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FOURIER TRANSFORM AND L_p–MIXED CENTROID BODIES

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Abstract. In this paper we introduce the concept of L_p-mixed centroid body of a convex body and consider a Busemann-Petty type problem whether $\Gamma_{-p,i}K \subseteq \Gamma_{-p,i}L$ implies $W_i(K) \leqslant W_i(L)$.

1. Introduction

The nature of the duality between the Brunn-Minkowski theory and the dual Brunn-Minkowski theory is subtle. Lutwak, Yang and Zhang [13] showed that there exists a new ellipsoid (John ellipsoid) $\Gamma_{-2}K$ associated with convex body K in the dual Brunn-Minkowski theory, which is the dual analog of the classical Legendre ellipsoid $\Gamma_2 K$ in the Brunn-Minkowski theory. More generally, they [14] introduced the concept of L_p-John ellipsoids. If K is a convex body which contains the origin in its interior and real $p > 0$, the L_p-John ellipsoid $\Gamma_{-p}K$ is defined by [14]

$$\rho(\Gamma_{-p}K,u)^{-p} = \frac{1}{\mathrm{vol}_n(K)}\int_{S^{n-1}} |u\cdot v|^p dS_p(K,v), \quad u \in S^{n-1}, \tag{1.1}$$

where $S_p(K,\cdot)$ is the L_p-surface area measure. If $p \geqslant 1$, the body $\Gamma_{-p}K$ is a convex body.

The main object of this article is the i-th L_p-mixed centroid body $\Gamma_{-p,i}K$. Let $\Gamma_{-p,i}K, i = 0,1,\cdots,n-1, p > 0$, denote the star body whose radial function is given by

$$\rho(\Gamma_{-p,i}K,\theta)^{-p} = \frac{1}{W_i(K)}\int_{S^{n-1}} |\theta\cdot u|^p dS_{p,i}(K,u), \forall\theta \in S^{n-1}. \tag{1.2}$$

Here $S_{p,i}(K,\cdot)$ is the i-th L_p-mixed surface area measure with $n-i-1$ copies of K and i copies of B (the unit ball). More precisely, the Borel measure $S_{p,i}(K,\cdot)$, on S^{n-1}, is defined by [12]

$$S_{p,i}(K,\omega) = \int_\omega h_K^{1-p}(u)dS_i(K,u), \tag{1.3}$$

for each Borel $\omega \subset S^{n-1}$.

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If $i=0$, $S_{p,i}(K,\cdot)$ is just $S_p(K,\cdot)$. Obviously, $\Gamma_{-p,0}K=\Gamma_{-p}K$. In this article, we consider the following Busemann-Petty type problem for L_p-mixed centroid bodies. Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$ and $i=0,1,\cdots,n-1,p\geqslant 1$. Suppose

$$\Gamma_{-p,i}K\subseteq\Gamma_{-p,i}L.$$

Does it follow that

$$W_i(K)\leqslant W_i(L)?$$

By using the Fourier analytic formula for the L_p-mixed centroid body, we will obtain the following results.

THEOREM 1. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, $i=0,1,\cdots,$ $n-1$ and $p\geqslant 1, p\neq n-i, p$ is not an even integer. Suppose that the support function h_K is infinitely smooth and the functions $C_p\widehat{h_K^p}(\theta)\geqslant 0$ for all $\theta\in S^{n-1}$. If*

$$\Gamma_{-p,i}K\subseteq\Gamma_{-p,i}L,$$

then

$$W_i(K)\leqslant W_i(L).$$

THEOREM 2. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$, and $i=0,1,\cdots,$ $n-1$, $p\geqslant 1, p\neq n-i, p$ is not an even integer. If the mixed curvature function $f_i(K,\cdot)$ is positive on S^{n-1} and $C_p\widehat{h_K^p}(\theta)$ is negative on an open subset of S^{n-1}, then there exists an origin-symmetric convex body D so that*

$$\Gamma_{-p,i}D\subseteq\Gamma_{-p,i}K,$$

but

$$W_i(D)>W_i(K).$$

2. Notation and preliminaries

2.1. L_p-mixed quermassintegrals and L_p-mixed curvature functions

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, we write $\mathscr{K}_0^n$. A compact convex set K is uniquely determined by its support function $h(K,\cdot)$ on the Euclidean unit sphere S^{n-1}, defined by

$$h(K,u)=h_K(u)=\max\{u\cdot x:\ x\in K\}. \tag{2.1}$$

The radial function $\rho(L,\cdot)$ of a compact, star-shaped L (about the origin) is defined by

$$\rho(L,u)=max\{\lambda\geqslant 0:\lambda u\in L\},\quad u\in S^{n-1}. \tag{2.2}$$

We call L a star body if $\rho(L,\cdot)$ is continuous on S^{n-1} and L contains the origin in its interior.

