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HARDY INEQUALITIES FOR SOME NON–CONVEX DOMAINS

WALEED ABUELELA

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Abstract. Considering two different geometrical conditions, we obtain some new Hardy-type inequalities for non-convex domains in $\mathbb{R}^n$. In order to do so, we study the three-dimensional case and then generalise the approach to the n-dimensional case.

1. Introduction

We study high dimension variants of the classical integral Hardy-type inequality ([8])

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leqslant \mu \int_0^\infty f^p(x)dx, \tag{1}$$

where $p > 1$, $f(x) \geqslant 0$, and $F(x) = \int_0^x f(t)dt$ with constant μ. Inequality (1) with its improvements have played a fundamental role in the development of many mathematical branches such as spectral theory and PDE's, see for instance [2], [3], [4], [5], [7] and [10]. We centre our attention on the multi-dimensional version of (1) for $p = 2$, which takes the following form (see for example [6]):

$$\mu \int_\Omega \frac{|f(x)|^2}{d(x)^2} dx \leqslant \int_\Omega |\nabla f|^2 \, dx, \quad f \in \mathscr{C}_c^\infty(\Omega), \tag{2}$$

where

$$d(x) := \min\{|x-y| : y \notin \Omega\}. \tag{3}$$

For convex domains $\Omega \subset \mathbb{R}^n$, the sharp constant μ in (2) has been shown to equal $\frac{1}{4}$, see for instance [5] and [10]. However, the sharp constant for non-convex domains is unknown, although for arbitrary planar simply-connected domains $\Omega \subset \mathbb{R}^2$, A. Ancona ([1]) proved, using the Koebe one-quarter Theorem, that the constant μ in (2) is greater than or equal to $\frac{1}{16}$. Later A. Laptev and A. Sobolev ([9]) considered, under certain geometrical conditions, classes of domains for which there is a stronger version of the Koebe Theorem, this implied better estimates for the constant μ. Other specific examples of non-convex domains were presented by E. B. Davies ([6]).

Our main goal is to obtain new Hardy-type inequalities for some non-convex domains in $\mathbb{R}^n$, $n \geqslant 3$, which satisfy certain geometrical conditions, focusing on obtaining upper bounds for μ. In fact we have two different conditions introduced in the following section.

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2. Notations and conditions

In this section we present two 'non-convexity measures' for domains $\Omega \subset \mathbb{R}^n$; $n \geqslant 3$. The n-dimensional open ball with centre a and radius R is defined by

$$B^n(a,R) = \{y \in \mathbb{R}^n : |y-a| < R\}.$$

CONDITION 2.1. (Exterior Ball Condition) *We say that* $\Omega \subset \mathbb{R}^n$ *satisfies the Exterior Ball Condition if there exists a number* $R > 0$ *such that for each* $w \in \partial\Omega$, *one can find a point* $a \in \mathbb{R}^n$ *with* $|w-a| = R$ *such that*

$$B^n(a,R) \cap \Omega = \emptyset.$$

Condition 2.1 means that one can touch every point on $\partial\Omega$ with a ball of some radius R. Let Π be a k-dimensional subspace of $\mathbb{R}^n$ and let $R > 0$. We refer to the set

$$\mathscr{X}(\Pi,R) = \{y \in \mathbb{R}^n : \operatorname{dist}(y,\Pi) < R\},$$

as an $(n,k)-$cylinder of radius R.

CONDITION 2.2. ($(n,k)-$Cylinder Condition) *We say that* $\Omega \subset \mathbb{R}^n$ *satisfies the* $(n,k)-$*Cylinder Condition if there exists a number* $R > 0$ *such that for each* $w \in \partial\Omega$ *there exists a* k*-dimensional subspace* Π *of* $\mathbb{R}^n$ *such that*

$$w \in \mathscr{X}(\Pi,R) \text{ and } \mathscr{X}(\Pi,R) \cap \Omega = \emptyset.$$

Observe that in the $(n,k)-$Cylinder Condition, if $k = 0$, then this condition is equivalent to the Exterior Ball Condition and if $k = n-1$, this condition is equivalent to the convexity of Ω. Therefore, we may suppose that $1 \leqslant k \leqslant n-2$ in our analysis.

Suppose that the domain Ω satisfies one of Conditions 2.1 and 2.2. For a fixed $x \in \Omega$, choose w, a mutual point of $\partial\Omega$ and $\partial\mathscr{B}$, to be such that $d(x) = |x-w|$. Denote by $\mathscr{B}$ the appropriate test domain, i.e. a ball (Condition 2.1) or an $(n,k)-$cylinder (Condition 2.2). Furthermore, by $d_u(x)$ we mean the distance from $x \in \Omega$ to $\partial\Omega$ in the direction u, i.e.

$$d_u(x) := \min\{|s| : x + su \notin \Omega\}, \tag{4}$$

and $\tilde{d}_u(x)$ the distance from $x \in \Omega$ to $\partial\mathscr{B}$, in the direction u, i.e.

$$\tilde{d}_u(x) := \min\{|s| : x + su \in \partial\mathscr{B}\}.$$

Finally, denote by $\theta_0 \in \left(0, \frac{\pi}{2}\right)$ the angle at which the line segment representing $\tilde{d}_u(x)$ leaves $\partial\mathscr{B}$ to infinity.

