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SOME INTEGRAL INEQUALITIES IN TWO INDEPENDENT VARIABLES ON TIME SCALES

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Abstract. The aim of the present paper is to investigate some integral inequalities in two independent variable on time scales, which unify and extend some integral inequalities and their corresponding discrete analogues. The inequalities given here can be used as handy tools to study the properties of certain partial dynamic equations on time scales.

1. Introduction

Following Hilger's landmark papers [1], there have been plenty of references focused on the theory of time scales in order to unify continuous and discrete analysis. Recently, many authors have extended some useful dynamic equations on time scale, for example $[2-10]$ and $[12,13]$, and the references therein. In this paper, we investigate some integral inequalities in two independent variable on time scales, which extended some discrete inequalities by Li [10] and Meng and Ji [11] to arbitrary time scales. The inequalities given here can be used as handy tools in the qualitative theory of certain classes of delay dynamic equations on time scales.

Throughout this paper, a knowledge and understanding of time scale notation is assumed. For an excellent introduction to the calculus on time scales, we refer the reader to monographs [2,3].

2. Some preliminaries

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0,\infty)$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}_0=\{0,1,2,...\}$ denotes the set of nonnegative integers, $\mathbb{T}$ is an arbitrary time scale, C_{rd} denotes the set of rd-continuous, $\mathscr{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathscr{R}^+ = \{p\in\mathscr{R} : 1+\mu(t)p(t)>0, t\in\mathbb{T}\}$. Throughout this paper, we always assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales, $t_0\in\mathbb{T}_1$, $s_0\in\mathbb{T}_2$, $t\geqslant t_0$, $s\geqslant s_0$ and $\Omega=\mathbb{T}_1\times\mathbb{T}_2$.

The following lemmas are very useful in our main results.

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LEMMA 2.1. ([2]) *Suppose* $u(t), b(t) \in \mathscr{C}_{rd}, a(t) \in \mathscr{R}^+$, *then*

$$u^{\Delta}(t) \leqslant a(t)u(t) + b(t), \quad t \in \mathbb{T},$$

implies

$$u(t) \leqslant u(t_0)e_a(t,t_0) + \int_{t_0}^{t} e_a(t,\sigma(\tau))b(\tau)\Delta\tau, \quad t \in \mathbb{T}.$$

LEMMA 2.2. ([9]) *Let* $u(t), f(t), g(t) \in \mathscr{C}_{rd}$, $u(t), f(t)$ *and* $g(t)$ *be nonnegative. If* $f(t)$ *is nondecreasing, then*

$$u(t) \leqslant f(t) + \int_{t_0}^{t} g(\tau)u(\tau)\Delta\tau, \ \ t \in \mathbb{T},$$

implies

$$u(t) \leqslant f(t)e_g(t,t_0), \ \ t \in \mathbb{T}.$$

LEMMA 2.3. ([11]) *Assume that* $p \geqslant q > 0, a \geqslant 0$, *then*

$$a^{\frac{q}{p}} \leqslant \frac{q}{p}k^{\frac{q-p}{p}}a + \frac{p-q}{p}k^{\frac{q}{p}}, \quad k > 0.$$

3. Main results

In this section, we study some integral inequalities on time scales. We always assume that p,q,r are constants and $p \geqslant q > 0$, $p \geqslant r > 0$.

THEOREM 3.1. *Assume that* $u(t,s)$, $a(t,s)$, $b(t,s)$, $f(t,s)$, $g(t,s)$, $h(t,s)$ *are nonnegative functions that are right-dense continuous for* $(t,s) \in \Omega$ *and* $a(t,s)$ *is nondecreasing. Then*

$$\begin{aligned} u^p(t,s) \leqslant a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}[f(x,y)u^q(x,y) \\ + g(x,y)u^r(x,y) + h(x,y)]\Delta y\Delta x, \ \ (t,s) \in \Omega, \end{aligned} \tag{1}$$

implies

$$u(t,s) \leqslant \left[a(t,s) + b(t,s)\int_{t_0}^{t} B(t)e_{A(\cdot,s)}(t,\sigma(x))\Delta x\right]^{\frac{1}{p}}, \ \ (t,s) \in \Omega, \tag{2}$$

where

$$\begin{aligned} A(t,s) &= \int_{s_0}^{s}\left(\frac{q}{p}k^{\frac{q-p}{p}}f(t,y) + \frac{r}{p}k^{\frac{r-p}{p}}g(t,y)\right)b(t,y)\Delta y, \\ B(t,s) &= \int_{s_0}^{s}\left[f(t,y)\left(\frac{q}{p}k^{\frac{q-p}{p}}a(t,y) + \frac{p-q}{p}k^{\frac{q}{p}}\right)\right. \\ &\left. +g(t,y)\left(\frac{r}{p}k^{\frac{r-p}{p}}a(t,y) + \frac{p-r}{p}k^{\frac{r}{p}}\right) + h(t,y)\right]\Delta y. \qquad (t,s) \in \Omega. \end{aligned} \tag{3}$$