For $K,L \in \mathscr{K}^n$, and $\varepsilon > 0$, the Minkowski linear combination $K+L \in \mathscr{K}^n$ is defined by

$$K+L = \{x+y|\ x\in K,\ y\in L\}. \tag{2.3}$$

It is easy to check that

$$h(K+\varepsilon L,\cdot) = h(K,\cdot)+\varepsilon h(L,\cdot). \tag{2.4}$$

For $K,L \in \mathscr{K}_0^n, p \geqslant 1$, and $\varepsilon > 0$, the Firey L_p-combination $K +_p \varepsilon \cdot L \in \mathscr{K}_0^n$ is defined by (see [1,12])

$$h(K +_p \varepsilon \cdot L,\cdot)^p = h(K,\cdot)^p + \varepsilon h(L,\cdot)^p, \tag{2.5}$$

where "$\cdot$" in $\varepsilon \cdot L$ denotes the Firey scalar multiplication, i.e., $\varepsilon \cdot L = \varepsilon^{\frac{1}{p}} L$.

Let $W_i(K,L)$ denote the mixed volume $V(K,\ldots,K,L,B)$ with $n-i-1$ copies of K, i copies of the unit ball B, and one L $(i = 0,1,\cdots,n-1)$. In particular, $W_i(K,K)$ is just the quermassintegral $W_i(K)$.

For $K \in \mathscr{K}^n$ and $i = 0,1,\cdots,n-1$, let $S_i(K,\cdot)$ denote the mixed surface area measure $S(K,\ldots,K,B,\ldots,B\cdot)$ with $n-i-1$ copies of K, i copies of B (see [11]). The mixed quermassintegral $W_i(K,L)$ has the following integral representation:

$$W_i(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)dS_i(K,u) \tag{2.6}$$

for all $L \in \mathscr{K}^n$.

Suppose that $\mathbb{R}$ is the set of real numbers. A convex body $K \in \mathscr{K}^n$ is said to have a continuous i-th curvature function $f_i(K,\cdot): S^{n-1} \to \mathbb{R}$, if its mixed surface area measure $S_i(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, and has the Radon-Nikodym derivative

$$\frac{dS_i(K,\cdot)}{dS} = f_i(K,\cdot). \tag{2.7}$$

For $K,L \in \mathscr{K}_0^n$ and $p \geqslant 1, i = 0,1,\cdots,n-1$, the i-th L_p-mixed quermassintegral $W_{p,i}(K,L)$ with $n-i-1$ copies of K, i copies of B is defined by [12]

$$\frac{n-i}{p}W_{p,i}(K,L) = \lim_{\varepsilon\to 0^+}\frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}.$$

Moreover, Lutwak [12] proved there exists a regular Borel measure $S_{p,i}(K,\cdot)$, such that the L_p-mixed quermassintegral $W_{p,i}(K,L)$ has the following integral representation:

$$W_{p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p dS_{p,i}(K,u) \tag{2.8}$$

for all $L \in \mathscr{K}_0^n$. And the measure $S_{p,i}(K,\cdot)$ is absolutely continuous with respect to $S_i(K,\cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K,\cdot)}{dS_i(K,\cdot)} = h(K,\cdot)^{1-p}. \tag{2.9}$$

For $K,L \in \mathscr{K}_0^n$ and $p \geqslant 1, i = 0,1,\cdots,n-1$, the L_p-mixed curvature function $f_{p,i}(K,\cdot)$ is defined by

$$f_{p,i}(K,\cdot) = \frac{dS_{p,i}(K,\cdot)}{dS}. \tag{2.10}$$

If the mixed surface area measure $S_i(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure S, we have

$$f_{p,i}(K,u) = f_i(K,u)h(K,u)^{1-p}. \tag{2.11}$$

2.2. Fourier transform and Parseval's formula

Koldobsky's book [7] is an excellent general reference for the Fourier transform. Some basic notions and the background material are required. As usual, we denote by $S(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable functions (test functions) on $\mathbb{R}^n$, and by $S'(\mathbb{R}^n)$ the space of distributions over $S(\mathbb{R}^n)$. Every locally integrable real valued function f on $\mathbb{R}^n$ with power growth at infinity represents a distribution acting by integration: for any $\phi \in S(\mathbb{R}^n), \langle f,\phi\rangle = \int_{\mathbb{R}^n} f(x)\phi(x)dx$.