3. Main results and discussion

In this section we state and discuss our main theorems which will be proved in Section 4.

3.1. Results related to the Exterior Ball Condition

The following two theorems are related to the Exterior Ball Condition.

THEOREM 3.1. *Suppose that the domain* $\Omega \subset \mathbb{R}^3$ *satisfies Condition* 2.1 *with constant* $R > 0$. *Then for any function* $f \in \mathscr{C}_c^\infty(\Omega)$ *the following Hardy-type inequality holds:*

$$\int_\Omega \mu(x,R) \frac{|f(x)|^2}{d(x)^2} dx \leqslant \int_\Omega |\nabla f(x)|^2 dx, \tag{5}$$

where

$$\mu(x,R) = \frac{(R - d(x)) \sqrt{d(x)^2 + 2Rd(x)} + d(x)^2}{4(R + d(x)) \sqrt{d(x)^2 + 2Rd(x)}}. \tag{6}$$

REMARK 3.2. The function $\mu(x,R)$ given by (6), can be written as powers of $\frac{d(x)}{R}$ as follows

$$\mu(x,R) = \frac{1}{4} - \frac{d(x)}{2R} + \mathscr{O}\left(\left(\frac{d(x)}{R}\right)^{\frac{3}{2}}\right), \tag{7}$$

which, for convex domains i.e. when $R \to \infty$, tends, linearly in $\frac{d(x)}{R}$, to $\frac{1}{4}$.

For higher dimensions we have not been able to find a simple analytic expression for the function $\mu(x,R)$, hence we content ourselves with the the asymptotic result stated in the following theorem.

THEOREM 3.3. *Suppose that the domain* $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, *satisfies Condition* 2.1 *with constant* $R > 0$. *Then for any function* $f \in \mathscr{C}_c^\infty(\Omega)$ *the following Hardy-type inequality holds:*

$$\int_\Omega \mu(x,R) \frac{|f(x)|^2}{d(x)^2} dx \leqslant \int_\Omega |\nabla f(x)|^2 dx, \tag{8}$$

where if x is such that $\frac{d(x)}{R} \leqslant \varepsilon$ *with some* $\varepsilon \in (0,1)$, *then*

$$\mu(x,R) = \frac{1}{4} - \left(\frac{n-1}{4}\right) \frac{d(x)}{R} + \mathscr{O}\left(\left(\frac{d(x)}{R}\right)^{\frac{3}{2}}\right). \tag{9}$$

In particular, if the in-radius $\delta_{\text{in}} < R$, *then*

$$\mu(x,R) \geqslant \frac{1}{4} - \left(\frac{n-1}{4}\right)\frac{\delta_{\text{in}}}{R} + \mathscr{O}\left(\left(\frac{\delta_{\text{in}}}{R}\right)^{\frac{3}{2}}\right),$$

uniformly for all $x \in \Omega$.

REMARK 3.4. Using (9) with $n = 3$ gives the asymptotic form (7) of $\mu(x,R)$ immediately.

3.2. Results related to the $(n,k)-$ Cylinder Condition

This section is devoted to domains which satisfy the $(n,k)-$Cylinder Condition.

THEOREM 3.5. *Suppose that the domain* $\Omega \subset \mathbb{R}^3$ *satisfies Condition* 2.2 *for* $k = 1$ *with constant* $R > 0$. *Then for any function* $f \in \mathscr{C}_c^\infty(\Omega)$ *the following Hardy-type inequality holds:*

$$\int_\Omega \mu(x,R)\frac{|f(x)|^2}{d(x)^2}dx \leqslant \int_\Omega |\nabla f(x)|^2\,dx, \tag{10}$$

where

$$\mu(x,R) = \frac{R\left[\pi R + 2\sqrt{d(x)(d(x)+2R)} + 2R\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)\right]}{2\pi(d(x)+2R)^2}. \tag{11}$$

REMARK 3.6. The function $\mu(x,R)$, given by (11), has the following asymptotic expansion in powers of $\frac{d(x)}{R}$

$$\mu(x,R) = \frac{1}{4} - \frac{d(x)}{4R} + \mathscr{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right). \tag{12}$$

In comparison with the asymptotic form (7) obtained for $\mu(x,R)$ under the Exterior Ball Condition, we can conclude that the $(n,k)-$Cylinder Condition gives a better result, since the coefficient of the second term in (12) is $\frac{1}{4}$ instead of $\frac{1}{2}$ in (7).

Again for higher dimensions we content ourselves with the asymptotic form of the function $\mu(x,R)$ stated in the following theorem.

