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OPERATOR FUNCTIONS ON CHAOTIC ORDER INVOLVING ORDER PRESERVING OPERATOR INEQUALITIES

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The fourth anniversary of
Professor Masahiro Nakamura's passing

(*Communicated by M. Fujii*)

Abstract. An operator T is said to be *positive* (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all vectors x in a Hilbert space, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. Let $\log A \geqslant \log B$ and $r_1, r_2, \ldots, r_n \geqslant 0$ and any fixed $\delta \geqslant 0$, and

$$p_1 \geqslant \delta, \quad p_2 \geqslant \frac{\delta + r_1}{p_1 + r_1}, \ldots, p_k \geqslant \frac{\delta + r_1 + r_2 + \ldots + r_{k-1}}{\mathfrak{q}[k-1]}, \ldots, p_n \geqslant \frac{\delta + r_1 + r_2 + \ldots + r_{n-1}}{\mathfrak{q}[n-1]}.$$

Let $\mathfrak{F}_n(p_n, r_n)$ be defined by

$$\mathfrak{F}_n(p_n, r_n) = A^{\frac{-r_n}{2}} \mathbb{C}_{A,B}[n]^{\frac{\delta + r_1 + r_2 + \ldots + r_n}{\mathfrak{q}[n]}} A^{\frac{-r_n}{2}}.$$

Then the following inequalities (i), (ii) and (iii) hold:

(i) $A^{\frac{p_{k-1}}{2}} \mathfrak{F}_{k-1}(p_{k-1}, r_{k-1}) A^{\frac{p_{k-1}}{2}} \geqslant \mathfrak{F}_k(p_k, r_k)$ for k such that $1 \leqslant k \leqslant n$,

(ii)
$$\begin{aligned} B^{\delta} &\geqslant A^{\frac{-r_1}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{\frac{\delta + r_1}{p_1 + r_1}} A^{\frac{-r_1}{2}} \\ &\geqslant A^{\frac{-(r_1+r_2)}{2}} \{A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}\}^{\frac{\delta + r_1 + r_2}{(p_1+r_1)p_2 + r_2}} A^{\frac{-(r_1+r_2)}{2}} \\ &\geqslant A^{\frac{-(r_1+r_2+r_3)}{2}} \{A^{\frac{r_3}{2}} [A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}]^{p_3} A^{\frac{r_3}{2}}\}^{\frac{\delta + r_1 + r_2 + r_3}{\{(p_1+r_1)p_2+r_2\}p_3 + r_3}} A^{\frac{-(r_1+r_2+r_3)}{2}} \\ &\;\;\vdots \\ &\geqslant A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}} \mathbb{C}_{A,B}[n]^{\frac{\delta + r_1 + r_2 + \ldots + r_n}{\mathfrak{q}[n]}} A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}}, \end{aligned}$$

(iii) $\mathfrak{F}_n(p_n, r_n)$ is a decreasing function of both $r_n \geqslant 0$ and $p_n \geqslant \frac{\delta + r_1 + r_2 + \ldots + r_{n-1}}{\mathfrak{q}[n-1]}$, where $\mathbb{C}_{A,B}[n]$ and $\mathfrak{q}[n]$ are defined as follows:

$$\mathbb{C}_{A,B}[n] = A^{\frac{r_n}{2}} \left\{A^{\frac{r_{n-1}}{2}} [\ldots A^{\frac{r_3}{2}} \{A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}\}^{p_3} A^{\frac{r_3}{2}} \ldots]^{p_{n-1}} A^{\frac{r_{n-1}}{2}} \right\}^{p_n} A^{\frac{r_n}{2}}$$

and

$$\mathfrak{q}[n] = [\ldots\{(p_1 + r_1)p_2 + r_2\}p_3 + \ldots r_{n-1}]p_n + r_n.$$

We remark that (ii) can be considered as *"a satellite inequality to chaotic order"*.

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1. Introduction

An operator T is said to be *positive* (denoted by $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all vectors x in a Hilbert space, and T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

THEOREM LH. (Löwner-Heinz inequality, denoted by **LH** briefly).

$$\textit{If } A \geqslant B \geqslant 0 \textit{ holds, then } A^{\alpha} \geqslant B^{\alpha} \textit{ for any } \alpha \in [0,1]. \tag{LH}$$

This inequality LH was originally proved in [27] and then in [21]. Many nice proofs of LH are known. We mention [28] and [3]. Although LH asserts that $A \geqslant B \geqslant 0$ *ensures* $A^{\alpha} \geqslant B^{\alpha}$ *for any* $\alpha \in [0,1]$, unfortunately $A^{\alpha} \geqslant B^{\alpha}$ *does not always hold for* $\alpha > 1$. The following result has been obtained from this point of view.

THEOREM A. *If* $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$,
(i) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$
and
(ii) $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$
hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with*

$$(1+r)q \geqslant p+r.$$

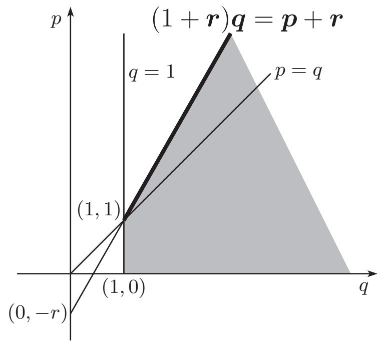

Fig. 1. Domain on p, q *and* r *for Theorem A*

The original proof of Theorem A is shown in [10], an elementary one-page proof is in [11] and alternative ones are in [4], [24]. It is shown in [29] that the conditions p, q and r in FIGURE 1 are best possible.

THEOREM B. (e.g., [12], [6], [24], [25], [19]) *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$, $p \geqslant 1$ *and* $r \geqslant 0$.

$$G_{A,B}(p,r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}A^{\frac{-r}{2}}$$

is a decreasing function of p *and* r, *and* $G_{A,A}[p,r] \geqslant G_{A,B}[p,r]$ *holds, that is,*

$$A^{1+r} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \textit{holds for } p \geqslant 1 \textit{ and } r \geqslant 0. \tag{1.1}$$

We write $A \gg B$ if $\log A \geqslant \log B$ for $A,B > 0$, which is called the chaotic order.

THEOREM C. *For* $A,B > 0$, *the following* (i) *and* (ii) *hold:*

(i) $A \gg B$ *holds if and only if* $A^r \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for* $p,r \geqslant 0$.

(ii) $A \gg B$ *holds if and only if for any fixed* $\delta \geqslant 0$, $F_{A,B}(p,r) = A^{\frac{-r}{2}}(A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{\delta+r}{p+r}}A^{\frac{-r}{2}}$ *is a decreasing function of* $p \geqslant \delta$ *and* $r \geqslant 0$.

(i) in Theorem C is shown in [12], [6] and an excellent proof in [31] and a proof in the case $p = r$ in [1], and (ii) in [12], [6] and etc.

LEMMA D. [13] *Let X be a positive invertible operator and Y be an invertible operator. For any real number λ,*

$$(YXY^*)^{\lambda} = YX^{\frac{1}{2}}(X^{\frac{1}{2}}Y^*YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$

We state the following result on the chaotic order which inspired us (see detail, §7).

THEOREM FKN-2. [9] *If $A \gg B$ for $A,B>0$, then*

$$A^{t-r}\sharp_{\frac{1+r-t}{(p-t)s+r}}(A^t\natural_s B^p) \leqslant A^t\sharp_{\frac{1-t}{p-t}}B^p \leqslant B$$

holds for $p\geqslant 1,\ s\geqslant 1,\ r\geqslant 0$ and $t\leqslant 0$.

We shall discuss further extensions of Theorem B, Theorem C and Theorem FKN-2.

The purpose of this paper is to emphasize that the chaotic order $A \gg B$ is sometimes more convenient and more useful than the usual order $A\geqslant B\geqslant 0$ for discussing some order preserving operator inequalities.