The Fourier transform $\hat{f}$ of a distribution $f \in S'(\mathbb{R}^n)$ is defined by $\langle \hat{f},\hat{\phi}\rangle = (2\pi)^n\langle f,\phi\rangle$ for every test function ϕ, where

$$\hat{\phi}(y) = \int \phi(x)\exp(-i\langle x,y\rangle)dx. \tag{2.12}$$

A distribution f is called even homogeneous of degree $p \in \mathbb{R}$ if $\langle f,\phi(\cdot/\alpha)\rangle = |\alpha|^{n+p}\langle f,\phi\rangle$ for every $\alpha \in \mathbb{R}, \alpha \neq 0$. The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$. A distribution f is called positive if $\langle f,\phi\rangle \geqslant 0$ for every $\phi \geqslant 0$, implying that f is necessarily a non-negative Borel measure on $\mathbb{R}^n$. We use Schwartz's generalization of Bochner's theorem (see [3]) as a definition, and call a homogeneous distribution positive definite if its Fourier transform is a positive distribution.

Let μ be a finite Borel measure on the unit sphere S^{n-1}. We extend μ to a homogeneous distribution of degree $-n-p$. A distribution $\mu_{p,e}$ is called the L_p extended measure of μ if, for every even test function $\phi \in S(\mathbb{R}^n)$,

$$\langle \mu_{p,e},\phi\rangle = \int_{S^{n-1}} \langle r_+^{-1-p},\phi(r\xi)\rangle d\mu(\xi). \tag{2.13}$$

In most cases we are only interested in even test functions supported outside of the origin, for which

$$\langle r_+^{-1-p},\phi(r\xi)\rangle = \int_{\mathbb{R}} r_+^{-1-p}\phi(r\xi)dr = \frac{1}{2}\int_{\mathbb{R}} |r|^{-1-p}\phi(r\xi)dr, \tag{2.14}$$

(see [2]) for the general definition of $\langle r_+^{-1-p},\phi(r\xi)\rangle$.

If μ is absolutely continuous with density $g \in L_1(S^{n-1})$, we define the extension $g(x), x \in \mathbb{R}^n\backslash\{0\}$ as a homogeneous function of degree $-n-p: g(x) = |x|^{-n-p}g(x/|x|)$, and identify $\widehat{\mu_{p,e}}$ with $\hat{g}$.

Since Koldobsky found the Fourier analytic characterization of intersection bodies, the Fourier analytic approach to Busemann-Petty type problems has recently been developed and has led to many results (see [4–9, 15–18]).

3. Main results

In order to prove our main results, the following results are required.

LEMMA 3.1. [7] *Let* $p > -1$, $p \neq 2k$, $k \in N \cup \{0\}$. *For every* $\theta \in S^{n-1}$,

$$\widehat{\mu_{p,e}}(\theta) = \frac{1}{4\pi C_p} \int_{S^{n-1}} |\theta \cdot y|^p d\mu(y), \tag{3.1}$$

where the constant

$$C_p = \frac{2^{p+1}\sqrt{\pi}\Gamma((p+1)/2)}{\Gamma(-p/2)}$$

is positive for each $p \in (4k-2, 4k)$ *and negative for each* $p \in (4k, 4k+2)$.

LEMMA 3.2. [12] *If* $K, L \in \mathscr{K}_0^n$, $i = 0, 1, \cdots, n-1$ *and* $p > 1$, *then*

$$W_{p,i}(K,L)^{n-i} \geqslant W_i(K)^{n-i-p} W_i(L)^p, \tag{3.2}$$

with equality if and only if K *and* L *are dilates.*

The following statement follows from (1.2) and Lemma 3.1.