THEOREM 3.7. *Suppose that the domain* $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, *satisfies Condition* 2.2 *with* $1 \leqslant k \leqslant n-2$. *Then for any function* $f \in \mathscr{C}_c^\infty(\Omega)$ *the following Hardy-type inequality holds:*

$$\int_\Omega \mu(x,R)\frac{|f(x)|^2}{d(x)^2}dx \leqslant \int_\Omega |\nabla f(x)|^2\,dx, \tag{13}$$

where if x is such that $\frac{d(x)}{R} \leqslant \varepsilon$ with some $\varepsilon \in (0,1)$, then

$$\mu(x,R) = \frac{1}{4} - \left(\frac{n-k-1}{4}\right)\frac{d(x)}{R} + \mathscr{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right). \tag{14}$$

In particular, if the in-radius $\delta_{\text{in}} < R$, then

$$\mu(x,R) \geqslant \frac{1}{4} - \left(\frac{n-k-1}{4}\right)\frac{\delta_{\text{in}}}{R} + \mathscr{O}\left(\left(\frac{\delta_{\text{in}}}{R}\right)^{3/2}\right),$$

uniformly for all $x \in \Omega$.

REMARK 3.8. If $n=3$ and $k=1$ then (14) leads to (12). Comparing the asymptotic expression (14) of $\mu(x,R)$ with the corresponding expression (9) under the Exterior Ball Condition, we easily see that (14) gives a better result with respect to the coefficient of the second term.

The key ingredient in proving Theorems 3.1, 3.3, 3.5 and 3.7 is the following proposition.

PROPOSITION 3.9. (E. B. Davies, [4, 7]) *Let Ω be a domain in $\mathbb{R}^n$ and let $f \in \mathscr{C}_c^\infty(\Omega)$. Then*

$$\frac{n}{4}\int_\Omega \frac{|f(x)|^2}{m(x)^2}\,dx \leqslant \int_\Omega |\nabla f(x)|^2\,dx,$$

where $m(x)$ is given by

$$\frac{1}{m(x)^2} := \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2}\,dS(u), \tag{15}$$

and

$$d_u(x) := \min\{|t| : x + tu \notin \Omega\},$$

for every unit vector $u \in \mathbb{S}^{n-1}$ and $x \in \Omega$. Here $\left|\mathbb{S}^{n-1}\right| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the unit sphere in $\mathbb{R}^n$.

In addition, we need the following proposition to prove Theorem 3.7.

PROPOSITION 3.10. *Let $n-k \geqslant 2$. For any function $f(\zeta) = f(\zeta_1,\zeta_2)$ of the angular variable $\zeta \in \mathbb{S}^{n-1}$, where ζ_1 and ζ_2 are projections of ζ on the $(n-k)$-dimensional subspace V (which we identify with $\mathbb{R}^{n-k}$) and its orthogonal complement (which we identify with $\mathbb{R}^k$) respectively, the following formula holds:*

$$\int_{\mathbb{S}^{n-1}} f(\zeta)d\zeta = \int_{\mathbb{S}^{k-1}}\int_{\mathbb{S}^{n-k-1}}\int_0^{\frac{\pi}{2}} f(\eta\sin\phi,\xi\cos\phi)\sin^{n-k-1}\phi\cos^{k-1}\phi\,d\phi\,d\eta\,d\xi.$$

Proof. Let $g(x) = f(x|x|^{-1})$ for all non-zero $x \in \mathbb{R}^n$. We use the following formula:

$$n^{-1}\int_{\mathbb{S}^{n-1}} f(\zeta)\,d\zeta = \int_{|x|<1} g(x)\,dx =: I.$$

Represent $x = (y,z)$ with $y \in \mathbb{R}^{n-k}$ and $z \in \mathbb{R}^k$, so that

$$I = \int_{|z|<1}\int_{|y|<\sqrt{1-|z|^2}} g(y,z)\,dy\,dz.$$

Introduce spherical coordinates:

$$y = (\rho,\eta), z = (t,\xi), \rho = |y|, t = |z|, \eta \in \mathbb{S}^{n-k-1}, \xi \in \mathbb{S}^{k-1}.$$

Thus

$$I = \int_{\mathbb{S}^{k-1}}\int_{\mathbb{S}^{n-k-1}}\int_{t<1}\int_{\rho<\sqrt{1-t^2}} g(\rho\eta,t\xi)\,\rho^{n-k-1}t^{k-1}\,d\rho\,dt\,d\eta\,d\xi.$$

Now we view the variables (t,ρ) as coordinates on the plane and introduce the polar coordinates:

$$v = \sqrt{\rho^2+t^2}, \rho = v\sin\phi, t = v\cos\phi, \phi \in (0,\pi/2),$$

so

$$I = \int_{\mathbb{S}^{k-1}}\int_{\mathbb{S}^{n-k-1}}\int_0^1 v^{n-1}\int_0^{\frac{\pi}{2}} g(v\eta\sin\phi, v\xi\cos\phi)\sin^{n-k-1}\phi\cos^{k-1}\phi\,dv\,d\phi\,d\eta\,d\xi.$$

By definition of g,

$$g(v\eta\sin\phi, v\xi\cos\phi) = f(\eta\sin\phi, \xi\cos\phi),$$

and hence this function is independent of v. Integrating in v, we get

$$I = \frac{1}{n}\int_{\mathbb{S}^{k-1}}\int_{\mathbb{S}^{n-k-1}}\int_0^{\frac{\pi}{2}} f(\eta\sin\phi, \xi\cos\phi)\sin^{n-k-1}\phi\cos^{k-1}\phi\,d\phi\,d\eta\,d\xi,$$

which leads to the required formula. □

Our strategy to prove Theorems 3.1, 3.3, 3.5 and 3.7 is to obtain lower bounds for the function $\frac{1}{m(x)^2}$ given by (15), containing $d(x)$, then apply Proposition 3.9.