2. Definitions of $\mathbb{C}_{A,B}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$, (denoted by $\mathbb{C}_{A,B}[n]$ or $\mathbb{C}_{[n]}$ briefly sometime) and $\mathfrak{q}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$ (denoted by $\mathfrak{q}[n]$ briefly)

Let $A,B\geqslant 0,\ p_1,p_2,..,p_n\geqslant 0$ and $r_1,r_2,..,r_n\geqslant 0$ for a natural number n.
Let $\mathbb{C}_{A,B}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$ be defined by

$$\begin{aligned}&\mathbb{C}_{A,B}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]\\ &= A^{\frac{r_n}{2}}\left\{A^{\frac{r_{n-1}}{2}}[...A^{\frac{r_3}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{p_3}A^{\frac{r_3}{2}}...]^{p_{n-1}}A^{\frac{r_{n-1}}{2}}\right\}^{p_n}A^{\frac{r_n}{2}}. \qquad (2.1)\end{aligned}$$

Denote $\mathbb{C}_{A,B}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$ by $\mathbb{C}_{A,B}[n]$ briefly.
For examples,

$$\mathbb{C}_{A,B}[1] = A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}} \quad \text{and} \quad \mathbb{C}_{A,B}[2] = A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}$$

and

$$\mathbb{C}_{A,B}[4] = A^{\frac{r_4}{2}}\left[A^{\frac{r_3}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{p_3}A^{\frac{r_3}{2}}\right]^{p_4}A^{\frac{r_4}{2}}.$$

Particularly put $A=B$ in $\mathbb{C}_{A,B}[n]$ in (2.1). Then

$$\begin{aligned}&\mathbb{C}_{A,A}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]\\ &= A^{\frac{r_n}{2}}\left\{A^{\frac{r_{n-1}}{2}}[...A^{\frac{r_3}{2}}\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}A^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{p_3}A^{\frac{r_3}{2}}...]^{p_{n-1}}A^{\frac{r_{n-1}}{2}}\right\}^{p_n}A^{\frac{r_n}{2}} \qquad (2.2)\\ &= A^{[...\{(p_1+r_1)p_2+r_2\}p_3+...r_{n-1}]p_n+r_n}. \qquad (2.3)\end{aligned}$$

Next let $\mathfrak{q}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$ be defined by

$$\begin{aligned}&\mathfrak{q}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]\\&=\text{the exponential power of } A \text{ in (2.3)}\\&=[...\{(p_1+r_1)p_2+r_2\}p_3+...r_{n-1}]p_n+r_n.\end{aligned}\tag{2.4}$$

$\mathfrak{q}\left[n;p_1,p_2,..,p_{n-1},p_n|r_1,r_2,..,r_{n-1},r_n\right]$ denoted by $\mathfrak{q}[p_1,p_2,..,p_{n-1},p_n]$ or denoted by $\mathfrak{q}[r_1,r_2,..,r_{n-1},r_n]$ for simplicity or sometimes denoted by $\mathfrak{q}[n]$ briefly.

For examples, $\mathfrak{q}[1]=p_1+r_1$ and $\mathfrak{q}[2]=(p_1+r_1)p_2+r_2$ and

$$\mathfrak{q}[4]=[\{(p_1+r_1)p_2+r_2\}p_3+r_3]p_4+r_4.$$

For the sake of convenience, we define

$$\mathbb{C}_{A,B}[0]=B \quad \text{and} \quad \mathfrak{q}[0]=1 \tag{2.5}$$

and these definitions in (2.5) may be reasonable by (2.1) and (2.4).

LEMMA 2.1. *For $A,B\geqslant 0$ and any natural number n, the following* (i) *and* (ii) *hold.*

(i) $$\mathbb{C}_{A,B}[n]=A^{\frac{r_n}{2}}\mathbb{C}_{A,B}[n-1]^{p_n}A^{\frac{r_n}{2}}.$$

(ii) $$\mathfrak{q}[n]=\mathfrak{q}[n-1]p_n+r_n.$$

Proof. (i) and (ii) can be easily obtained by the definitions (2.1) and (2.4). □

We state two examples using these notations of $\mathbb{C}_{A,B}[n]$ and $\mathfrak{q}[n]$ for reader's convenience.

$$A^r\geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}} \Longleftrightarrow A^r\geqslant \mathbb{C}_{A,B}[1]^{\frac{r}{\mathfrak{q}[1]}},$$
$$A^{1+r}\geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \Longleftrightarrow A^{1+r}\geqslant \mathbb{C}_{A,B}[1]^{\frac{1+r}{\mathfrak{q}[1]}}.$$

REMARK 2.1. We remark that quite similar definitions to $\mathbb{C}_{A,B}[n]$ and $\mathfrak{q}[n]$ are given in [18] and related results are discussed in [18], [22], [23], [34] and etc.

3. Basic results associated with $\mathbb{C}_{A,B}[n]$ and $\mathfrak{q}[n]$

THEOREM 3.1. *Let $A\gg B$ and $r_1,r_2,...,r_n\geqslant 0$ for a natural number n. Then the following inequality holds,*

$$A^{r_1+r_2...+r_n}=\mathbb{C}_{A,A}[n]^{\frac{r_1+r_2...+r_n}{\mathfrak{q}[n]}}\geqslant \mathbb{C}_{A,B}[n]^{\frac{r_1+r_2...+r_n}{\mathfrak{q}[n]}} \tag{3.1}$$

for $p_1,p_2,...,p_n$ satisfying

$$p_j\geqslant \frac{r_1+r_2+...+r_{j-1}}{\mathfrak{q}[j-1]} \quad \textit{for} \quad j=1,2,...,n \qquad (r_0=0 \quad \textit{and} \quad \mathfrak{q}[0]=1), \tag{3.2}$$

that is,

$$p_1\geqslant 0, \quad p_2\geqslant \frac{r_1}{p_1+r_1}, \quad p_3\geqslant \frac{r_1+r_2}{(p_1+r_1)p_2+r_2}, \ ..., \ p_n\geqslant \frac{r_1+r_2+...+r_{n-1}}{\mathfrak{q}[n-1]},$$

where $\mathbb{C}_{A,B}[n]$ *is defined in* (2.1) *and* $\mathfrak{q}[n]$ *is defined in* (2.4).

Proof. We shall show (3.1) by Mathematical Induction. In the case $n=1$,

$$A \gg B \quad \text{implies} \quad A^{r_1} \geqslant (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}} = \mathbb{C}_{A,B}[1]^{\frac{r_1}{\mathfrak{q}[1]}}$$

holds for any $p_1 \geqslant 0$ and $r_1 \geqslant 0$ by (i) of Theorem C. Whence (3.1) for $n=1$,

Assume (3.1). We shall show (3.1) for $r_1, r_2, .., r_n, r_{n+1} \geqslant 0$ and $p_1, p_2, ..., p_n, p_{n+1}$ which satisfies (3.2) for $n+1$: that is,

$$p_1 \geqslant 0, \quad p_2 \geqslant \frac{r_1}{p_1+r_1}, \ ..., \ p_n \geqslant \frac{r_1+r_2+...+r_{n-1}}{\mathfrak{q}[n-1]}, \quad \text{and} \quad p_{n+1} \geqslant \frac{r_1+r_2+...+r_n}{\mathfrak{q}[n]}. \tag{3.2$'$}$$

Put $A_1 = A^{r_1+r_2...+r_n}$ and $B_1 = \mathbb{C}_{A,B}[n]^{\frac{r_1+r_2...+r_n}{\mathfrak{q}[n]}}$ in (3.1). Then $A_1 \geqslant B_1 \geqslant 0$ by the assumption (3.1). Theorem B ensures

$$A_1^{1+t} \geqslant (A_1^{\frac{t}{2}} B_1^s A_1^{\frac{t}{2}})^{\frac{1+t}{s+t}} \quad \text{for any } s \geqslant 1 \text{ and } t \geqslant 0,$$

that is,

$$A^{(r_1+r_2...+r_n)(1+t)} \geqslant \left\{ A^{\frac{(r_1+r_2...+r_n)t}{2}} \mathbb{C}_{A,B}[n]^{(\frac{r_1+r_2...+r_n}{\mathfrak{q}[n]})s} A^{\frac{(r_1+r_2...+r_n)t}{2}} \right\}^{\frac{1+t}{s+t}} \tag{3.3}$$

holds for any $s \geqslant 1$ and $t \geqslant 0$. Put $s = \left(\dfrac{\mathfrak{q}[n]}{r_1+r_2...+r_n} \right) p_{n+1} \geqslant 1$ since $\mathfrak{q}[n] p_{n+1} \geqslant r_1+r_2...+r_n$ by the last inequality in (3.2') and also put $(r_1+r_2...+r_n)t = r_{n+1}$ in (3.3).

Then the exponential power $\dfrac{1+t}{s+t}$ of the right hand side of (3.3) can be written as follows;

$$\begin{aligned} \frac{1+t}{s+t} &= \frac{(1+t)(r_1+r_2...+r_n)}{\mathfrak{q}[n]p_{n+1}+(r_1+r_2...+r_n)t} \\ &= \frac{r_1+r_2...+r_n+r_{n+1}}{\mathfrak{q}[n]p_{n+1}+r_{n+1}} \\ &= \frac{r_1+r_2...+r_n+r_{n+1}}{\mathfrak{q}[n+1]} \quad \text{by (ii) of Lemma 2.1} \end{aligned} \tag{3.4}$$

and we have the following desired (3.5) by (3.3) and (3.4)

$$\begin{aligned} A^{r_1+r_2...+r_n+r_{n+1}} &\geqslant \left(A^{\frac{r_{n+1}}{2}} \mathbb{C}_{A,B}[n]^{p_{n+1}} A^{\frac{r_{n+1}}{2}}\right)^{\frac{r_1+r_2...+r_n+r_{n+1}}{\mathfrak{q}[n+1]}} \\ &= \mathbb{C}_{A,B}[n+1]^{\frac{r_1+r_2...+r_n+r_{n+1}}{\mathfrak{q}[n+1]}} \quad \text{by (i) of Lemma 2.1,} \end{aligned} \tag{3.5}$$

so that (3.5) shows that (3.1) holds for $p_1, p_2, .., p_n, p_{n+1}$ which satisfies (3.2') and $r_1, r_2, .., r_n, r_{n+1} \geqslant 0$ for a natural number n. $\square$

COROLLARY 3.2. *Let $A \gg B$ and $r_1, r_2, r_3 \geqslant 0$. Then*

(i) $$A^{r_1+r_2+r_3} \geqslant \{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{r_1+r_2+r_3}{[(p_1+r_1)p_2+r_2]p_3+r_3}}.$$

holds for $p_2 \geqslant \frac{r_1}{p_1+r_1}$ and $p_3 \geqslant \frac{r_1+r_2}{(p_1+r_1)p_2+r_2}$.