Proof. Define a function $v(t,s)$ by

$$v(t,s) = \int_{t_0}^{t}\int_{s_0}^{s}[f(x,y)u^q(x,y) + g(x,y)u^r(x,y) + h(x,y)]\Delta y\Delta x. \tag{4}$$

Then (3.1) can be restated as

$$u^p(t,s) \leqslant a(t,s) + b(t,s)v(t,s). \tag{5}$$

Using Lemma 2.3, for any $k > 0$ we easily obtain

$$\begin{aligned} u^q(t,s) &\leqslant [a(t,s)+b(t,s)v(t,s)]^{\frac{q}{p}} \leqslant \frac{q}{p}k^{\frac{q-p}{p}}[a(t,s)+b(t,s)v(t,s)] + \frac{p-q}{p}k^{\frac{q}{p}},\\ u^r(t,s) &\leqslant [a(t,s)+b(t,s)v(t,s)]^{\frac{r}{p}} \leqslant \frac{r}{p}k^{\frac{r-p}{p}}[a(t,s)+b(t,s)v(t,s)] + \frac{p-r}{p}k^{\frac{r}{p}}. \end{aligned} \tag{6}$$

It follows from (3.4) and (3.6) that

$$\begin{aligned} v^{\Delta t}(t,s) \leqslant \int_{s_0}^{s} \Big\{ & f(t,y)\left[\frac{q}{p}k^{\frac{q-p}{p}}\big(a(t,y)+b(t,y)v(t,y)\big) + \frac{p-q}{p}k^{\frac{q}{p}}\right] \\ & + g(t,y)\left[\frac{r}{p}k^{\frac{r-p}{p}}\big(a(t,y)+b(t,y)v(t,y)\big) + \frac{p-r}{p}k^{\frac{r}{p}}\right] + h(t,y)\Big\}\Delta y \\ \leqslant\, & B(t,s) + A(t,s)v(t,s), \quad (t,s)\in\Omega, \end{aligned} \tag{7}$$

where $A(t,s)$ and $B(t,s)$ are defined by (3.3) respectively. Obviously, $A(t,s)$, $B(t,s)$ $\in \mathscr{C}_{rd}$, $A(t,s)$ and $B(t,s)$ are nonnegative, $A(t,s)$ is nondecreasing. Noting $v(t_0,s)=0$, using Lemma 2.1, from (3.7) we have

$$v(t,s) \leqslant \int_{t_0}^{t} e_{A(\cdot,s)}(t,\sigma(x))B(x,s)\Delta x. \tag{8}$$

Therefore, the desired inequality (3.2) follows from (3.5) and (3.8). □

REMARK 3.1. Theorem 3.1 extends some known inequalities on time scales. If $b(t,s)=1$, $g(t,s)=h(t,s)=0$, then Theorem 3.1 reduces to [10, Theorem 2]. If $q=1$, $b(t,s)=1$, $h(t,s)=0$, then Theorem 3.1 reduces to the form of [10, Theorem 3].

COROLLARY 3.1. *Let $\mathbb{T}=\mathbb{R}$, assume that $u(t,s)$, $a(t,s)$, $b(t,s)$, $f(t,s)$, $g(t,s)$, $h(t,s)$ are nonnegative functions defined for $t,s\in\mathbb{R}_+$, then the following inequality*

$$u^p(t,s) \leqslant a(t,s) + b(t,s)\int_0^t\int_0^s [f(x,y)u^q(x,y) + g(x,y)u^r(x,y) + h(x,y)]dydx, \tag{9}$$

for $t,s\in\mathbb{R}_+$, implies

$$u(t,s) \leqslant \left[a(t,s) + b(t,s)\overline{B}(t,s)exp\left(\int_0^t \overline{A}(x,s)dx\right)\right]^{\frac{1}{p}}, \quad t,s\in\mathbb{R}_+, \tag{10}$$

where

$$\overline{A}(t,s)=\int_0^s\left(\frac{q}{p}k^{\frac{q-p}{p}}f(t,y)+\frac{r}{p}k^{\frac{r-p}{p}}g(t,y)\right)b(t,y)dy,$$
$$\overline{B}(t,s)=\int_0^t\int_0^s\left[f(x,y)\left(\frac{q}{p}k^{\frac{q-p}{p}}a(x,y)+\frac{p-q}{p}k^{\frac{q}{p}}\right)\right.$$
$$\left.+g(x,y)\left(\frac{r}{p}k^{\frac{r-p}{p}}a(x,y)+\frac{p-r}{p}k^{\frac{r}{p}}\right)+h(x,y)\right]dydx,\ \ t,s\in\mathbb{R}_+. \tag{11}$$