LEMMA 3.3. *Let* $p \geqslant 1$, p *is not an even integer and* $i = 0, 1, \cdots, n-1$. *Then for every* $\theta \in S^{n-1}$,

$$\widehat{S_{p,i}(K,\cdot)}(\theta) = \frac{W_i(K)}{4\pi C_p} \rho(\Gamma_{-p,i}K, \theta)^{-p}, \tag{3.3}$$

where C_p *is as above.*

In particular, if $S_{p,i}(K,)$ is absolutely continuous with respect to the spherical Lebesgue measure, then

$$\frac{\widehat{f_{p,i}(K,\cdot)}(\theta)}{W_i(K)} = \frac{1}{4\pi C_p} \rho(\Gamma_{-p,i}K, \theta)^{-p}. \tag{3.4}$$

Taking $i = 0$ to Lemma 3.3, we immediately obtain that

COROLLARY 3.1. *Let* $p \geqslant 1$, p *is not an even integral. If* $S_p(K,)$ *is absolutely continuous with respect to the spherical Lebesgue measure, then for every* $\theta \in S^{n-1}$,

$$\frac{\widehat{f_p(K,\cdot)}(\theta)}{\mathrm{vol}_n(K)} = \frac{1}{4\pi C_p} \rho(\Gamma_{-p}K, \theta)^{-p}. \tag{3.5}$$

THEOREM 3.1. *Let* K *and* L *be origin-symmetric convex bodies in* $\mathbb{R}^n$, $i = 0, 1, \cdots, n-1$ *and* $p \geqslant 1$, $p \neq n-i, p$ *is not an even integer. If*

$$\Gamma_{-p,i}K = \Gamma_{-p,i}L,$$

then

$$K = L.$$

Proof. Applying (1.2) and the uniqueness theorem of the Fourier transform, we have $S_{p,i,e}(K,\cdot) = S_{p,i,e}(L,\cdot)$. By homogeneity, $S_{p,i}(K,\cdot) = S_{p,i}(L,\cdot)$ is the same as $S_{p,i,e}(K,\cdot) = S_{p,i,e}(L,\cdot)$. It remains to use the uniqueness property of L_p-mixed surface area measures for $p \neq n-i$ (see [12]). □

REMARK. In the case $p = n-i$ and p is not an even integer, it follows that $\Gamma_{-(n-i),i}K = \Gamma_{-(n-i),i}L$ implies K and L are dilates. Theorem 3.1 is not true for even values of p. Indeed, one can perturb $S_{p,i}(K,\cdot)$ (i.e. to perturb a body K) without changing $\rho(\Gamma_{-p,i}K,\cdot)$ (see the following theorem).

THEOREM 3.2. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$, $i = 0,1,\cdots,n-1$ and $p \geqslant 1$, $p \neq n-i$. If p is an even integer, then there exists an origin-symmetric convex body L, such that*

$$\Gamma_{-p,i}K = \Gamma_{-p,i}L,$$

but

$$W_i(K) \neq W_i(L).$$

Proof. Then there exists a nonzero continuous even function g on S^{n-1} such that

$$\int_{S^{n-1}} |x\cdot\xi|^p g(x)dx = 0, \qquad \xi \in S^{n-1}. \tag{3.6}$$

Indeed, if $p = 2k$, then $|x\cdot\xi|^{2k}$ is a polynomial of degree $2k$ with coefficients depending on ξ. So, it is enough to construct a nontrivial even function g, satisfying

$$\int_{S^{n-1}} x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} g(x)dx = 0, \tag{3.7}$$

for all integer power $0 \leqslant i_j \leqslant 2k$ such that $\sum_{j=1}^n i_j = 2k$.

Taking $g(x) = \sum_{l=1}^n c_l x_1^{2l}$ and solving the system of linear equations, one can find a nontrivial solution $c_1,\cdots,c_m$ provided m is big enough. Consider an origin-symmetric convex body K in $\mathbb{R}^n$ with a strictly positive i-th L_p-mixed curvature function (i.e., $f_{p,i}(K,\xi) > 0$, for all $\xi \in S^{n-1}$). We may assume that

$$\int_{S^{n-1}} h_K^p(\xi)g(\xi)d\xi \geqslant 0, \tag{3.8}$$

(otherwise consider $-g(\xi)$ instead of $g(\xi)$). Choose $\varepsilon > 0$ such that

$$\frac{f_{p,i}(K,\xi)}{W_i(K)} - \varepsilon g(\xi) > 0. \tag{3.9}$$

Then we may use the existence theorem for L_p-mixed curvature functions to conclude that there exists an origin-symmetric convex body L in $\mathbb{R}^n$ such that

$$\frac{f_{p,i}(L,\xi)}{W_i(L)} = \frac{f_{p,i}(K,\xi)}{W_i(K)} - \varepsilon g(\xi). \tag{3.10}$$