(ii) $$A^{r_1+r_2} \geqslant \{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{\frac{r_1+r_2}{(p_1+r_1)p_2+r_2}}$$

holds for $p_1 \geqslant 0$ and $p_2 \geqslant \frac{r_1}{p_1+r_1}$.

Proof. Put $n = 3$ in Theorem 3.1 for (i) and put $n = 2$ in Theorem 3.1 for (ii). □

REMARK 3.1.

(i) implies (ii) by putting $r_3 = 0$, and (ii) implies $A^{r_1} \geqslant (A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}}$ by putting $r_2 = 0$ in (ii), which is (i) itself of Theorem C.

More precise estimation than Corollary 3.2 will be shown in Corollary 5.3.

We state the following Theorem 3.3 and Corollary 3.4 without proofs. In fact, by the almost same way as Theorem 3.1 we have Theorem 3.3. (see Remark 3.2.)

THEOREM 3.3. *Let $A \geqslant B \geqslant 0$ and $r_1, r_2, ..., r_n \geqslant 0$ for a natural number n. Then the following inequality holds,*

$$A^{1+r_1+r_2...+r_n} = \mathbb{C}_{A,A}[n]^{\frac{1+r_1+r_2...+r_n}{\mathfrak{q}[n]}} \geqslant \mathbb{C}_{A,B}[n]^{\frac{1+r_1+r_2...+r_n}{\mathfrak{q}[n]}} \tag{3.6}$$

for $p_1, p_2, ..., p_n$ satisfying

$$p_j \geqslant \frac{1+r_1+r_2+...+r_{j-1}}{\mathfrak{q}[j-1]} \quad \textit{for} \quad j = 1,2,...,n \qquad (r_0 = 0 \quad \textit{and} \quad \mathfrak{q}[0] = 1), \tag{3.7}$$

that is,

$$p_1 \geqslant 1, \quad p_2 \geqslant \frac{1+r_1}{p_1+r_1}, \quad p_3 \geqslant \frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}, \; ..., \; p_n \geqslant \frac{1+r_1+r_2+...+r_{n-1}}{\mathfrak{q}[n-1]}.$$

COROLLARY 3.4. *Let $A \geqslant B \geqslant 0$ and $r_1, r_2, r_3 \geqslant 0$. Then*

(i) $$A^{1+r_1+r_2+r_3} \geqslant \{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{1+r_1+r_2+r_3}{[(p_1+r_1)p_2+r_2]p_3+r_3}}.$$

holds for $p_1 \geqslant 1$, $p_2 \geqslant \frac{1+r_1}{p_1+r_1}$ and $p_3 \geqslant \frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}$.

(ii) $$A^{1+r_1+r_2} \geqslant \{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\}^{\frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}}$$

holds for $p_1 \geqslant 1$ and $p_2 \geqslant \frac{1+r_1}{p_1+r_1}$.

REMARK 3.2. We remark that Theorem 3.3 is a parallel result to Theorem 3.1 and also Corollary 3.4 is a parallel one to Corollary 3.2, and Theorem 3.1 is usually obtained from Theorem 3.3 by applying Uchiyama's nice technique [31] after proving Theorem 3.3.

Although many results on the chaotic order ($A \gg B$) have been derived from the corresponding results on the usual order ($A \geqslant B \geqslant 0$) by applying Uchiyama's nice method, we shall show Corollary 5.4 on the usual order ($A \geqslant B \geqslant 0$), which is a further extension of Theorem 3.3, *by using the corresponding result Corollary 5.2 on the chaotic order* ($A \gg B$) at the end of §5.

4. Monotonicity property on operator functions $\mathfrak{F}_k(p_k,r_k)=A^{\frac{-r_k}{2}}\mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+r_2+\ldots+r_k}{\mathfrak{q}[k]}}A^{\frac{-r_k}{2}}$

THEOREM 4.1. *Let $A \gg B$ and $r_1,r_2,\ldots,r_n \geqslant 0$ for a natural number n. For any fixed $\delta \geqslant 0$, let $p_1,p_2,\ldots,p_n$ be satisfied by*

$$p_j \geqslant \frac{\delta+r_1+r_2+\ldots+r_{j-1}}{\mathfrak{q}[j-1]} \quad \textit{for } j=1,2,\ldots,n, \tag{4.1}$$

that is,

$$p_1 \geqslant \delta, \quad p_2 \geqslant \frac{\delta+r_1}{p_1+r_1}, \ \ldots, \ p_k \geqslant \frac{\delta+r_1+r_2+\ldots+r_{k-1}}{\mathfrak{q}[k-1]}, \ \ldots, \ p_n \geqslant \frac{\delta+r_1+r_2+\ldots+r_{n-1}}{\mathfrak{q}[n-1]}.$$

The operator function $\mathfrak{F}_k(p_k,r_k)$ for any natural number k such that $1 \leqslant k \leqslant n$ is defined by

$$\mathfrak{F}_k(p_k,r_k)=A^{\frac{-r_k}{2}}\mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+r_2+\ldots+r_k}{\mathfrak{q}[k]}}A^{\frac{-r_k}{2}}. \tag{4.2}$$

Then the following inequality holds:

$$A^{\frac{r_{k-1}}{2}}\mathfrak{F}_{k-1}(p_{k-1},r_{k-1})A^{\frac{r_{k-1}}{2}} \geqslant \mathfrak{F}_k(p_k,r_k) \qquad (\mathfrak{F}_0(p_0,r_0)=B^{\delta}) \tag{4.3}$$

for every natural number k such that $1 \leqslant k \leqslant n$.

Proof. Since $\mathbb{C}_{A,B}[0]=B$, $\mathfrak{q}[0]=1$ in (2.5) and $p_0=r_0=0$ in (3.2), we may define $\mathfrak{F}_0(p_0,r_0)=B^{\delta}$ in (4.3). Let $A \gg B$. Then for any fixed $\delta \geqslant 0$,

$$B^{\delta} \geqslant A^{\frac{-r_1}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{\delta+r_1}{p_1+r_1}}A^{\frac{-r_1}{2}} \quad \text{for } p_1 \geqslant \delta \text{ and } r_1 \geqslant 0 \tag{4.4}$$

since $F_{A,B}(\delta,r_1) \geqslant F_{A,B}(p_1,r_1)$ holds by (ii) of Theorem C. And (4.4) can be expressed as

$$B^{\delta}=A^{\frac{r_0}{2}}\mathfrak{F}_0(p_0,r_0)A^{\frac{r_0}{2}} \geqslant A^{\frac{-r_1}{2}}\mathbb{C}_{A,B}[1]^{\frac{\delta+r_1}{\mathfrak{q}[1]}}A^{\frac{-r_1}{2}}=\mathfrak{F}_1(p_1,r_1) \quad \text{by (4.2).} \tag{4.5}$$

Since the condition (4.1) with $\delta \geqslant 0$ suffices (3.2) in Theorem 3.1, in fact, (3.2) is itself (4.1) without $\delta \geqslant 0$, we can apply Theorem 3.1 and we have the following (4.6) for natural number k such that $1 \leqslant k \leqslant n$

$$A^{r_1+r_2+\ldots+r_k} \geqslant \mathbb{C}_{A,B}[k]^{\frac{r_1+r_2+\ldots+r_k}{\mathfrak{q}[k]}}. \tag{4.6}$$

Since $X \geqslant Y > 0$ implies $X \gg Y$ and then $X^t \gg Y^t$ holds for any $t \geqslant 0$, (4.6) ensures

$$A^{\delta+r_1+r_2+\ldots+r_k} \gg \mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+r_2+\ldots+r_k}{\mathfrak{q}[k]}}.$$