COROLLARY 3.2. *Let $\mathbb{T}=\mathbb{Z}$ and assume that $u(t,s)$, $a(t,s)$, $b(t,s)$, $f(t,s)$, $g(t,s)$, $h(t,s)$ are nonnegative functions and $a(t,s)$ is nondecreasing function defined for $t,s\in\mathbb{N}_0$, then the following inequality*

$$u^p(t,s)\leqslant a(t,s)+b(t,s)\sum_{x=0}^{t-1}\sum_{y=0}^{s-1}[f(x,y)u^q(x,y)+g(x,y)u^r(x,y)+h(x,y)],\ \ t,s\in\mathbb{N}_0, \tag{12}$$

implies

$$u(t,s)\leqslant\left\{a(t,s)+b(t,s)\overline{\overline{B}}(t,s)\prod_{x=0}^{t-1}\left[1+\overline{\overline{A}}(x,s)\right]\right\}^{\frac{1}{p}},\ \ t,s\in\mathbb{N}_0, \tag{13}$$

where

$$\overline{\overline{A}}(t,s)=\sum_{y=0}^{s-1}\left(\frac{q}{p}k^{\frac{q-p}{p}}f(t,y)+\frac{r}{p}k^{\frac{r-p}{p}}g(t,y)\right)b(t,y),$$
$$\overline{\overline{B}}(t,s)=\sum_{x=0}^{t-1}\sum_{y=0}^{s-1}\left[f(x,y)\left(\frac{q}{p}k^{\frac{q-p}{p}}a(x,y)+\frac{p-q}{p}k^{\frac{q}{p}}\right)\right.$$
$$\left.+g(x,y)\left(\frac{r}{p}k^{\frac{r-p}{p}}a(x,y)+\frac{p-r}{p}k^{\frac{r}{p}}\right)+h(x,y)\right],\ \ t,s\in\mathbb{N}_0. \tag{14}$$

THEOREM 3.2. *Assume that $u(t,s)$, $a(t,s)$, $b(t,s)$, $f(t,s)$, $g(t,s)$, $h(t,s)$ are nonnegative functions that are right-dense continuous for $(t,s)\in\Omega$ and $a(t,s)$ is nondecreasing. Then*

$$u^p(t,s)\leqslant a(t,s)+\int_{t_0}^t b(x,s)u^p(x,s)\triangle x+\int_{t_0}^t\int_{s_0}^s[f(x,y)u^q(x,y)$$
$$+g(x,y)u^r(x,y)+h(x,y)]\Delta y\Delta x,\quad (t,s)\in\Omega, \tag{15}$$

implies

$$u(t,s)\leqslant R^{\frac{1}{p}}(t,s)\left[a(t,s)+\int_{t_0}^t e_{C(\cdot,s)}(t,\sigma(x))D(x,s)\Delta x\right]^{\frac{1}{p}},\quad (t,s)\in\Omega, \tag{16}$$

where

$$R(t,s)=e_{b(\cdot,s)}(t,t_0),\quad (t,s)\in\Omega, \tag{17}$$

$$C(t,s)=\int_{s_0}^{s}\left[\frac{q}{p}k^{\frac{q-p}{p}}f(t,y)R^{\frac{q}{p}}(t,y)+\frac{r}{p}k^{\frac{r-p}{p}}g(t,y)R^{\frac{r}{p}}(t,y)\right]\Delta y,$$
$$D(t,s)=\int_{s_0}^{s}\left[f(t,y)R^{\frac{q}{p}}(t,y)\left(\frac{q}{p}k^{\frac{q-p}{p}}a(t,y)+\frac{p-q}{p}k^{\frac{q}{p}}\right)\right. \tag{18}$$
$$\left.+g(t,y)R^{\frac{r}{p}}(t,y)\left(\frac{r}{p}k^{\frac{r-p}{p}}a(t,y)+\frac{p-r}{p}k^{\frac{r}{p}}\right)+h(t,y)\right]\Delta y,\quad (t,s)\in\Omega.$$

Proof. Define a function $v(t,s)$ by

$$v(t,s)=a(t,s)+\int_{t_0}^{t}\int_{s_0}^{s}[f(x,y)u^q(x,y)+g(x,y)u^r(x,y)+h(x,y)]\Delta y\Delta x. \tag{19}$$