Applying (1.2) (2.10) (3.6) and (3.10), we obtain that

$$\begin{aligned}\rho(\Gamma_{-p,i}L,\xi)^{-p} &= \frac{1}{W_i(L)}\int_{S^{n-1}}|\theta\cdot\xi|^p dS_{p,i}(L,\xi)\\ &= \frac{1}{W_i(L)}\int_{S^{n-1}}|\theta\cdot\xi|^p f_{p,i}(L,\xi)dS(\xi)\\ &= \frac{1}{W_i(K)}\int_{S^{n-1}}|\theta\cdot\xi|^p f_{p,i}(K,\xi)dS(\xi)-\varepsilon\int_{S^{n-1}}|\theta\cdot\xi|^p g(\xi)dS(\xi)\\ &= \frac{1}{W_i(K)}\int_{S^{n-1}}|\theta\cdot\xi|^p f_{p,i}(K,\xi)dS(\xi)\\ &= \frac{1}{W_i(K)}\int_{S^{n-1}}|\theta\cdot\xi|^p dS_{p,i}(K,\xi)\\ &= \rho(\Gamma_{-p,i}K,\xi)^{-p}.\end{aligned}$$

It is just to say

$$\Gamma_{-p,i}L=\Gamma_{-p,i}K. \tag{3.11}$$

But

$$\begin{aligned}1 &= \frac{W_{p,i}(K,K)}{W_i(K)}\\ &= \frac{1}{n}\int_{S^{n-1}}h_K^p(\xi)\frac{f_{p,i}(K,\xi)}{W_i(K)}d\xi\\ &= \frac{1}{n}\int_{S^{n-1}}h_K^p(\xi)\frac{f_{p,i}(L,\xi)}{W_i(L)}d\xi+\frac{\varepsilon}{n}\int_{S^{n-1}}h_K^p(\xi)g(\xi)d\xi \qquad (3.12)\\ &\geqslant \frac{1}{n}\int_{S^{n-1}}h_K^p(\xi)\frac{f_{p,i}(L,\xi)}{W_i(L)}d\xi\\ &= \frac{W_{p,i}(L,K)}{W_i(L)}.\end{aligned}$$

Hence

$$W_i(L)\geqslant W_{p,i}(L,K).$$

It follows from Lemma 3.2 that

$$W_i(L)\geqslant W_i(K).$$

So if $W_i(L)=W_i(K)$, then there is an equality in (3.2) and then L and K are dilates. This contradicts the construction of the body L. $\square$

Proof of Theorem 1. Noting that

$$\Gamma_{-p,i}K \subseteq \Gamma_{-p,i}L \Rightarrow C_p\frac{\widehat{f_{p,i}(K,\cdot)}(\theta)}{W_i(K)} \geqslant C_p\frac{\widehat{f_{p,i}(L,\cdot)}(\theta)}{W_i(L)}. \tag{3.13}$$

From $C_p\frac{\widehat{f_{p,i}(K,\cdot)}(\theta)}{W_i(K)} \geqslant C_p\frac{\widehat{f_{p,i}(L,\cdot)}(\theta)}{W_i(L)}$ and $C_p\widehat{h_K^p}(\theta) \geqslant 0,\ \forall\ \theta \in S^{n-1}$, we get

$$\int_{S^{n-1}} \widehat{h_K^p}(\theta)\frac{\widehat{f_{p,i}(K,\cdot)}(\theta)}{W_i(K)}d\theta \geqslant \int_{S^{n-1}} \widehat{h_K^p}(\theta)\frac{\widehat{f_{p,i}(L,\cdot)}(\theta)}{W_i(L)}d\theta = (*) \tag{3.14}$$