4. Proofs

Proof of Theorem 3.1. By (15) and the fact that $\tilde{d}_u(x) \geqslant d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{4\pi}\int\limits_{\mathbb{S}^2}\frac{1}{d_u(x)^2}dS(u) \geqslant \frac{1}{4\pi}\int\limits_{\mathbb{S}^2}\frac{1}{\tilde{d}_u(x)^2}dS(u). \tag{16}$$

Since the function $\tilde{d}_u(x)$ is symmetric, with respect to the ball $B^3(a,R)$, using spherical coordinates, (r,θ,ϕ) where $r \geqslant 0$, $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$, leads to $u = u(\theta,\phi)$, and that $\tilde{d}_u(x)$ depends on θ only. Thus, slightly abusing the notation, from this point on we write $\tilde{d}(x,\theta)$ instead of $\tilde{d}_u(x)$. Therefore, inequality (16) becomes

$$\frac{1}{m(x)^2} \geqslant \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{1}{\tilde{d}(x,\theta)^2}\sin\theta\, d\theta d\phi = \int_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2}\sin\theta\, d\theta. \tag{17}$$

Since $\Omega \subset \mathbb{R}^3$ satisfies Condition 2.1, then considering the two-dimensional cross section that contains $x \in \Omega$, and the line segments representing both $d(x)$ and $\tilde{d}(x,\theta)$, we have

$$\sin\theta_0 = \frac{R}{R+d(x)} = \frac{1}{1+\frac{d(x)}{R}}, \tag{18}$$

and the Cosine law gives

$$\frac{1}{\tilde{d}(x,\theta)^2} = \frac{1}{\cos^2\theta\left(R+d(x)-\sqrt{R^2-(d(x)^2+2Rd(x))\tan^2\theta}\right)^2}. \tag{19}$$

Therefore, inequality (17) takes the following form:

$$\frac{1}{m(x)^2} \geqslant \frac{1}{(R+d(x))^2}\int_0^{\theta_0} \frac{\sin\theta}{\cos^2\theta\left(1-\sqrt{\frac{R^2}{(R+d(x))^2}-\left(\frac{d(x)^2+2Rd(x)}{(R+d(x))^2}\right)\tan^2\theta}\right)^2} d\theta. \tag{20}$$

To compute the integral in (20), we use the following substitution:

$$\sec\theta = \frac{R+d(x)}{\sqrt{d(x)^2+2Rd(x)}}t,$$

which produces the following inequality:

$$\frac{1}{m(x)^2} \geqslant \frac{2}{(R+d(x))\sqrt{d(x)^2+2Rd(x)}} \times \left(I_1(x,R)-I_2(x,R)+I_3(x,R)\right), \tag{21}$$

where

$$I_1(x,R) = \int_{\frac{\sqrt{d(x)^2+2Rd(x)}}{R+d(x)}}^{1} \frac{dt}{t^4}, \quad I_2(x,R) = \int_{\frac{\sqrt{d(x)^2+2Rd(x)}}{R+d(x)}}^{1} \frac{dt}{2t^2},$$

$$\text{and } I_3(x,R) = \int_{\frac{\sqrt{d(x)^2+2Rd(x)}}{R+d(x)}}^{1} \frac{\sqrt{1-t^2}}{t^4}dt.$$

Concerning the first integral $I_1(x,R)$, we have

$$I_1(x,R) = \frac{(R+d(x))^3 - \left(d(x)^2 + 2Rd(x)\right)^{\frac{3}{2}}}{3\left(d(x)^2 + 2Rd(x)\right)^{\frac{3}{2}}}. \tag{22}$$

Moreover, the integral $I_2(x,R)$ gives

$$I_2(x,R) = \frac{R + d(x) - \sqrt{d(x)^2 + 2Rd(x)}}{2\sqrt{d(x)^2 + 2Rd(x)}}. \tag{23}$$

Finally, use the substitution $r = \frac{1}{t}$ for $I_3(x,R)$ to obtain

$$I_3(x,R) = \frac{R^3}{3\left(d(x)^2 + 2Rd(x)\right)^{\frac{3}{2}}}. \tag{24}$$

Using (22), (23), and (24) in (21), yields the following lower bound on $\frac{1}{m(x)^2}$:

$$\frac{1}{m(x)^2} \geqslant \frac{(R - d(x))\sqrt{d(x)^2 + 2Rd(x)} + d(x)^2}{3d(x)^2 (R + d(x))\sqrt{d(x)^2 + 2Rd(x)}}. \tag{25}$$