Then (3.15) can be restated as

$$u^p(t,s)\leqslant v(t,s)+\int_{t_0}^{t}b(x,s)u^p(x,s)\triangle x. \tag{20}$$

Noting that $v(t,s)$ is nondecreasing, by Lemma 2.2, from (3.20), we obtain

$$u^p(t,s)\leqslant R(t,s)v(t,s), \tag{21}$$

where $R(t,s)$ defined as (3.17). From (3.19) and (3.21), we obtain

$$u(t)\leqslant R^{\frac{1}{p}}(t,s)[a(t)+w(t)]^{\frac{1}{p}}, \tag{22}$$

where

$$w(t,s)=\int_{t_0}^{t}\int_{s_0}^{s}[f(x,y)u^q(x,y)+g(x,y)u^r(x,y)+h(x,y)]\Delta y\Delta x. \tag{23}$$

It follows from (3.22) and Lemma 2.3, we get

$$\begin{aligned}u^q(t,s)&\leqslant R^{\frac{q}{p}}(t,s)[a(t,s)+w(t,s)]^{\frac{q}{p}}\\&\leqslant R^{\frac{q}{p}}(t,s)\left[\frac{q}{p}k^{\frac{q-p}{p}}(a(t,s)+w(t,s))+\frac{p-q}{p}k^{\frac{q}{p}}\right],\\u^r(t,s)&\leqslant R^{\frac{r}{p}}(t,s)[a(t,s)+w(t,s)]^{\frac{r}{p}}\\&\leqslant R^{\frac{r}{p}}(t,s)\left[\frac{r}{p}k^{\frac{r-p}{p}}(a(t,s)+w(t,s))+\frac{p-r}{p}k^{\frac{r}{p}}\right].\end{aligned} \tag{24}$$

It follows (3.23) and (3.24) that

$$\begin{aligned}w^{\Delta t}(t,s)\leqslant&\int_{s_0}^{s}\left\{f(t,y)R^{\frac{q}{p}}(t,y)\left[\frac{q}{p}k^{\frac{q-p}{p}}(a(t,y)+w(t,y))+\frac{p-q}{p}k^{\frac{q}{p}}\right]\right.\\&\left.+g(t,y)R^{\frac{r}{p}}(t,y)\left[\frac{r}{p}k^{\frac{r-p}{p}}(a(t,y)+w(t,y))+\frac{p-r}{p}k^{\frac{r}{p}}\right]+h(t,y)\right\}\Delta y\\\leqslant&\, D(t,s)+C(t,s)w(t,s),\quad (t,s)\in\Omega,\end{aligned} \tag{25}$$

where $C(t,s)$ and $D(t,s)$ are defined by (3.18) respectively. Obviously, $C(t,s)$, $D(t,s)$ $\in \mathscr{C}_{rd}$, $C(t,s)$ and $D(t,s)$ are nonnegative, $C(t,s)$ is nondecreasing. Noting $w(t_0,s) = 0$, using Lemma 2.1, from (3.25) we have

$$w(t,s) \leqslant \int_{t_0}^{t} e_{C(\cdot,s)}(t,\sigma(x))D(x,s)\Delta x. \tag{26}$$

Therefore, the desired inequality (3.16) follows from (3.22) and (3.26). □

COROLLARY 3.3. *Let $\mathbb{T} = \mathbb{R}$, assume that $u(t,s)$, $a(t,s)$, $b(t,s)$, $f(t,s)$, $g(t,s)$, $h(t,s)$ are nonnegative functions defined for $t,s \in \mathbb{R}_+$, then the following inequality*

$$\begin{aligned} u^p(t,s) \leqslant a(t,s) &+ \int_0^t b(x,s)dx \\ &+ \int_0^t \int_0^s [f(x,y)u^q(x,y) + g(x,y)u^r(x,y) + h(x,y)]dydx, \end{aligned} \tag{27}$$

for $t,s \in \mathbb{R}_+$, implies

$$u(t,s) \leqslant H^{\frac{1}{p}}(t,s)\left[a(t,s) + b(t,s)\overline{D}(t,s)exp\left(\int_0^t \overline{C}(x,s)dx\right)\right]^{\frac{1}{p}}, \quad t,s \in \mathbb{R}_+, \tag{28}$$

where

$$H(t,s) = \exp\int_0^t b(x,s)dx, \quad t,s \in \mathbb{R}_+, \tag{29}$$

$$\begin{aligned} \overline{C}(t,s) = \int_0^s \Bigg(& f(t,y)H^{\frac{q}{p}}(t,y)\Big(\frac{q}{p}k^{\frac{q-p}{p}}a(t,y) + \frac{p-q}{p}k^{\frac{q}{p}}\Big) \\ & + g(t,y)H^{\frac{r}{p}}(t,y)\Big(\frac{r}{p}k^{\frac{r-p}{p}}a(t,y) + \frac{p-r}{p}k^{\frac{r}{p}}\Big) + h(t,y)\Bigg), \\ \overline{D}(t,s) = \int_0^t\int_0^s \Bigg(& f(x,y)H^{\frac{q}{p}}(x,y)\Big(\frac{q}{p}k^{\frac{q-p}{p}}a(x,y) + \frac{p-q}{p}k^{\frac{q}{p}}\Big) \\ & + g(x,y)H^{\frac{r}{p}}(x,y)\Big(\frac{r}{p}k^{\frac{r-p}{p}}a(x,y) + \frac{p-r}{p}k^{\frac{r}{p}}\Big) + h(x,y)\Bigg)dydx, \quad t,s \in \mathbb{R}_+. \end{aligned} \tag{30}$$