Put $A_1=A^{\delta+r_1+r_2+\ldots+r_k}$ and $B_1=\mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+r_2+\ldots+r_k}{\mathfrak{q}[k]}}$ and applying (4.4) for $\delta=1$ and $A_1 \gg B_1$, we have

$$B_1 \geqslant A_1^{\frac{-r}{2}}(A_1^{\frac{r}{2}}B_1^{p}A_1^{\frac{r}{2}})^{\frac{1+r}{p+r}}A_1^{\frac{-r}{2}} \quad \text{holds for } p \geqslant 1 \text{ and } r \geqslant 0. \tag{4.7}$$

Put $r_{k+1} = r(\delta + r_1 + r_2 + ... + r_k)$ in (4.7). Then (4.7) can be rewritten by

$$B_1 \geqslant A^{\frac{-r_{k+1}}{2}} (A^{\frac{r_{k+1}}{2}} \mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+...+r_k}{\mathfrak{q}[k]}p} A^{\frac{r_{k+1}}{2}})^{\frac{1+r}{p+r}} A^{\frac{-r_{k+1}}{2}}. \tag{4.8}$$

Put $p = \frac{\mathfrak{q}[k]}{\delta+r_1+...+r_k} p_{k+1} \geqslant 1$, that is, $p_{k+1} \geqslant \frac{\delta+r_1+...+r_k}{\mathfrak{q}[k]}$ in (4.8), then we have

$$\begin{aligned}
A^{\frac{r_k}{2}} \mathfrak{F}_k(p_k, r_k) A^{\frac{r_k}{2}} &= \mathbb{C}_{A,B}[k]^{\frac{\delta+r_1+r_2+...+r_k}{\mathfrak{q}[k]}} = B_1 \quad \text{by (4.2)} \\
&\geqslant A^{\frac{-r_{k+1}}{2}} (A^{\frac{r_{k+1}}{2}} \mathbb{C}_{A,B}[k]^{p_{k+1}} A^{\frac{r_{k+1}}{2}})^{\frac{\delta+r_1+r_2+...+r_k+r_{k+1}}{\mathfrak{q}[k]p_{k+1}+r_{k+1}}} A^{\frac{-r_{k+1}}{2}} \\
&= A^{\frac{-r_{k+1}}{2}} (\mathbb{C}_{A,B}[k+1])^{\frac{\delta+r_1+r_2+...+r_k+r_{k+1}}{\mathfrak{q}[k+1]}} A^{\frac{-r_{k+1}}{2}} \\
&\qquad \text{by (i) and (ii) of Lemma 2.1} \\
&= \mathfrak{F}_{k+1}(p_{k+1}, r_{k+1}) \qquad \text{by (4.2) for } k+1
\end{aligned} \tag{4.9}$$

and we have (4.3) for k such that $1 \leqslant k \leqslant n$ by (4.9) and (4.5) since (4.5) means (4.3) for $k = 1$. □

REMARK 4.1. We shall give an alternative proof of Theorem 4.1 in Remark 6.1 via Theorem 6.1 at the end of §6.

5. Order preserving operator inequalities via operator functions in §4

We shall give order preserving operator inequalities as an application of Theorem 4.1.

THEOREM 5.1.. *Let $A \gg B$ and $r_1, r_2, ..., r_n \geqslant 0$ for a natural number n. Then the following inequalities hold for any fixed $\delta \geqslant 0$:*

$$\begin{aligned}
B^{\delta} &\geqslant A^{\frac{-r_1}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{\frac{\delta+r_1}{p_1+r_1}} A^{\frac{-r_1}{2}} \\
&\geqslant A^{\frac{-(r_1+r_2)}{2}} \{A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}\}^{\frac{\delta+r_1+r_2}{(p_1+r_1)p_2+r_2}} A^{\frac{-(r_1+r_2)}{2}} \\
&\geqslant A^{\frac{-(r_1+r_2+r_3)}{2}} \{A^{\frac{r_3}{2}} [A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}]^{p_3} A^{\frac{r_3}{2}}\}^{\frac{\delta+r_1+r_2+r_3}{\{(p_1+r_1)p_2+r_2\}p_3+r_3}} A^{\frac{-(r_1+r_2+r_3)}{2}} \\
&\ \vdots \\
&\geqslant A^{\frac{-(r_1+r_2+...+r_n)}{2}} \mathbb{C}_{A,B}[n]^{\frac{\delta+r_1+r_2+...+r_n}{\mathfrak{q}[n]}} A^{\frac{-(r_1+r_2+...+r_n)}{2}}
\end{aligned} \tag{5.1}$$

for $p_1, p_2, ..., p_n$ satisfying

$$p_j \geqslant \frac{\delta + r_1 + r_2 + ... + r_{j-1}}{\mathfrak{q}[j-1]} \quad \text{for } j = 1, 2, ..., n, \tag{4.1}$$

that is,

$$p_1 \geqslant \delta, \quad p_2 \geqslant \frac{\delta + r_1}{p_1 + r_1}, \ ..., \ p_k \geqslant \frac{\delta + r_1 + r_2 + ... + r_{k-1}}{\mathfrak{q}[k-1]}, \ ..., \ p_n \geqslant \frac{\delta + r_1 + r_2 + ... + r_{n-1}}{\mathfrak{q}[n-1]},$$

where $\mathbb{C}_{A,B}[n]$ *is defined in* (2.1) *and* $\mathfrak{q}[n]$ *is defined in* (2.4).

Proof. Applying (4.3) of Theorem 4.1 for k such that $1 \leqslant k \leqslant n$, we have

$$\begin{aligned}
B^{\delta} &= A^{\frac{r_0}{2}}\mathfrak{F}_0(p_0,r_0)A^{\frac{r_0}{2}} \\
&\geqslant \mathfrak{F}_1(p_1,r_1) \quad \text{by } p_1 \geqslant \delta \text{ in (4.3) for } k=1 \\
&\geqslant A^{\frac{-r_1}{2}}\mathfrak{F}_2(p_2,r_2)A^{\frac{-r_1}{2}} \quad \text{by } p_2 \geqslant \frac{\delta+r_1}{p_1+r_1} \text{ in (4.3) for } k=2 \\
&\geqslant A^{\frac{-r_1}{2}}A^{\frac{-r_2}{2}}\mathfrak{F}_3(p_3,r_3)A^{\frac{-r_2}{2}}A^{\frac{-r_1}{2}} \quad \text{by } p_3 \geqslant \frac{\delta+r_1+r_2}{(p_1+r_1)p_2+r_2} \text{ in (4.3) for } k=3 \\
&\ \vdots \\
&\geqslant A^{\frac{-(r_1+r_2+\ldots+r_{n-1})}{2}}\mathfrak{F}_n(p_n,r_n)A^{\frac{-(r_1+r_2+\ldots+r_{n-1})}{2}} \quad \text{for } p_n \geqslant \frac{\delta+r_1+r_2+\ldots+r_{n-1}}{\mathfrak{q}[n-1]} \text{ in (4.3)} \\
&= A^{\frac{-(r_1+r_2+\ldots+r_{n-1})}{2}}\left(A^{\frac{-r_n}{2}}\mathbb{C}_{[n]}^{\frac{\delta+r_1+r_2+\ldots+r_n}{\mathfrak{q}[n]}}A^{\frac{-r_n}{2}}\right)A^{\frac{-(r_1+r_2+\ldots+r_{n-1})}{2}} \quad \text{by (4.2)} \\
&= A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}}\mathbb{C}_{[n]}^{\frac{\delta+r_1+r_2+\ldots+r_n}{\mathfrak{q}[n]}}A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}}
\end{aligned}$$

and these inequalities yield the concrete inequalities in (5.1). □

COROLLARY 5.2.. *Let* $A \gg B$ *and* $r_1,r_2,\ldots,r_n \geqslant 0$ *for a natural number* n*. Then the following* (i) *and* (ii) *hold.*

(i)

$$\begin{aligned}
B &\geqslant A^{\frac{-r_1}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}A^{\frac{-r_1}{2}} \\
&\geqslant A^{\frac{-(r_1+r_2)}{2}}\left\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\right\}^{\frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}}A^{\frac{-(r_1+r_2)}{2}} \\
&\geqslant A^{\frac{-(r_1+r_2+r_3)}{2}}\{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{1+r_1+r_2+r_3}{\{(p_1+r_1)p_2+r_2\}p_3+r_3}}A^{\frac{-(r_1+r_2+r_3)}{2}} \\
&\ \vdots \\
&\geqslant A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}}\mathbb{C}_{A,B}[n]^{\frac{1+r_1+r_2+\ldots+r_n}{\mathfrak{q}[n]}}A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}} \qquad (5.2)
\end{aligned}$$

holds for $p_1,p_2,\ldots,p_n$ *satisfying* (3.7)*, that is,*

$$p_1 \geqslant 1, \quad p_2 \geqslant \frac{1+r_1}{p_1+r_1}, \quad p_3 \geqslant \frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}, \ \ldots, \ p_n \geqslant \frac{1+r_1+r_2+\ldots+r_{n-1}}{q[n-1]},$$

(ii) (5.2) *holds for* $p_1,p_2,\ldots,p_n \geqslant 1$*, where* $\mathbb{C}_{A,B}[n]$ *is defined in* (2.1) *and* $\mathfrak{q}[n]$ *is defined in* (2.4).

