_\gamma\left(\frac{x}{s}\right)|f(s)-f(x)|\frac{ds}{s}\right)^p dx\right)^{\frac1p}\\
&= \limsup_{\gamma\to\infty}\left(\int_0^a \left(x^\alpha\int_{\frac{x}{a}}^1 \Psi_\gamma(u)\left|f\left(\frac{x}{u}\right)-f(x)\right|\frac{du}{u}\right)^p dx\right)^{\frac1p}\\
&\leqslant \limsup_{\gamma\to\infty}\int_0^1 \Psi_\gamma(u)\left(\int_0^{ua}\left(x^\alpha\left|f\left(\frac{x}{u}\right)-f(x)\right|\right)^p dx\right)^{\frac1p}\frac{du}{u}\\
&= \limsup_{\gamma\to\infty}\int_0^\theta \frac{\Psi_\gamma(u)}{u}\left(\int_0^{ua}\left(x^\alpha\left|f\left(\frac{x}{u}\right)-f(x)\right|\right)^p dx\right)^{\frac1p}du\\
&\quad + \limsup_{\gamma\to\infty}\int_\theta^1 \frac{\Psi_\gamma(u)}{u}\left(\int_0^{ua}\left(x^\alpha\left|f\left(\frac{x}{u}\right)-f(x)\right|\right)^p dx\right)^{\frac1p}du
\end{aligned}$$

We consider the last term. By Theorem 2, for every $\varepsilon>0$, there exists $0<\theta_\varepsilon<1$ such that for $\theta_\varepsilon<u<1$,

$$\left(\int_0^a (x^\alpha|f(ux)-f(x)|)^p\,dx\right)^{\frac1p} < \frac{\varepsilon}{C\max\{1,\theta_\varepsilon^{\alpha-\frac{1}{p'}}\}}$$

and so

$$\begin{aligned}
&\limsup_{\gamma\to\infty}\int_{\theta_\varepsilon}^1 \frac{\Psi_\gamma(u)}{u}\left(\int_0^{ua}\left(x^\alpha\left|f\left(\frac{x}{u}\right)-f(x)\right|\right)^p dx\right)^{\frac1p}du\\
&= \limsup_{\gamma\to\infty}\int_{\theta_\varepsilon}^1 u^{\alpha-\frac{1}{p'}}\Psi_\gamma(u)\left(\int_0^a (s^\alpha|f(us)-f(s)|)^p\,ds\right)^{\frac1p}du\\
&\leqslant \frac{\varepsilon}{C\max\{1,\theta_\varepsilon^{\alpha-\frac{1}{p'}}\}}\limsup_{\gamma\to\infty}\int_{\theta_\varepsilon}^1 u^{\alpha-\frac{1}{p'}}\Psi_\gamma(u)\,du \leqslant \varepsilon.
\end{aligned}$$

For the first term,

$$\begin{aligned}
&\limsup_{\gamma\to\infty}\int_0^{\theta_\varepsilon}\frac{\Psi_\gamma(u)}{u}\left(\int_0^{ua}\left(x^\alpha\left|f\left(\frac{x}{u}\right)-f(x)\right|\right)^p dx\right)^{\frac{1}{p}} du\\
&\leqslant \limsup_{\gamma\to\infty}\int_0^{\theta_\varepsilon}\frac{\Psi_\gamma(u)}{u}\left(\int_0^{ua}\left(x^\alpha\left|f\left(\frac{x}{u}\right)\right|\right)^p dx\right)^{\frac{1}{p}} du\\
&\quad+\limsup_{\gamma\to\infty}\int_0^{\theta_\varepsilon}\frac{\Psi_\gamma(u)}{u}\left(\int_0^{ua}(x^\alpha|f(x)|)^p dx\right)^{\frac{1}{p}} du\\
&\leqslant \limsup_{\gamma\to\infty}\int_0^{\theta_\varepsilon}u^{\alpha-\frac{1}{p'}}\Psi_\gamma(u)\left(\int_0^{a}(s^\alpha|f(s)|)^p ds\right)^{\frac{1}{p}} du\\
&\quad+\limsup_{\gamma\to\infty}\int_0^{\theta_\varepsilon}\frac{\Psi_\gamma(u)}{u}\left(\int_0^{a}(x^\alpha|f(x)|)^p dx\right)^{\frac{1}{p}} du\\
&=\|f\|_{L^p_{x^\alpha}}\limsup_{\gamma\to\infty}\left(\int_0^{\theta_\varepsilon}u^{\alpha-\frac{1}{p'}}\Psi_\gamma(u)\,du+\int_0^{\theta_\varepsilon}\frac{\Psi_\gamma(u)}{u}\,du\right)=0. \qquad \square
\end{aligned}$$

Taking $\phi_\gamma(x)=x^\gamma$, we obtain:

COROLLARY 4. *For $f\in L^p_{x^\alpha}(0,a)$, $1\leqslant p<\infty$, $\alpha\in\mathbf{R}$,*

$$\lim_{\gamma\to\infty}\|(\tilde{A}_\gamma-I)f\|_{L^p_{x^\alpha}(0,a)}=0.$$

4. Appendix

It is easy to see that $\{x^\gamma\}\in\mathfrak{W}$. The question arises whether powers are the only functions in $\mathfrak{W}$. One can easily verify that

$$\phi_\gamma(x)=\begin{cases}x^\gamma & \text{for } 0<x\leqslant 1\\ x^{\gamma+1} & \text{for } x\geqslant 1\end{cases}$$

satisfies the conditions of all the theorems and so shows that all results apply to some $\phi_\gamma(x)$ which are not powers of x. We shall construct additional functions in $\mathfrak{W}$. Given $\{\phi_\gamma\}\in\mathfrak{W}$, let

$$\tilde{\phi}_\gamma(x)=\begin{cases}\phi_\gamma(x)x\log\frac{e}{x} & \text{for } 0<x\leqslant 1\\ \phi_\gamma(x) & \text{for } x\geqslant 1.\end{cases}$$

We shall see that $\{\tilde{\phi}_\gamma\}\in\mathfrak{W}$. Moreover, we shall see that should $\{\phi_\gamma\}$ satisfy the hypothesis of any theorem in Section 2, then $\{\tilde{\phi}_\gamma\}$ satisfies that same hypothesis.