COROLLARY 3.4. *Let $\mathbb{T} = \mathbb{Z}$ and assume that $u(t,s)$, $a(t,s)$, $b(t,s)$, $f(t,s)$, $g(t,s)$, $h(t,s)$ are nonnegative functions and $a(t,s)$ is nondecreasing function defined for $t,s \in \mathbb{N}_0$, then the following inequality*

$$\begin{aligned} u^p(t,s) \leqslant & a(t,s) + \sum_{x=0}^{t-1} b(x,s)u^p(x,s) + \sum_{x=0}^{t-1}\sum_{y=0}^{s-1}[f(x,y)u^q(x,y) \\ & + g(x,y)u^r(x,y) + h(x,y)], \quad t,s \in \mathbb{N}_0, \end{aligned} \tag{31}$$

implies

$$u(t,s) \leqslant \overline{R}^{\frac{1}{p}}(t,s)\left\{a(t,s) + \overline{\overline{D}}(t,s)\prod_{x=0}^{t-1}\left[1 + \overline{\overline{C}}(x,s)\right]\right\}^{\frac{1}{p}}, \quad t,s \in \mathbb{N}_0, \tag{32}$$

where

$$\overline{R} = \prod_{x=0}^{t-1}[1+b(x,s)], \quad t,s \in \mathbb{N}_0, \tag{33}$$

$$\overline{\overline{C}}(t,s) = \sum_{y=0}^{s-1}\left[\frac{q}{p}k^{\frac{q-p}{p}}\overline{R}^{\frac{q}{p}}(t,y)f(t,y) + \frac{r}{p}k^{\frac{r-p}{p}}\overline{R}^{\frac{r}{p}}(t,y)g(t,y)\right],$$

$$\begin{aligned}\overline{\overline{D}}(t,s) = \sum_{x=0}^{t-1}\sum_{y=0}^{s-1}\Bigg[&\overline{R}^{\frac{q}{p}}(x,y)f(x,y)\left(\frac{q}{p}k^{\frac{q-p}{p}}a(x,y) + \frac{p-q}{p}k^{\frac{q}{p}}\right) \\ &+\overline{R}^{\frac{r}{p}}(x,y)g(x,y)\left(\frac{r}{p}k^{\frac{r-p}{p}}a(x,y) + \frac{p-r}{p}k^{\frac{r}{p}}\right) + h(x,y)\Bigg], \quad t,s \in \mathbb{N}_0.\end{aligned} \tag{34}$$

THEOREM 3.3. *Assume that* $u(t,s)$, $a(t,s)$, $b(t,s)$, $f(t,s)$, $g(t,s)$, $h(t,s)$ *are nonnegative functions that are right-dense continuous for* $(t,s) \in \Omega$ *and* $a(t,s)$ *is non-decreasing. If* $L : \mathbb{T}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous function such that*

$$0 \leqslant L(t,s,u) - L(t,s,v) \leqslant M(t,s,v)(u-v) \tag{35}$$

for $(t,s) \in \Omega$ *and* $u \geqslant v \geqslant 0$, *where* $M : \mathbb{T}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ *is a nonnegative continuous function. Then*

$$\begin{aligned}u^p(t,s) \leqslant a(t,s) + \int_{t_0}^{t} b(x,s)u^p(x,s)\triangle x + \int_{t_0}^{t}\int_{s_0}^{s} L\big(&x,y,f(x,y)u^q(x,y) \\ &+ g(x,y)u^r(x,y) + h(x,y)\big)\Delta y\Delta x, \quad (t,s) \in \Omega,\end{aligned} \tag{36}$$

implies

$$u(t,s) \leqslant R^{\frac{1}{p}}(t,s)\left[a(t,s) + \int_{t_0}^{t} e_{E(\cdot,s)}(t,\sigma(x))F(x,s)\Delta x\right]^{\frac{1}{p}}, \quad (t,s) \in \Omega, \tag{37}$$