Proof. (i) We have only to put $\delta = 1$ in Theorem 5.1.

(ii) If $p_1,p_2,\ldots,p_n \geqslant 1$, then $w_j = \frac{1+r_1+r_2+\ldots r_{j-1}}{\mathfrak{q}[j-1]} \in [0,1]$ in (i) holds for j such that $2 \leqslant j \leqslant n$. Assume $p_j \geqslant w_j = \frac{1+r_1+r_2+\ldots r_{j-1}}{\mathfrak{q}[j-1]} \in [0,1]$ for j. In fact $w_2 = \frac{1+r_1}{p_1+r_1} \in$

$[0,1]$ by $p_1 \geqslant 1$. Then we have

$$\begin{aligned} w_{j+1} &= \frac{1+r_1+r_2+...r_{j-1}+r_j}{\mathfrak{q}[j]} \\ &= \frac{1+r_1+r_2+...r_{j-1}+r_j}{\mathfrak{q}[j-1]p_j+r_j} \leqslant 1 \end{aligned}$$

because the equality follows by (ii) of Lemma 2.1 and the inequality follows by the assumption. Whence if $p_1, p_2, ..., p_n \geqslant 1$, then $w_j = \frac{1+r_1+r_2+...r_{j-1}}{\mathfrak{q}[j-1]} \in [0,1]$ in (i) holds for $j = 2...,n$ and (5.2) holds by (i). $\square$

COROLLARY 5.3. *Let $A \gg B$ and $r_1, r_2, ..., r_n \geqslant 0$ for a natural number n. Then*

$$\begin{aligned} I &\geqslant A^{\frac{-r_1}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{r_1}{p_1+r_1}}A^{\frac{-r_1}{2}} \\ &\geqslant A^{\frac{-(r_1+r_2)}{2}}\left\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\right\}^{\frac{r_1+r_2}{(p_1+r_1)p_2+r_2}}A^{\frac{-(r_1+r_2)}{2}} \\ &\geqslant A^{\frac{-(r_1+r_2+r_3)}{2}}\{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{r_1+r_2+r_3}{\{(p_1+r_1)p_2+r_2\}p_3+r_3}}A^{\frac{-(r_1+r_2+r_3)}{2}} \\ &\;\;\vdots \\ &\geqslant A^{\frac{-(r_1+r_2+...+r_n)}{2}}\mathbb{C}_{A,B}[n]^{\frac{r_1+r_2+...+r_n}{\mathfrak{q}[n]}}A^{\frac{-(r_1+r_2+...+r_n)}{2}} \end{aligned} \tag{5.3}$$

holds for $p_1, p_2, ..., p_n$ satisfying (3.2)*, that is,*

$$p_1 \geqslant 0, \quad p_2 \geqslant \frac{r_1}{p_1+r_1}, \quad p_3 \geqslant \frac{r_1+r_2}{(p_1+r_1)p_2+r_2}, \; ..., \; p_n \geqslant \frac{r_1+r_2+...+r_{n-1}}{q[n-1]}.$$

Proof. We have only to put $\delta = 0$ in Theorem 5.1. $\square$

COROLLARY 5.4. *Let $A \geqslant B \geqslant 0$ and $r_1, r_2, ..., r_n \geqslant 0$ for a natural number n. Then*

$$\begin{aligned} A \geqslant B &\geqslant A^{\frac{-r_1}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}}A^{\frac{-r_1}{2}} \\ &\geqslant A^{\frac{-(r_1+r_2)}{2}}\left\{A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}\right\}^{\frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}}A^{\frac{-(r_1+r_2)}{2}} \\ &\geqslant A^{\frac{-(r_1+r_2+r_3)}{2}}\{A^{\frac{r_3}{2}}[A^{\frac{r_2}{2}}(A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{p_2}A^{\frac{r_2}{2}}]^{p_3}A^{\frac{r_3}{2}}\}^{\frac{1+r_1+r_2+r_3}{\{(p_1+r_1)p_2+r_2\}p_3+r_3}}A^{\frac{-(r_1+r_2+r_3)}{2}} \\ &\;\;\vdots \\ &\geqslant A^{\frac{-(r_1+r_2+...+r_n)}{2}}\mathbb{C}_{A,B}[n]^{\frac{1+r_1+r_2+...+r_n}{\mathfrak{q}[n]}}A^{\frac{-(r_1+r_2+...+r_n)}{2}} \end{aligned} \tag{5.4}$$

holds for $p_1, p_2, ..., p_n$ satisfying (3.7)*, that is,*

$$p_1 \geqslant 1, \quad p_2 \geqslant \frac{1+r_1}{p_1+r_1}, \quad p_3 \geqslant \frac{1+r_1+r_2}{(p_1+r_1)p_2+r_2}, \; ..., \; p_n \geqslant \frac{1+r_1+r_2+...+r_{n-1}}{q[n-1]},$$

where $\mathbb{C}_{A,B}[n]$ *is defined in* (2.1) *and* $\mathfrak{q}[n]$ *is defined in* (2.4).

Proof. The hypothesis $A \geqslant B \geqslant 0$ implies $A \gg B$ and the proof follows by (i) in Corollary 5.2 and the hypothesis $A \geqslant B \geqslant 0$. □

REMARK 5.1. Corollary 5.2 is a further extension of [24], [17], [19], [33] and Theorem FKN-2 in [9]. Corollary 5.3 is more precise estimation than Corollary 3.2.

We would like to emphasize that Corollary 5.4 is a further extension of Theorem 3.3 since (5.4) *easily implies* (3.6) *in Theorem 3.3 and moreover the essential part of* (5.4) *in Corollary 5.4 on the usual order* $(A \geqslant B \geqslant 0)$ *is derived from Corollary 5.2 on the chaotic order* $(A \gg B)$.

6. Further extensions of Theorem B and Theorem C

Further extensions of Theorem B and Theorem C are given by using the operator function

$$\mathfrak{F}_n(p_n, r_n) = A^{\frac{-r_n}{2}} \mathbb{C}_{A,B}[n]^{\frac{\delta+r_1+r_2\ldots+r_n}{\mathfrak{q}[n]}} A^{\frac{-r_n}{2}} \text{ in } \S 4.$$

THEOREM 6.1. *Let* $A \gg B$ *and* $r_1, r_2, \ldots, r_n \geqslant 0$ *for a natural number* n. *For any fixed* $\delta \geqslant 0$, *let* $p_1, p_2, \ldots, p_n$ *be satisfied by*

$$p_j \geqslant \frac{\delta + r_1 + r_2 + \ldots + r_{j-1}}{\mathfrak{q}[j-1]} \quad \textit{for } j = 1, 2, \ldots, n, \tag{4.1}$$

that is,

$$p_1 \geqslant \delta, \quad p_2 \geqslant \frac{\delta+r_1}{p_1+r_1}, \ \ldots, \ p_k \geqslant \frac{\delta+r_1+r_2+\ldots+r_{k-1}}{q[k-1]}, \ \ldots, \ p_n \geqslant \frac{\delta+r_1+r_2+\ldots+r_{n-1}}{\mathfrak{q}[n-1]}.$$

Then

$$\mathfrak{F}_n(p_n, r_n) = A^{\frac{-r_n}{2}} \mathbb{C}_{A,B}[n]^{\frac{\delta+r_1+r_2+\ldots+r_n}{\mathfrak{q}[n]}} A^{\frac{-r_n}{2}} \tag{6.1}$$

is a decreasing function of both $r_n \geqslant 0$ *and* p_n *which satisfies*

$$p_n \geqslant \frac{\delta + r_1 + r_2 + \ldots + r_{n-1}}{\mathfrak{q}[n-1]}. \tag{6.2}$$

Proof. Since the condition (4.1) with $\delta \geqslant 0$ suffices (3.2) in Theorem 3.1, in fact, (3.2) is itself (4.1) without $\delta \geqslant 0$, we have the following (3.1) by Theorem 3.1:

$$A^{r_1+r_2\ldots+r_n} = \mathbb{C}_{A,A}[n]^{\frac{r_1+r_2\ldots+r_n}{\mathfrak{q}[n]}} \geqslant \mathbb{C}_{A,B}[n]^{\frac{r_1+r_2\ldots+r_n}{\mathfrak{q}[n]}}. \tag{3.1}$$

We state the following important inequality (6.3) for the forthcoming discussion thanks to (6.2) which is the inequality in (4.1) for $j = n$:

$$\mathfrak{q}[n] = \mathfrak{q}[n-1]p_n + r_n \geqslant \delta + r_1 + r_2 + \ldots + r_{n-1} + r_n \tag{6.3}$$

because the equality in (6.3) follows by (ii) of Lemma 2.1 and the inequality follows by $\mathfrak{q}[n-1]p_n \geqslant \delta + r_1 + r_2 + \ldots + r_{n-1}$ obtained by (6.2).