Applying Proposition 3.9 to (25) returns the Hardy-type inequality (5) with $\mu(x,R)$ given by (6). $\square$

Proof of Theorem 3.3. By (15) and the relation between $d(x)$ and $\tilde{d}_u(x)$, we obtain

$$\frac{1}{m(x)^2} = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2} dS(u) \geqslant \frac{1}{|\mathbb{S}^{n-1}|} \times J_n(x), \tag{26}$$

where

$$J_n(x) = \int_{\mathbb{S}^{n-1}} \frac{1}{\tilde{d}_u(x)^2} dS(u). \tag{27}$$

Now the aim is to estimate $J_n(x)$. Using the spherical coordinates leads to

$$J_n(x) = \int_0^{\pi} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta \int_{\mathbb{S}^{n-2}} d\omega = b \int_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(x,\theta)^2} \sin^{n-2}\theta d\theta;\ b = 2\left|\mathbb{S}^{n-2}\right|.$$

Since $\Omega \subset \mathbb{R}^n$ satisfies Condition 2.1, then using (19) implies that

$$J_n(x) = \frac{b}{R^2} \int_0^{\theta_0} \frac{\sin^{n-2}\theta}{\cos^2\theta \left(1 + \frac{d(x)}{R} - \sqrt{1 - \left(\frac{d(x)^2}{R^2} + 2\frac{d(x)}{R}\right)\tan^2\theta}\right)^2} d\theta. \tag{28}$$

Equation (28) can be simplified as follows:

$$J_n(x) = \frac{b}{R^2}\int\limits_0^{\theta_0} \frac{\sin^{n-2}\theta\left(1+\alpha+\sqrt{1-(\alpha^2+2\alpha)\tan^2\theta}\right)^2}{\cos^2\theta\left(1+2\alpha+\alpha^2-1+(\alpha^2+2\alpha)\tan^2\theta\right)^2}d\theta;\ \alpha=\frac{d(x)}{R}$$
$$= \frac{b}{R^2}\underbrace{\frac{1}{(2\alpha+\alpha^2)^2}\int\limits_0^{\theta_0}\cos^2\theta\sin^{n-2}\theta\left(1+\alpha+\sqrt{1-(\alpha^2+2\alpha)\tan^2\theta}\right)^2 d\theta}_{I_1(\alpha)}. \tag{29}$$

Now let us estimate $I_1(\alpha)$: since $\theta_0\in(0,\frac{\pi}{2})$ then we can write $\theta_0=\frac{\pi}{2}-\tau_1$ where $c_1\sqrt{\alpha}\leqslant\tau_1\leqslant c_2\sqrt{\alpha}$ with some positive constants $c_1,c_2;c_1<c_2$. Moreover, since

$$0\leqslant\left(\alpha^2+2\alpha\right)\tan^2\theta\leqslant 1,$$

using the expansion

$$\sqrt{1-t}=1-\frac{t}{2}+\mathscr{O}(t^2);\ \ 0\leqslant t\leqslant 1,$$

the integral $I_1(\alpha)$ can be rewritten as

$$I_1(\alpha) = \frac{1}{(2\alpha+\alpha^2)^2}\int\limits_0^{\theta_0}\cos^2\theta\sin^{n-2}\theta\left(1+\alpha+1-\left(\frac{\alpha^2}{2}+\alpha\right)\tan^2\theta\right.$$
$$\left.+\mathscr{O}\left(\left(\alpha\tan^2\theta\right)^2\right)\right)^2 d\theta$$
$$= \frac{1}{\alpha^2}\int\limits_0^{\theta_0}\cos^2\theta\sin^{n-2}\theta\left(1-\alpha\tan^2\theta\right)d\theta$$
$$+\underbrace{\frac{1}{\alpha^2(\alpha+2)^2}\int\limits_0^{\theta_0}\cos^2\theta\sin^{n-2}\theta\ \mathscr{O}\left(\left(\alpha\tan^2\theta\right)^2\right)d\theta}_{I_2(\alpha)}. \tag{30}$$

The second term in (30), $I_2(\alpha)$, can be estimated as follows: since $\frac{1}{\alpha^2(\alpha+2)^2}\leqslant\frac{1}{\alpha^2}$, and $|\sin\theta|\leqslant 1$, then

$$I_2(\alpha)\leqslant\frac{1}{\alpha^2}\int\limits_0^{\theta_0}\alpha^2\frac{\sin^{n+2}\theta}{\cos^2\theta}d\theta\leqslant\int\limits_0^{\theta_0}\frac{1}{\cos^2\theta}d\theta=\int\limits_0^{\theta_0}\frac{1}{\sin^2\left(\frac{\pi}{2}-\theta\right)}d\theta.$$

However, for $0 \leqslant x \leqslant \frac{\pi}{2}$, we have $\sin x \geqslant \frac{2x}{\pi}$, hence,

$$I_2(\alpha) \leqslant \frac{\pi^2}{4} \int_0^{\theta_0} \frac{1}{\left(\frac{\pi}{2}-\theta\right)^2} d\theta \leqslant \frac{\pi^2}{4} \cdot \frac{1}{\underbrace{\frac{\pi}{2}-\theta_0}_{\tau_1}} \leqslant \frac{\pi^2}{4} \cdot \frac{1}{c_1\sqrt{\alpha}} = \mathscr{O}\left(\frac{1}{\sqrt{\alpha}}\right).$$