where $R(t,s)$ *defined as* (3.17) *and*

$$\begin{aligned}E(t,s) = \int_{s_0}^{s} M\Bigg(&t,y,f(t,y)R^{\frac{q}{p}}(t,y)\Big(\frac{q}{p}k^{\frac{q-p}{p}}a(t,y) + \frac{p-q}{p}k^{\frac{q}{p}}\Big) \\ &+g(t,y)R^{\frac{r}{p}}(t,y)\Big(\frac{r}{p}k^{\frac{r-p}{p}}a(t,y) + \frac{p-r}{p}k^{\frac{r}{p}}\Big) + h(t,y)\Bigg) \\ &\times\left[\frac{q}{p}k^{\frac{q-p}{p}}f(t,y)R^{\frac{q}{p}}(t,y) + \frac{r}{p}k^{\frac{r-p}{p}}g(t,y)R^{\frac{r}{p}}(t,y)\right]\Delta y, \\ F(t,s) = \int_{s_0}^{s} L\Bigg(&t,y,f(t,y)R^{\frac{q}{p}}(t,y)\Big(\frac{q}{p}k^{\frac{q-p}{p}}a(t,y) + \frac{p-q}{p}k^{\frac{q}{p}}\Big) \\ &+g(t,y)R^{\frac{r}{p}}(t,y)\Big(\frac{r}{p}k^{\frac{r-p}{p}}a(t,y) + \frac{p-r}{p}k^{\frac{r}{p}}\Big) + h(t,y)\Bigg)\Delta y, \quad (t,s) \in \Omega.\end{aligned} \tag{38}$$

Proof. Define a function $v(t,s)$ by

$$v(t,s) = a(t,s) + w(t,s), \tag{39}$$

where

$$w(t,s)=\int_{t_0}^{t}\int_{s_0}^{s}L\big(x,y,f(x,y)u^q(x,y)+g(x,y)u^r(x,y)+h(x,y)\big)\Delta y\Delta x. \tag{40}$$

Then (3.36) can be restated as (3.20). Similarly we have (3.21), (3.22) and (3.24). Noting the hypotheses on L, from (3.24) and (3.40) that

$$\begin{aligned}
w^{\Delta t}(t,s)\leqslant \int_{s_0}^{s}\Big\{&L\Big(t,y,f(t,y)R^{\frac{q}{p}}(t,y)\big(\frac{q}{p}k^{\frac{q-p}{p}}(a(t,y)+w(t,y))+\frac{p-q}{p}k^{\frac{q}{p}}\big)\\
&+g(t,y)R^{\frac{r}{p}}(t,y)\big(\frac{r}{p}k^{\frac{r-p}{p}}(a(t,y)+w(t,y))+\frac{p-r}{p}k^{\frac{r}{p}}\big)+h(t,y)\Big)\\
&-L\Big(t,y,f(t,y)R^{\frac{q}{p}}(t,y)\big(\frac{q}{p}k^{\frac{q-p}{p}}a(t,y)+\frac{p-q}{p}k^{\frac{q}{p}}\big)\\
&+g(t,y)R^{\frac{r}{p}}(t,y)\big(\frac{r}{p}k^{\frac{r-p}{p}}a(t,y)+\frac{p-r}{p}k^{\frac{r}{p}}\big)+h(t,y)\Big)\\
&+L\Big(t,y,f(t,y)R^{\frac{q}{p}}(t,y)\big(\frac{q}{p}k^{\frac{q-p}{p}}a(t,y)+\frac{p-q}{p}k^{\frac{q}{p}}\big)\\
&+g(t,y)R^{\frac{r}{p}}(t,y)\big(\frac{r}{p}k^{\frac{r-p}{p}}a(t,y)+\frac{p-r}{p}k^{\frac{r}{p}}\big)+h(t,y)\Big)\Big\}\Delta y\\
\leqslant F(t,s)&+E(t,s)w(t,s),\quad (t,s)\in\Omega,
\end{aligned} \tag{41}$$

where $E(t,s)$ and $F(t,s)$ are defined by (3.28) respectively. Obviously, $G(t,s)$, $H(t,s)$ $\in\mathscr{C}_{rd}$, $G(t,s)$ and $H(t,s)$ are nonnegative, $G(t,s)$ is nondecreasing. Noting $w(t_0,s)=0$, using Lemma 2.2, from (3.41) we have

$$w(t,s)\leqslant\int_{t_0}^{t}e_{E(\cdot,s)}(t,\sigma(x))F(x,s)\Delta x,\quad (t,s)\in\Omega. \tag{42}$$