For simplicity, let $\mathbb{C}_{A,B}[n]$ *be denoted by* $\mathbb{C}_{[n]}$ in the proof.

(a) *Proof of the result that* $\mathfrak{F}_{A,B}(p_n, r_n)$ *is a decreasing function of* p_n.

Raise each side of (3.1) to the power $\dfrac{r_n}{r_1 + r_2 \ldots + r_n} \in [0,1]$ by LH, then

$$A^{r_n} \geqslant \mathbb{C}_{[n]}^{\frac{r_n}{\mathfrak{q}[n]}} = (A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{p_n} A^{\frac{r_n}{2}})^{\frac{r_n}{\mathfrak{q}[n]}} \qquad \text{by (i) of Lemma 2.1} \tag{6.4}$$

$$= A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \left(\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{r_n} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \right)^{\frac{r_n - q[n]}{\mathfrak{q}[n]}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{\frac{r_n}{2}} \text{ by Lemma D} \tag{6.5}$$

and (6.5) implies

$$\left(\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{r_n} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \right)^{\frac{\mathfrak{q}[n] - r_n}{\mathfrak{q}[n]}} \geqslant \mathbb{C}_{[n-1]}^{p_n}. \tag{6.6}$$

Put

$$\alpha = \frac{w}{p_n} \in [0,1] \qquad \text{for } p_n \geqslant w \geqslant 0. \tag{6.7}$$

Let $\mathfrak{q}[n|p_1, p_2, \ldots, p_n + w|r_1, r_2, \ldots, r_n]$ be denoted by $\mathfrak{q}[p_1, p_2, \ldots, p_n + w]$ for simplicity. Then we have

$$\begin{aligned} &\mathfrak{q}[p_1, p_2, \ldots, p_n + w] - \mathfrak{q}[n] \\ &= \mathfrak{q}[n|p_1, p_2, \ldots, p_n + w|r_1, r_2, \ldots, r_n] \mathfrak{q}[n|p_1, p_2, \ldots, p_n|r_1, r_2, \ldots, r_n] \\ &= (\mathfrak{q}[n] - r_n)\alpha \text{ by (2.4) and (6.7).} \end{aligned} \tag{6.8}$$

Raise each side of (6.6) to the power $\alpha = \dfrac{w}{p_n} \in [0,1]$ in (6.7), then

$$\left(\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{r_n} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \right)^{\frac{(\mathfrak{q}[n] - r_n)\alpha}{\mathfrak{q}[n]}} \geqslant \mathbb{C}_{[n-1]}^{w}. \tag{6.9}$$

Whence we have

$$\begin{aligned} f(p_n) &= \mathbb{C}_{[n]}^{\frac{\delta + r_1 + r_2 \ldots + r_n}{\mathfrak{q}[n]}} \\ &= (A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{p_n} A^{\frac{r_n}{2}})^{\frac{\delta + r_1 + r_2 \ldots + r_n}{\mathfrak{q}[p_1, p_2, \ldots p_n]}} \qquad \text{by (i) of Lemma 2.1} \qquad (6.10) \\ &= \left\{ (A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{p_n} A^{\frac{r_n}{2}})^{\frac{\mathfrak{q}[p_1, p_2, \ldots, p_n + w]}{\mathfrak{q}[n]}} \right\}^{\frac{\delta + r_1 + r_2 \ldots + r_n}{\mathfrak{q}[p_1, p_2, \ldots, p_n + w]}} \\ &= \left\{ A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} (\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{r_n} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}})^{\frac{\mathfrak{q}[p_1, p_2, \ldots, p_n + w] - \mathfrak{q}[n]}{\mathfrak{q}[n]}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{\frac{r_n}{2}} \right\}^{\frac{\delta + r_1 + r_2 \ldots + r_n}{\mathfrak{q}[p_1, p_2, \ldots, p_n + w]}} \\ &\qquad \text{by Lemma D} \\ &= \{ A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} (\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{r_n} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}})^{\frac{(\mathfrak{q}(n) - r_n)\alpha}{\mathfrak{q}[n]}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{\frac{r_n}{2}} \}^{\frac{\delta + r_1 + r_2 \ldots + r_n}{\mathfrak{q}[p_1, p_2, \ldots, p_n + w]}} \text{ by (6.8)} \\ &\geqslant \{ A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \mathbb{C}_{[n-1]}^{w} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{\frac{r_n}{2}} \}^{\frac{\delta + r_1 + r_2 \ldots + r_n}{\mathfrak{q}[p_1, p_2, \ldots, p_n + w]}} \\ &= \left(A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{p_n + w} A^{\frac{r_n}{2}} \right)^{\frac{\delta + r_1 + r_2 \ldots + r_n}{\mathfrak{q}[p_1, p_2, \ldots, p_n + w]}} \\ &= f(p_n + w) \qquad \text{by (6.10)} \qquad (6.11) \end{aligned}$$

and the last inequality holds by LH because (6.9) and $\frac{\delta+r_1+r_2...+r_n}{\mathfrak{q}[p_1,p_2,...,p_n+w]} \in [0,1]$, which is ensured by (6.3) and $\mathfrak{q}[p_1,p_2,...,p_n+w] \geqslant \mathfrak{q}[p_1,p_2,...,p_n]$ by (2.4), so that $\mathfrak{F}_n(p_n,r_n) = A^{\frac{-r_n}{2}} f(p_n) A^{\frac{-r_n}{2}}$ is a decreasing function of p_n by (6.11).

(b) *Proof of the result that* $\mathfrak{F}_n(p_n,r_n)$ *is a decreasing function of* r_n.

Raise each side of (6.4) to the power $\dfrac{u}{r_n} \in [0,1]$ for $r_n \geqslant u \geqslant 0$ by LH, then

$$A^u \geqslant (A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{p_n} A^{\frac{r_n}{2}})^{\frac{u}{\mathfrak{q}[n]}}. \tag{6.12}$$

By (2.4) of the definition $\mathfrak{q}[n]$ we have

$$\begin{aligned}\mathfrak{q}[n]+u &= \mathfrak{q}[n|p_1,p_2,...,p_n|r_1,r_2,...,r_n]+u \\ &= \mathfrak{q}[n|p_1,p_2,...,p_n|r_1,r_2,...,r_n+u] \quad \text{(denoted by } \mathfrak{q}[r_1,r_2,...,r_n+u]).\end{aligned} \tag{6.13}$$

We state the following (6.14) by (ii) of Lemma 2.1,

$$\begin{aligned}\delta+r_1+r_2+...+r_{n-1}+r_n-\mathfrak{q}[n] &= \delta+r_1+r_2+...+r_{n-1}+r_n-(\mathfrak{q}[n-1]p_n+r_n) \\ &= \delta+r_1+r_2+...+r_{n-1}-\mathfrak{q}[n-1]p_n.\end{aligned} \tag{6.14}$$

Recall that the right hand side of (6.14) *is the numerator of the exponential power in the forthcoming* (6.16) *which does not contain the term "* r_n *".*

Then we have

$$\begin{aligned}
\mathfrak{F}_n(p_n,r_n) &= A^{\frac{-r_n}{2}} \mathbb{C}_{[n]}^{\frac{\delta+r_1+r_2...r_{n-1}+r_n}{\mathfrak{q}[n]}} A^{\frac{-r_n}{2}} && (6.1)\\
&= A^{\frac{-r_n}{2}} (A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{p_n} A^{\frac{r_n}{2}})^{\frac{\delta+r_1+r_2...r_{n-1}+r_n}{\mathfrak{q}[n]}} A^{\frac{-r_n}{2}} \quad \text{by (i) of Lemma 2.1} \\
&= \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} (\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{r_n} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}})^{\frac{\delta+r_1+r_2...r_{n-1}+r_n-\mathfrak{q}[n]}{\mathfrak{q}[n]}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \text{ by Lemma D} \\
&= \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} (\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{r_n} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}})^{\frac{\delta+r_1+r_2...+r_{n-1}-\mathfrak{q}[n-1]p_n}{\mathfrak{q}[[r_1,r_2,...r_n]}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \text{ by (6.14)} && (6.15)\\
&= \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \{(\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{r_n} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}})^{\frac{\mathfrak{q}[n]+u}{\mathfrak{q}[n]}}\}^{\frac{\delta+r_1+r_2...+r_{n-1}-\mathfrak{q}[n-1]p_n}{\mathfrak{q}[n]+u}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \\
&= \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \{(\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{r_n} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}})^{\frac{\mathfrak{q}[n]+u}{\mathfrak{q}[n]}}\}^{\frac{\delta+r_1+r_2...+r_{n-1}-\mathfrak{q}[n-1]p_n}{\mathfrak{q}[r_1,r_2,...,r_n+u]}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \quad \text{by (6.13)} \\
&= \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \left\{\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{\frac{r_n}{2}} \left(A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{p_n} A^{\frac{r_n}{2}}\right)^{\frac{u}{\mathfrak{q}[n]}} A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}}\right\}^{\frac{\delta+r_1+r_2...+r_{n-1}-\mathfrak{q}[n-1]p_n}{\mathfrak{q}[r_1,r_2,...,r_n+u]}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \\
&\qquad \text{by Lemma D} \\
&\geqslant \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} (\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{\frac{r_n}{2}} A^u A^{\frac{r_n}{2}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}})^{\frac{\delta+r_1+r_2...+r_{n-1}-\mathfrak{q}[n-1]p_n}{\mathfrak{q}[r_1,r_2,...,r_n+u]}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \\
&= \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} \left(\mathbb{C}_{[n-1]}^{\frac{p_n}{2}} A^{r_n+u} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}}\right)^{\frac{\delta+r_1+r_2...+r_{n-1}-\mathfrak{q}[n-1]p_n}{\mathfrak{q}[r_1,r_2,...,r_n+u]}} \mathbb{C}_{[n-1]}^{\frac{p_n}{2}} && (6.16)\\
&= \mathfrak{F}_n(p_n,r_n+u) \quad \text{by (6.15)} && (6.17)
\end{aligned}$$