Thus, the integral $I_1(\alpha)$ takes the following form:

$$\begin{aligned} I_1(\alpha) &= \frac{1}{\alpha^2}\int_0^{\frac{\pi}{2}} \cos^2\theta \sin^{n-2}\theta d\theta - \frac{1}{\alpha^2}\int_{\theta_0}^{\frac{\pi}{2}} \cos^2\theta \sin^{n-2}\theta d\theta \\ &\quad - \frac{1}{\alpha}\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta + \frac{1}{\alpha}\int_{\theta_0}^{\frac{\pi}{2}} \sin^n\theta d\theta + \mathscr{O}\left(\frac{1}{\sqrt{\alpha}}\right). \end{aligned}$$

However, the second and fourth terms in the above equation have the following estimates: since for any $\theta \in (0,\theta_0)$ we have $\theta = \frac{\pi}{2} - \tau_2$ with $c_1\sqrt{\alpha} \leqslant \tau_2 \leqslant \tau_1 \leqslant c_2\sqrt{\alpha}$ which implies $\cos\theta = \sin\tau_2 \leqslant c_2\sqrt{\alpha}$, we obtain $\cos^2\theta \leqslant c\alpha$, so

$$\left|\frac{1}{\alpha^2}\int_{\theta_0}^{\frac{\pi}{2}} \cos^2\theta \sin^{n-2}\theta d\theta\right| \leqslant \frac{c\alpha}{\alpha^2}\int_{\theta_0}^{\frac{\pi}{2}} d\theta = \frac{c}{\alpha}\left(\frac{\pi}{2}-\theta_0\right) = \frac{c\tau_1}{\alpha} \leqslant \frac{cc_2}{\sqrt{\alpha}} = \mathscr{O}\left(\frac{1}{\sqrt{\alpha}}\right),$$

and

$$\left|\frac{1}{\alpha}\int_{\theta_0}^{\frac{\pi}{2}} \sin^n\theta d\theta\right| \leqslant \frac{1}{\alpha}\int_{\theta_0}^{\frac{\pi}{2}} d\theta = \frac{1}{\alpha}\left(\frac{\pi}{2}-\theta_0\right) \leqslant \mathscr{O}\left(\frac{1}{\sqrt{\alpha}}\right).$$

Therefore, $I_1(\alpha)$ becomes

$$I_1(\alpha) = \frac{1}{\alpha^2}\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta - \left(\frac{1}{\alpha^2}+\frac{1}{\alpha}\right)\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta + \mathscr{O}\left(\frac{1}{\sqrt{\alpha}}\right). \tag{31}$$

Thus, using (31) and the fact that $\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\, d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$ in (29), we obtain the following form for $J_n(x)$:

$$\begin{aligned} J_n(x) &= 2\frac{\left|\mathbb{S}^{n-2}\right|}{d(x)^2}\left[\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta - \int_0^{\frac{\pi}{2}} \sin^n\theta d\theta - \frac{d(x)}{R}\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta + \mathscr{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right] \\ &= \frac{1}{nd(x)^2}\cdot\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\left[1-(n-1)\cdot\frac{d(x)}{R} + \mathscr{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right]. \end{aligned} \tag{32}$$

Now use (32) in (26) to obtain the following lower bound on the function $\frac{1}{m(x)^2}$:

$$\frac{1}{m(x)^2} \geqslant \frac{1}{n\,d(x)^2}\left[1-(n-1)\cdot\frac{d(x)}{R}+\mathscr{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right]. \tag{33}$$

Apply Proposition 3.9 on (33), to obtain the Hardy-type inequality (8) with $\mu(x,R)$ as in (9). □

Proof of Theorem 3.5. By (15) and the fact that $\tilde{d}_u(x) \geqslant d_u(x)$, we have

$$\frac{1}{m(x)^2} = \frac{1}{4\pi}\int\limits_{\mathbb{S}^2}\frac{1}{d_u(x)^2}dS(u) \geqslant \frac{1}{4\pi}\int\limits_{\mathbb{S}^2}\frac{1}{\tilde{d}_u(x)^2}dS(u). \tag{34}$$