We consider the case $1\leqslant a\leqslant\infty$. The case $0<a<1$ requires only trivial changes in the argument. Let $g(x)=x\log\frac{e}{x}$. Then $g'(x)=-\log x\geqslant 0$ for $0<x\leqslant 1$, and so $g(x)$ is nondecreasing on $(0,1]$. Since $\tilde{\phi}_\gamma(x)=\phi_\gamma(x)$ for $x\geqslant 1$, ϕ_γ is a nondecreasing function on $(0,a)$. Also, for $0<x\leqslant 1$, $\tilde{\phi}_\gamma(x)=\phi_\gamma(x)x\log\frac{e}{x}\leqslant\phi_\gamma(x)$.

For $0 < x \leqslant 1$, $u \in (0,1)$,

$$\begin{aligned}
\tilde{\Phi}_\gamma(x) &= \int_0^x \tilde{\phi}_\gamma(s)\,ds = \int_0^x \phi_\gamma(s) s \log\frac{e}{s}\,ds \\
&\geqslant \int_{\frac{ux}{2}}^x \phi_\gamma(s) s \log\frac{e}{s}\,ds \geqslant \frac{ux}{2}\left(\log\frac{2e}{ux}\right)\int_{\frac{ux}{2}}^x \phi_\gamma(s)\,ds \\
&= \frac{ux}{2}\left(\log\frac{2e}{ux}\right)\frac{(2-u)x}{2}\frac{2}{(2-u)x}\int_{\frac{ux}{2}}^x \phi_\gamma(s)\,ds \\
&\geqslant \frac{ux}{2}\left(\log\frac{2e}{ux}\right)\frac{x}{2}\frac{1}{x}\int_0^x \phi_\gamma(s)\,ds \\
&= \frac{ux}{4}\left(\log\frac{2e}{ux}\right)\Phi_\gamma(x)
\end{aligned}$$

while for $x > 1$,

$$\begin{aligned}
\tilde{\Phi}_\gamma(x) &= \int_0^x \tilde{\phi}_\gamma(s)\,ds = \int_0^1 \phi_\gamma(s) s \log\frac{e}{s}\,ds + \int_1^x \phi_\gamma(s)\,ds \\
&\geqslant \int_{\frac{1}{2}}^1 \phi_\gamma(s) s \log\frac{e}{s}\,ds + \int_1^x \phi_\gamma(s)\,ds \\
&\geqslant \frac{1}{2}(\log 2e)\int_{\frac{1}{2}}^1 \phi_\gamma(s)\,ds + \int_1^x \phi_\gamma(s)\,ds \\
&\geqslant \frac{1}{4}(\log 2e)\int_0^1 \phi_\gamma(s)\,ds + \int_1^x \phi_\gamma(s)\,ds \\
&\geqslant \frac{1}{4}(\log 2e)\int_0^x \phi_\gamma(s)\,ds = \frac{1}{4}(\log 2e)\Phi_\gamma(x).
\end{aligned}$$

Thus, for $u \in (0,1)$, for $0 < x \leqslant 1$,

$$\limsup_{\gamma\to\infty}\frac{\tilde{\phi}_\gamma(ux)}{\tilde{\Phi}_\gamma(x)} \leqslant \frac{4\log\frac{e}{ux}}{\log\frac{2e}{ux}}\limsup_{\gamma\to\infty}\frac{\phi_\gamma(ux)}{\Phi_\gamma(x)} = 0$$

and if

$$\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)} \leqslant \psi_\gamma(u)$$

then

$$\frac{x\tilde{\phi}_\gamma(ux)}{\tilde{\Phi}_\gamma(x)} \leqslant \frac{4x\phi_\gamma(ux)ux\log\frac{e}{ux}}{ux\left(\log\frac{2e}{ux}\right)\Phi_\gamma(x)} \leqslant \frac{4x\phi_\gamma(ux)}{\Phi_\gamma(x)} \leqslant 4\psi_\gamma(u).$$

Also, for $x > 1$,

$$\limsup_{\gamma\to\infty}\frac{\tilde{\phi}_\gamma(ux)}{\tilde{\Phi}_\gamma(x)} \leqslant \frac{4}{\log 2e}\limsup_{\gamma\to\infty}\frac{\phi_\gamma(ux)}{\Phi_\gamma(x)} = 0$$

and if

$$\frac{x\phi_\gamma(ux)}{\Phi_\gamma(x)} \leqslant \psi_\gamma(u)$$

then

$$\frac{x\tilde{\phi}_\gamma(ux)}{\tilde{\Phi}_\gamma(x)} \leqslant \frac{4x\phi_\gamma(ux)}{(\log 2e)\Phi_\gamma(x)} \leqslant \frac{4}{\log 2e}\psi_\gamma(u).$$

We were unable to prove that $\tilde{\phi}_\gamma$ satisfies (8).

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