and the last inequality holds by LH because (6.12) and

$$\frac{\delta+r_1+r_2...+r_{n-1}-\mathfrak{q}[n-1]p_n}{\mathfrak{q}[r_1,r_2,...,r_n+u]}=\frac{-\{\mathfrak{q}[n]-(\delta+r_1+r_2...+r_n)\}}{\mathfrak{q}[n]+u}\in[-1,0],$$

which is shown by (6.14), (6.13) and $\mathfrak{q}[n]\geqslant\delta+r_1+r_2...+r_n$ in (6.3), and taking inverses of both sides, so that $\mathfrak{F}_n(p_n,r_n)$ is a decreasing function of r_n by (6.17). □

COROLLARY 6.2. *Let $A\gg B$ and $r_1,r_2,...,r_n\geqslant 0$ and also $p_1,p_2,...,p_n\geqslant 1$ for a natural number n. Then*

$$\mathfrak{F}_n(p_n,r_n)=A^{\frac{-r_n}{2}}\mathbb{C}_{A,B}[n]^{\frac{1+r_1+r_2+...+r_n}{\mathfrak{q}[n]}}A^{\frac{-r_n}{2}} \tag{6.1$'$}$$

is a decreasing function of both $r_n\geqslant 0$ and $p_n\geqslant 1$.

Proof. Put $\delta=1$ in Theorem 6.1. It is shown in the proof of (ii) in Corollary 5.2 that if $p_1,p_2,...,p_n\geqslant 1$, then each of the right hand in (4.1), $\frac{1+r_1+r_2+...+r_{j-1}}{\mathfrak{q}[j-1]}\in[0,1]$ for $j=1,2,...,n$, and we have the conclusion by Theorem 6.1. □

REMARK 6.1. *An alternative proof of Theorem 4.1 via Theorem 6.1.*
Assume all the conditions in Theorem 4.1. Then

$$\begin{aligned}\mathfrak{F}_k(p_k,r_k)&=A^{\frac{-r_k}{2}}\mathbb{C}_{[k]}^{\frac{\delta+r_1+r_2+...+r_k}{\mathfrak{q}[k]}}A^{\frac{-r_k}{2}} &(4.2)\\ &=A^{\frac{-r_k}{2}}(A^{\frac{r_k}{2}}\mathbb{C}_{[k-1]}^{p_k}A^{\frac{r_k}{2}})^{\frac{\delta+r_1+r_2+...+r_{k-1}+r_k}{\mathfrak{q}[k-1]p_k+r_k}}A^{\frac{-r_k}{2}}\\ &\qquad\text{by (i) and (ii) of Lemma 2.1} &(6.18)\end{aligned}$$

and by putting $r_k=0$ in (4.2) and (6.18) stated above, for every natural number k such that $1\leqslant k\leqslant n$, we have

$$\mathfrak{F}_k(p_k,0)=(\mathbb{C}_{[k-1]}^{p_k})^{\frac{\delta+r_1+r_2+...+r_{k-1}}{\mathfrak{q}[k-1]p_k}}=\mathbb{C}_{[k-1]}^{\frac{\delta+r_1+r_2+...+r_{k-1}}{\mathfrak{q}[k-1]}} \tag{6.19}$$

and

$$\begin{aligned}\mathfrak{F}_{k-1}(p_{k-1},r_{k-1})&=A^{\frac{-r_{k-1}}{2}}\mathbb{C}_{[k-1]}^{\frac{\delta+r_1+r_2+...+r_{k-1}}{\mathfrak{q}[k-1]}}A^{\frac{-r_{k-1}}{2}} \quad \text{by (4.2) for } k-1\\ &=A^{\frac{-r_{k-1}}{2}}\mathfrak{F}_k(p_k,0)A^{\frac{-r_{k-1}}{2}} \quad \text{by (6.19)}\\ &\geqslant A^{\frac{-r_{k-1}}{2}}\mathfrak{F}_k(p_k,r_k)A^{\frac{-r_{k-1}}{2}} &(6.20)\end{aligned}$$

and the last inequality follows by $\mathfrak{F}_k(p_k,0)\geqslant\mathfrak{F}_k(p_k,r_k)$ for $p_k\geqslant\frac{\delta+r_1+r_2+...+r_{k-1}}{\mathfrak{q}[k-1]}$ by applying Theorem 6.1 for every k such that $1\leqslant k\leqslant n$ and we have (4.3) in Theorem 4.1 by (6.20). □

REMARK 6.2. Theorem 6.1 is a further extensions of (ii) in Theorem C. In fact, (ii) of Theorem C is just Theorem 6.1 in the case $n=1$. Moreover Theorem 6.1 is a further extension of Theorem B since the hypothesis $A\gg B$ in Theorem 6.1 is weaker than the hypothesis $A\geqslant B\geqslant 0$ in Theorem B.

7. Concluding remarks

Let us state the background of this paper. At first we state the following operator inequality (G).

If $A \geqslant B \geqslant 0$ with $A > 0$, then for $t \in [0,1]$ and $p \geqslant 1$

$$A^{1+r-t} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}} \tag{G}$$

holds for $s \geqslant 1$ and $r \geqslant t$.

This operator inequality (G) is shown in [13], [5] and [14], and the best possibility of the exponential power $\frac{1+r-t}{(p-t)s+r}$ is shown in [30], [32], [7] and see related papers (e,g., [16], [19], [20], [25]).

This inequality (G) interpolates Theorem A and an inequality equivalent to the main result of log majorization by Ando-Hiai [2].

If we replace the hypothesis $A \geqslant B \geqslant 0$ with $A > 0$ in (G) *by weaker condition $A \gg B$, what can we obtain the corresponding result to* (G)*?* $(\diamondsuit)$

Motivated by $(\diamondsuit)$ and LH, we posed the following question in [15].

F-QUESTION. *For $A,B > 0$, $A \gg B$ if and only if*

$$A^{r-t} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}} \tag{Q}$$

holds for all $p \geqslant 1, s \geqslant 1$, $t \in [0,1]$ and $r \geqslant t$?

By a nicely applying Kantorovich type operator inequality, Fujii et al. [8] obtained the following interesting result as an answer to F-Question.

THEOREM FKN-1. [8] *For $A,B > 0$, $A \geqslant B$ if and only if*

$$A^{r-t} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^{p}A^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$$

holds for all $p \geqslant 1, s \geqslant 1$, $t \in [0,1]$ and $r \geqslant t$.

Fujii et al. [9] have been considering to discuss on the operator inequality (Q) under the chaotic order. Among others, they obtained the following result by using a mean theoretic idea in [26].

THEOREM FKN-2. [9] *If $A \gg B$ for $A,B > 0$, then*

$$A^{t-r}\sharp_{\frac{1+r-t}{(p-t)s+r}}(A^{t}\natural_{s}B^{p}) \leqslant A^{t}\sharp_{\frac{1-t}{p-t}}B^{p} \leqslant B$$

holds for $p \geqslant 1$, $s \geqslant 1$, $r \geqslant 0$ and $t < 0$.

Inspired by Theorem FKN-2, we obtain the following results.