Consider a two dimensional cross section of the cylinder $\mathscr{Z}(\ell,R)$, here ℓ is a straight line (cylinder's axis), and the domain Ω by the plane Λ, which is orthogonal to ℓ and containing x. Let $u^{'} \in \Lambda$ be the projection of u onto Λ. Therefore, we now have a planar 'picture' in which we have the point x, a disk of radius R with centre that belongs to ℓ, and the line segments representing the distance from x to that disk as well as the distance from x to the boundary of that disk in the direction $u^{'}$. Let $\tilde{d}_{u'}(x)$ be the distance from x to the the boundary of that disk in the direction $u^{'}$. Let θ, ϕ be the standard spherical coordinates of the vector u such that $\theta \in [-\pi,\pi]$ is the angle in the plane Λ and $\phi \in [0,\pi]$. Because of the planar 'picture' depicted, we have

$$\tilde{d}_{u'}(x) = \tilde{d}(x,\theta) = \cos\theta\left(R+d(x)-\sqrt{R^2-(d(x)^2+2Rd(x))\tan^2\theta}\right), \tag{35}$$

where $-\theta_0 \leqslant \theta \leqslant \theta_0$ with th $\theta_0 \in \left(0,\frac{\pi}{2}\right)$ is the angle at which the line segment representing $\tilde{d}_{u'}(x,\theta)$ leaves the boundary of the two dimensional disk to infinity. On the other hand, it is clear that

$$\tilde{d}_u(x)^2 = \tilde{d}(x,\theta,\phi)^2 = \tilde{d}(x,\theta)^2+\tilde{d}(x,\theta)^2\cot^2\phi, \tag{36}$$

where $0 \leqslant \phi \leqslant \pi$. Therefore, using the spherical coordinates with (36) in (34) implies

$$\frac{1}{m(x)^2} \geqslant \frac{1}{2\pi}\int\limits_0^{\pi}\int\limits_{-\theta_0}^{\theta_0}\frac{\sin\phi}{\tilde{d}(x,\theta)^2\,(1+\cot^2\phi)}d\theta d\phi. \tag{37}$$

Now use (35) in (37) to obtain

$$\frac{1}{m(x)^2} \geqslant \frac{1}{2}\int\limits_0^{\pi}\sin^3\phi\,d\phi\cdot I(x,R) = \frac{2}{3}\cdot I(x,R),$$

where

$$I(x,R) = \frac{1}{\pi}\int\limits_{-\theta_0}^{\theta_0}\frac{d\theta}{\cos^2\theta\left(R+d(x)-\sqrt{R^2-(d(x)^2+2Rd(x))\tan^2\theta}\right)^2},$$

To evaluate $I(x)$, use the substitution

$$s = \frac{\sqrt{a(x)^2 - R^2}}{R}\tan\theta, \quad \text{where } a(x) = R + d(x)$$

to have

$$\begin{aligned} I(x,R) &= \frac{1}{\pi}\frac{R}{\sqrt{a(x)^2-R^2}}\int_{-s_0}^{s_0}\frac{1}{\left(a(x)-R\sqrt{1-s^2}\right)^2}ds \\ &= \frac{1}{\pi}\frac{R}{\sqrt{a(x)^2-R^2}}\left(I_1(x,R)+I_2(x,R)+I_3(x,R)\right), \end{aligned}$$

where

$$\begin{aligned} I_1(x,R) &= \int_{-s_0}^{s_0}\frac{2a(x)R\sqrt{1-s^2}}{\left(R^2s^2+a(x)^2-R^2\right)^2}ds, \\ I_2(x,R) &= \int_{-s_0}^{s_0}\frac{2\left(R^2-R^2s^2\right)ds}{\left(R^2s^2+a(x)^2-R^2\right)^2}, \text{and} \\ I_3(x,R) &= \int_{-s_0}^{s_0}\frac{ds}{R^2s^2+a(x)^2-R^2}. \end{aligned}$$

Use the substitutions

$$v_1 = \frac{\sqrt{1+A}}{\sqrt{1-s^2}}s, \quad \text{where } A = \frac{R^2}{a(x)^2-R^2},$$

for $I_1(x,R)$ and

$$v_{2,3} = \frac{Rs}{\sqrt{a(x)^2-R^2}},$$

for the integrals $I_2(x,R), I_3(x,R)$, and then use the fact that $s_0 = \frac{\sqrt{d(x)^2+2Rd(x)}}{R}\tan\theta_0$ $= 1$, see (18), to have

$$\frac{1}{m(x)^2} \geqslant \frac{2R\left[\pi R + 2\sqrt{d(x)(d(x)+2R)} + 2R\tan^{-1}\left(\frac{R}{\sqrt{d(x)(d(x)+2R)}}\right)\right]}{3d(x)^2\pi(d(x)+2R)^2}. \quad (38)$$

Now Proposition 3.9 with (38) to obtain (10) with $\mu(x,R)$ given by (11). □

Proof of Theorem 3.7. As illustrated before, $\frac{1}{m(x)^2}$ has the following lower bound

$$\frac{1}{m(x)^2} = \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\mathbb{S}^{n-1}} \frac{1}{d_u(x)^2}dS(u) \geqslant \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\mathbb{S}^{n-1}} \frac{1}{\tilde{d}_u(x)^2}dS(u). \tag{39}$$

Now let us consider an $(n-k)$-dimensional cross section of the cylinder $\mathscr{Z}(\Pi,R)$ and the domain Ω by the $(n-k)$ plane Λ, which is orthogonal to $\mathbb{R}^k$ and containing x. Then this cross section is exactly a sphere $\mathbb{S}^{n-k-1}$.