Therefore, the desired inequality (3.37) follows from (3.22) and (3.42). □

COROLLARY 3.5. *Let $\mathbb{T}=\mathbb{R}$, assume that $u(t,s)$, $a(t,s)$, $b(t,s)$, $f(t,s)$, $g(t,s)$, $h(t,s)$ are nonnegative functions defined for $t,s\in\mathbb{R}_+$. $L,M\in\mathscr{C}(\mathbb{R}_+^2,\mathbb{R}_+)$ satisfied the inequality as* (3.35)*, then the inequality*

$$\begin{aligned}
u^p(t,s)\leqslant a(t)+\int_0^t b(x,s)u^p(x,s)dx+\int_0^t\int_0^s L\big(x,y,f(x,y)u^q(x,y)\\
+g(x,y)u^r(x,y)+h(x,y)\big)dydx,\quad t,s\in\mathbb{R}_+,
\end{aligned} \tag{43}$$

implies

$$u(t,s)\leqslant H^{\frac{1}{p}}(t,s)\left[a(t,s)+\overline{F}(t,s)\exp\left(\int_0^t\overline{E}(x,s)dx\right)\right]^{\frac{1}{p}},t,s\in\mathbb{R}_+, \tag{44}$$

where $H(t,s)$ *defined as* (3.29) *and*

$$\begin{aligned}\overline{E}(t,s) = \int_0^s M\Big(& t,y,f(t,y)H^{\frac{q}{p}}(t,y)\big(\frac{q}{p}k^{\frac{q-p}{p}}a(t,y)+\frac{p-q}{p}k^{\frac{q}{p}}\big)\\ &+g(t,y)H^{\frac{r}{p}}(t,y)\big(\frac{r}{p}k^{\frac{r-p}{p}}a(t,y)+\frac{p-r}{p}k^{\frac{r}{p}}\big)+h(t,y)\Big)\\ &\times\Big[\frac{q}{p}k^{\frac{q-p}{p}}f(t,y)H^{\frac{q}{p}}(t,y)+\frac{r}{p}k^{\frac{r-p}{p}}g(t,y)H^{\frac{r}{p}}(t,y)\Big]dy,\\ \overline{F}(t,s) = \int_0^t\int_0^s L\Big(& x,y,f(x,y)H^{\frac{q}{p}}(x,y)\big(\frac{q}{p}k^{\frac{q-p}{p}}a(x,y)+\frac{p-q}{p}k^{\frac{q}{p}}\big)\\ &+g(x,y)H^{\frac{r}{p}}(x,y)\big(\frac{r}{p}k^{\frac{r-p}{p}}a(x,y)+\frac{p-r}{p}k^{\frac{r}{p}}\big)+h(x,y)\Big)dydx,\quad t,s\in\mathbb{R}_+.\end{aligned} \tag{45}$$

COROLLARY 3.6. *Let* $\mathbb{T}=\mathbb{Z}$ *and assume that* $u(t,s)$, $a(t,s)$, $b(t,s)$, $f(t,s)$, $g(t,s)$, $h(t,s)$ *are nonnegative functions and* $a(t,s)$ *is nondecreasing function defined for* $t,s\in\mathbb{N}_0$, $L:\mathbb{N}_0\times\mathbb{R}_+\to\mathbb{R}_+$ *is a function that satisfies the condition as* (3.35), *then the following inequality*

$$\begin{aligned}u^p(t,s) \leqslant a(t,s)+\sum_{x=0}^{t-1}b(x,s)u^p(x,s)+\sum_{x=0}^{t-1}\sum_{y=0}^{s-1}L\big(&x,y,f(x,y)u^q(x,y)\\ &+g(x,y)u^r(x,y)+h(x,y)\big),\quad t,s\in\mathbb{N}_0,\end{aligned} \tag{46}$$

implies

$$u(t,s) \leqslant \overline{R}^{\frac{1}{p}}(t,s)\left\{a(t,s)+\overline{\overline{F}}(t,s)\prod_{x=0}^{t-1}\Big[1+\overline{\overline{E}}(x,s)\Big]\right\}^{\frac{1}{p}},\quad t,s\in\mathbb{N}_0, \tag{47}$$

where $\overline{R}(t,s)$ *defined as* (3.29) *and*

$$\begin{aligned}\overline{\overline{E}}(t,s) = \sum_{y=0}^{s-1} M\Big(& t,y,\overline{R}^{\frac{q}{p}}(t,y)f(t,y)\big(\frac{q}{p}k^{\frac{q-p}{p}}a(t,y)+\frac{p-q}{p}k^{\frac{q}{p}}\big)\\ &+\overline{R}^{\frac{r}{p}}(t,y)g(t,y)\big(\frac{r}{p}k^{\frac{r-p}{p}}a(t,y)+\frac{p-r}{p}k^{\frac{r}{p}}\big)+h(t,y)\Big)\\ &\times\Big[\frac{q}{p}k^{\frac{q-p}{p}}\overline{R}^{\frac{q}{p}}(t,y)f(t,y)+\frac{r}{p}k^{\frac{r-p}{p}}\overline{R}^{\frac{r}{p}}(t,y)g(t,y)\Big],\\ \overline{\overline{F}}(t,s) = \sum_{x=0}^{t-1}\sum_{y=0}^{s-1} L\Big(& x,y,\overline{R}^{\frac{q}{p}}(x,y)f(x,y)\big(\frac{q}{p}k^{\frac{q-p}{p}}a(x,y)+\frac{p-q}{p}k^{\frac{q}{p}}\big)\\ &+\overline{R}^{\frac{r}{p}}(x,y)g(x,y)\big(\frac{r}{p}k^{\frac{r-p}{p}}a(x,y)+\frac{p-r}{p}k^{\frac{r}{p}}\big)+h(x,y)\Big),\quad t,s\in\mathbb{N}_0.\end{aligned} \tag{48}$$