Let $A \gg B$ and $r_1, r_2, \ldots, r_n \geqslant 0$ and any fixed $\delta \geqslant 0$, let

$$p_1 \geqslant \delta, \quad p_2 \geqslant \frac{\delta+r_1}{p_1+r_1}, \ \ldots, \ p_k \geqslant \frac{\delta+r_1+r_2+\ldots+r_{k-1}}{\mathfrak{q}[k-1]}, \ \ldots, \ p_n \geqslant \frac{\delta+r_1+r_2+\ldots+r_{n-1}}{\mathfrak{q}[n-1]}.$$

Let $\mathfrak{F}_n(p_n, r_n)$ *be defined by*

$$\mathfrak{F}_n(p_n, r_n) = A^{\frac{-r_n}{2}} \mathbb{C}_{A,B}[n]^{\frac{\delta + r_1 + r_2 + \ldots + r_n}{\mathfrak{q}[n]}} A^{\frac{-r_n}{2}}.$$

Then the following inequalities (i), (ii) *and* (iii) *hold*:

(i) $A^{\frac{p_{k-1}}{2}} \mathfrak{F}_{k-1}(p_{k-1}, r_{k-1}) A^{\frac{p_{k-1}}{2}} \geqslant \mathfrak{F}_k(p_k, r_k)$ *for* k *such that* $1 \leqslant k \leqslant n$,

(ii)

$$\begin{aligned}
B^{\delta} &\geqslant A^{\frac{-r_1}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{\frac{\delta + r_1}{p_1 + r_1}} A^{\frac{-r_1}{2}} \\
&\geqslant A^{\frac{-(r_1+r_2)}{2}} \{A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}\}^{\frac{\delta + r_1 + r_2}{(p_1+r_1)p_2 + r_2}} A^{\frac{-(r_1+r_2)}{2}} \\
&\geqslant A^{\frac{-(r_1+r_2+r_3)}{2}} \{A^{\frac{r_3}{2}} [A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} B^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}]^{p_3} A^{\frac{r_3}{2}}\}^{\frac{\delta + r_1 + r_2 + r_3}{\{(p_1+r_1)p_2+r_2\}p_3 + r_3}} A^{\frac{-(r_1+r_2+r_3)}{2}} \\
&\vdots \\
&\geqslant A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}} \mathbb{C}_{A,B}[n]^{\frac{\delta + r_1 + r_2 + \ldots + r_n}{\mathfrak{q}[n]}} A^{\frac{-(r_1+r_2+\ldots+r_n)}{2}},
\end{aligned}$$

(iii) $\mathfrak{F}_n(p_n, r_n)$ *is a decreasing function of both* $r_n \geqslant 0$ *and* $p_n \geqslant \frac{\delta + r_1 + r_2 + \ldots + r_{n-1}}{\mathfrak{q}[n-1]}$, *where* $\mathbb{C}_{A,B}[n]$ *] and* $\mathfrak{q}[n]$ *are defined as follows*:

$$\mathbb{C}_{A,B}[n] = A^{\frac{r_n}{2}} \left\{ A^{\frac{r_{n-1}}{2}} [\ldots A^{\frac{r_3}{2}} \{A^{\frac{r_2}{2}} (A^{\frac{r_1}{2}} A^{p_1} A^{\frac{r_1}{2}})^{p_2} A^{\frac{r_2}{2}}\}^{p_3} A^{\frac{r_3}{2}} \ldots]^{p_{n-1}} A^{\frac{r_{n-1}}{2}} \right\}^{p_n} A^{\frac{r_n}{2}}$$

and

$$\mathfrak{q}[n] = [\ldots\{(p_1 + r_1)p_2 + r_2\}p_3 + \ldots r_{n-1}]p_n + r_n.$$

In fact, (ii) can be considered as *satellite inequalities to the chaotic order* and a further extension of Theorem FKN-2.

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REFERENCES

[1] T. ANDO, *On some operator inequality*, Math. Ann. **279** (1987), 157–159.

[2] T. ANDO AND F. HIAI, *Log majorization and complementary Golden-Thompson type inequalities,* Linear Alg. and Its Appl. **197, 198** (1994), 113–131.

[3] R. BHATIA, *Positive Definite Matrices,* Princeton Univ. Press, 2007.

[4] M. FUJII, *Furuta's inequality and its mean theoretic approach,* J. Operator Theory **23** (1990), 67–72.

[5] M. FUJII AND E. KAMEI, *Mean theoretic approach to the grand Furuta inequality,* Proc. Amer. Math. Soc. **124** (1996), 2751–2756.

[6] M. FUJII, T. FURUTA AND E. KAMEI, *Furuta's inequality and its application to Ando's theorem,* Linear Alg. Appl. **179** (1993), 161–169.

[7] M. FUJII, A. MATSUMOTO AND R. NAKAMOTO, *A short proof of the best possibility for the grand Furuta inequality,* J. of Inequal. and Appl. **4** (1999), 339–344.

[8] M. FUJII, E. KAMEI AND R. NAKAMOTO, *On a question of Furuta on chaotic order,* Linear Algebra Appl. **341** (2002), 119–127.
[9] M. FUJII, E. KAMEI AND R. NAKAMOTO, *On a question of Furuta on chaotic order* II, Math. J. Okayama Univ. **45** (2003), 123–131.
[10] T. FURUTA, *$A \geqslant B \geqslant 0$ assures $(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ with $(1+2r)q \geqslant p+2r$,* Proc. Amer. Math. Soc. **101** (1987), 85–88.
[11] T. FURUTA, *Elementary proof of an order preserving inequality,* Proc. Japan Acad. **65** (1989), 126.
[12] T. FURUTA, *Applications of order preserving operator inequalities,* Operator Theory: Advances and Applications **59** (1992), 180–190.
[13] T. FURUTA, *An extension of the Furuta inequality and Ando-Hiai log majorization,* Linear Alg. and Its Appl. **219** (1995), 139–155.
[14] T. FURUTA, *Simplified proof of an order preserving operator inequality,* Proc. Japan Acad. **74**, Ser. A (1998), 114.
[15] T. FURUTA, *Results under $\log A \geqslant \log B$ can be derived from ones under $A \geqslant B \geqslant 0$ by Uchiyama's method associated with Furuta and Kantorovich type operator inequalities,* Math. Inequal. Appl. **3** (2000), 423–436.
[16] T. FURUTA, *Invitation to Linear Operators,* Taylor & and Francis, **2001**, London.
[17] T. FURUTA, *$A \geqslant B > 0$ assures $A^{1+r-t} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})A^{\frac{r}{2}}\}^{\frac{1+r-t}{(p-t)s+r}}$ for $t \in [0,1]$, $r \geqslant t$, $p \geqslant 1$, $s \geqslant 1$ and related inequalities,* Archives of Inequalities and Applications **2** (2004), 141–158.
[18] T. FURUTA, *Further extension of an order preserving operator inequality,* J. Math. Inequal. **2** (2008), 465–472.
[19] T. FURUTA AND M. HASHIMOTO AND M. ITO, *Equivalence relation between generalized Furuta inequality and related operator functions,* Scientiae Mathematicae **1** (1998), 257–259.
[20] T. FURUTA, M. YANAGIDA AND T. YAMAZAKI, *Operator functions implying Furuta inequality,* Math. Inequal. Appl. **1** (1998), 123–130.
[21] E. HEINZ, *Beiträge zur Störungsteorie der Spektralzerlegung,* Math. Ann. **123** (1951), 415–438.
[22] M. ITO AND E. KAMEI, *Mean theoretic approach to a further extension of grand Furuta inequality,* J.Math. Inequal. **4** (2010), 325–333.
[23] S. IZUMINO, N. NAKAMURA AND M. TOMINAGA, *Mean theoretic operator functions for extensions of the grand Furuta inequality,* Sci. Math. Jpn. **72** (2010), 157–163.
[24] E. KAMEI, *A satelite to Furuta's inequality,* Math. Japon. **33** (1988), 883–886.
[25] E. KAMEI, *Parametrized grand Furuta inequality,* Math. Japon. **50** (1999), 79–83.
[26] F. KUBO AND T. ANDO, *Means of positive linear operators,* Math. Ann. **246** (1980), 205–224.
[27] K. LÖWNER, *Über monotone Matrixfunktionen,* Math. Z. **38** (1934), 177–216.
[28] G. K. PEDERSEN, *Some operator monotone functions,* Proc. Amer. Math. Soc. **36** (1972), 309–310.
[29] K. TANAHASHI, *Best possibility of the Furuta inequality,* Proc. Amer. Math. Soc. **124** (1996), 141–146.
[30] K. TANAHASHI, *The best possibility of the grand Furuta inequality,* Proc. Amer. Math. Soc. **128** (2000), 511–519.
[31] M. UCHIYAMA, *Some exponential operator inequalities,* Math. Inequal. Appl. **2** (1999), 469–471.
[32] T. YAMAZAKI, *Simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality,* Math. Inequal. Appl. **2** (1999), 473–477.
[33] J. YUAN AND Z. GAO, *Classified construction of generalized Furuta type operator functions,* Math. Inequal. Appl. **11** (2008), 189–202.
[34] C. YANG AND Y. WANG, *Further extension of Furuta inequality,* J. Math. Inequal. **4** (2010), 391–398.

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