Considering the above cross section, the vector $\underline{\tilde{d}}_u(x) = \gamma u; \gamma = |\underline{\tilde{d}}_u(x)|$, representing the distance $\tilde{d}_u(x)$, can be written as a sum of two orthogonal vectors. One is the projection of $\underline{\tilde{d}}_u(x)$ onto the k-dimensional subspace $\mathscr{V}$, which is parallel to the subspace $\mathbb{R}^k$, and the other is in the orthogonal subspace $\mathscr{V}^\perp$, which is parallel to $\mathbb{R}^{n-k}$. Accordingly, we have

$$\left|\underline{\tilde{d}}_u(x)\right|^2 = \left|\left(\underline{\tilde{d}}_u(x)\right)_{\mathscr{V}}\right|^2 + \left|\left(\underline{\tilde{d}}_u(x)\right)_{\mathscr{V}^\perp}\right|^2.$$

In order to evaluate $\left|\left(\underline{\tilde{d}}_u(x)\right)_{\mathscr{V}}\right|^2$ and $\left|\left(\underline{\tilde{d}}_u(x)\right)_{\mathscr{V}^\perp}\right|^2$, we decompose the vector $u \in \mathbb{S}^{n-1}$ into two orthogonal components as follows:

$$u = (\eta\, \sin\phi, \xi\, \cos\phi)\,;\, \xi \in \mathbb{S}^{k-1} \subset \mathscr{V},\, \eta \in \mathbb{S}^{n-k-1} \subset \mathscr{V}^\perp \,\text{and}\, \phi \in \left(0, \frac{\pi}{2}\right).$$

Denote $\tilde{d}_\eta(x) = \left|\left(\underline{\tilde{d}}_u(x)\right)_{\mathscr{V}}\right|$. Clearly this is the distance from x to a sphere $\mathbb{S}^{n-k-1}$ in the direction η (which is the projection of u onto the $(n-k)$ subspace).

Since, the two coordinates, $\eta\, \sin\phi$ and $\xi\, \cos\phi$, representing the vector u are orthogonal coordinates, in the $(n-k)$-dimensional subspace and the k-dimensional subspace respectively, we have $|\gamma\eta\, \sin\phi| = \gamma \sin\phi$ is the distance from x to the $(n-k)$-dimensional cross section of the cylinder, i.e., to $\mathbb{S}^{n-k-1}$. Therefore, we have

$$\gamma^2 = \frac{\tilde{d}_\eta(x)^2}{\sin^2\phi}. \tag{40}$$

Consequently, inequality (39), using Proposition 3.10 with relation (40), produces the following bound:

$$\begin{aligned}\frac{1}{m(x)^2} &\geqslant \frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{k-1}}\int_{\mathbb{S}^{n-k-1}}\int_0^{\frac{\pi}{2}} \frac{1}{\tilde{d}(\eta\sin\phi, \xi\cos\phi)^2}\sin^{n-k-1}\phi\cos^{k-1}\phi\, d\phi\, d\eta\, d\xi\\ &= I_{n,k}\times J_{n-k}(x),\end{aligned} \tag{41}$$

where

$$I_{n,k} = \frac{|\mathbb{S}^{k-1}|}{|\mathbb{S}^{n-1}|}\int_0^{\frac{\pi}{2}} \sin^{n-k+1}\phi\cos^{k-1}\phi d\phi = \frac{|\mathbb{S}^{k-1}|}{2|\mathbb{S}^{n-1}|}\beta\left(\frac{k}{2}, \frac{n-k+2}{2}\right),$$

β is the Beta function, and

$$J_{n-k}(x) = \int\limits_{\mathbb{S}^{n-k-1}} \frac{1}{\tilde{d}_\eta(x)^2}d\eta.$$

Recall that, $|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ and $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, so we have

$$I_{n,k} = \frac{\Gamma(n/2)\,\Gamma(\frac{n-k+2}{2})}{2\Gamma(\frac{n+2}{2})}\pi^{\frac{k-n}{2}}. \tag{42}$$

On the other hand, for $J_{n-k}(x)$, we follow the same argument applied to $J_n(x)$ defined in (27), which results in

$$J_{n-k}(x) = \frac{1}{(n-k)\,d(x)^2}\cdot\frac{2\pi^{\frac{n-k}{2}}}{\Gamma\left(\frac{n-k}{2}\right)}\left[1-(n-k-1)\frac{d(x)}{R}+\mathscr{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right]. \tag{43}$$

Use (43) and (42) in (41) to obtain

$$\frac{1}{m(x)^2} \geqslant \frac{1}{n\,d(x)^2}\left[1-(n-k-1)\frac{d(x)}{R}+\mathscr{O}\left(\left(\frac{d(x)}{R}\right)^{3/2}\right)\right]. \tag{44}$$

Apply Proposition 3.9 on the lower bound (44) to obtain the Hardy-type inequality (13) with $\mu(x,R)$ given by (14). □

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Waleed Abuelela
Mathematics Department, Faculty of Science
Al-Azhar University
Nasr City 11884
Cairo, Egypt
e-mail: waleed_abu_elela@hotmail.com