4. Some applications

In this section, we present two applications of our main result. Consider the following partial dynamic equation on time scales

$$(u^p(t))^{\Delta t\Delta s} = F(t,s,u(t,s)), \quad (t,s) \in \Omega \tag{1}$$

with boundary conditions

$$u^p(t_0,s) = \alpha(t), \quad u^p(t,s_0) = \beta(s), \quad u^p(t_0,s_0) = \gamma, \tag{2}$$

where $F : \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{R} \to \mathbb{R}$ is right-dense continuous on Ω and continuous on $\mathbb{R}$, $\alpha : \mathbb{T}_1 \to \mathbb{R}$ and $\beta : \mathbb{T}_2 \to \mathbb{R}$ are right-dense continuous, and $\gamma \in \mathbb{R}$ is a constant.

EXAMPLE 4.1. Assume that

$$\begin{aligned} &F(t,s,u(t,s)) \leqslant f(t,s)u^q(t,s) + g(t,s)u^r(t,s), \\ &|\, \alpha(t)+\beta(s)-\gamma \,| \leqslant K, \quad (t,s) \in \Omega. \end{aligned} \tag{3}$$

where $f(t,s), g(t,s)$ are nonnegative right-dense continuous function for $(t,s) \in \Omega$. $p \geqslant q > 0$, $p \geqslant r > 0$, $K > 0$ are constants.

If every solution $u(t)$ of (4.1) satisfying the boundary condition (4.2), implies

$$|\, u(t,s) \,| \leqslant \left[|\, K \,| + \int_{t_0}^{t} B(x,s) e_{A(\cdot,s)}(t,\sigma(x)) \right]^{\frac{1}{p}}, \quad (t,s) \in \Omega. \tag{4}$$

where $A(t)$, $B(t)$ are defined as in (3.3) and (3.4) with $a(t,s) = K$, $b(t,s) = 1$, $h(t,s) = 0$.

Indeed, the solution $u(t)$ of (4.1) satisfies the following equivalent equation

$$u^p(t,s) = \alpha(t) + \beta(s) - \gamma + \int_{t_0}^{t}\int_{s_0}^{s} F(x,y,u(x,y))\Delta y \Delta x, \quad (t,s) \in \Omega. \tag{5}$$

It follows from (4.3) and (4.5) that

$$\begin{aligned} |\, u(t) \,|^p &= |\, \alpha(t)+\beta(s)-\gamma \,| + \int_{t_0}^{t}\int_{s_0}^{s} |\, F(x,y,u(x,y)) \,| \Delta y \Delta x \\ &\leqslant K + \int_{t_0}^{t}\int_{s_0}^{s} [f(x,y) \,|\, u(x,y) \,|^q + g(x,y) \,|\, u(x,y) \,|^r] \Delta y \Delta x, \quad (t,s) \in \Omega. \end{aligned} \tag{6}$$

Using Theorem 3.1, the inequality (4.4) is obtained from (4.6).

EXAMPLE 4.2. Assume that

$$| F(t,s,u_1) - F(t,s,u_2) | \leqslant f(t,s) \mid u_1^p(t,s) - u_2^p(t,s) \mid, \quad (t,s) \in \Omega. \tag{7}$$

where $f(t,s)$ is defined as in Example 4.1. Then the problems (4.1) and (4.2) has at most one solution on Ω.

Indeed, let $u_1(t,s)$ and $u_2(t,s)$ be two solution of the problems (4.1) with (4.2). It follows from (4.5) that

$$| u_1^p(t,s) - u_2^p(t,s) | \leqslant \int_{t_0}^{t} \int_{s_0}^{s} f(x,y) \mid u_1^p(x,y) - u_2^p(x,y) \mid \Delta y \Delta x, \quad (t,s) \in \Omega. \tag{8}$$

By Theorem 3.1, from (4.8) we have $| u_1^p(t,s) - u_2^p(t,s) | \equiv 0$, which implies that the problems (4.1) with (4.2) has at most one solution on Ω. This completes the proof of example 4.2.

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