Using Parseval's formula on the sphere, one can have

$$\begin{aligned}(*) &= \frac{(2\pi)^n}{W_i(L)}\int_{S^{n-1}} h_K^p(\theta)f_{p,i}(L,\theta)d\theta \\ &= \frac{(2\pi)^n}{W_i(L)}\int_{S^{n-1}} h_K^p(\theta)dS_{p,i}(L,\theta) \\ &= n(2\pi)^n\frac{W_{p,i}(L,K)}{W_i(L)}\end{aligned} \tag{3.15}$$

But

$$\begin{aligned}\int_{S^{n-1}} \widehat{h_K^p}(\theta)\frac{\widehat{f_{p,i}(K,\cdot)}(\theta)}{W_i(K)}(\theta)d\theta &= \frac{(2\pi)^n}{W_i(K)}\int_{S^{n-1}} h_K^p(\theta)f_{p,i}(K,\theta)d\theta \\ &= n(2\pi)^n\frac{W_{p,i}(K,K)}{W_i(K)} \\ &= n(2\pi)^n.\end{aligned} \tag{3.16}$$

Thus

$$W_{p,i}(L,K) \leqslant W_i(L). \tag{3.17}$$

Applying the Lemma 3.2, we get

$$W_i(K) \leqslant W_i(L). \qquad \square \tag{3.18}$$

Proof of Theorem 2. Let $\Omega = \{\theta \in S^{n-1} : C_p\widehat{h_K^p}(\theta) < 0\}$. Consider a function $\nu \in C^\infty(S^{n-1})$ such that $C_p\nu$ is a positive even function supported on Ω, ν is not identically zero. We extend ν to a homogeneous function $r^p\nu(\theta)$ of degree p on $\mathbb{R}^n$. Then the Fourier transform of $r^p\nu(\theta)$ is a homogeneous function of degree $-n-p$: $\widehat{r^p\nu(\theta)} = r^{-n-p}g(\theta)$, where g is an infinitely smooth function on S^{n-1}. Since g is bounded on S^{n-1} and $f_{p,i}(K,\theta)=h_K^{1-p}(\theta)f_i(K,\theta)>0$, one can choose a small $\varepsilon>0$ so that, for every $\theta\in S^{n-1}$ and $r>0$,

$$\frac{f_{p,i}(D,r\theta)}{W_i(D)}=\frac{f_{p,i}(K,r\theta)}{W_i(K)}+\varepsilon r^{-n-p}g(\theta)>0. \tag{3.19}$$

By Lutwak's extension of the Minkowski's existence theorem, $f_{p,i}(D,\theta)$ defines a convex body $D\in\mathbb{R}^n$. By the definition of the function ν, one can obtain that

$$C_p\frac{\widehat{f_{p,i}(D,\cdot)}(r\theta)}{W_i(D)}=C_p\frac{\widehat{f_{p,i}(K,\cdot)}(r\theta)}{W_i(K)}+\varepsilon r^pC_p\nu(\theta)\geqslant C_p\frac{\widehat{f_{p,i}(K,\cdot)}(r\theta)}{W_i(K)}, \tag{3.20}$$

or equivalently

$$\Gamma_{-p,i}D\subseteq\Gamma_{-p,i}K.$$

Next, since $C_p\nu$ is supported and is positive in the set where $C_p\widehat{h_K^p}<0$,

$$\begin{aligned}
&\int_{S^{n-1}}\widehat{h_K^p}(\theta)\frac{\widehat{f_{p,i}(D,\cdot)}(\theta)}{W_i(D)}d\theta\\
&=\int_{S^{n-1}}\widehat{h_K^p}(\theta)\frac{\widehat{f_{p,i}(K,\cdot)}(\theta)}{W_i(K)}d\theta+\int_{S^{n-1}}\widehat{h_K^p}(\theta)\varepsilon\nu(\theta)d\theta\\
&<\int_{S^{n-1}}\widehat{h_K^p}(\theta)\frac{\widehat{f_{p,i}(K,\cdot)}(\theta)}{W_i(K)}d\theta=(*)
\end{aligned} \tag{3.21}$$

Now the Parseval's formula gives

$$\begin{aligned}
(*)&=\frac{(2\pi)^n}{W_i(K)}\int_{S^{n-1}}h_K^p(\theta)f_{p,i}(K,\theta)d\theta\\
&=\frac{n(2\pi)^n}{W_i(K)}\int_{S^{n-1}}h_K^p(\theta)dS_{p,i}(K,\theta)\\
&=n(2\pi)^n\frac{W_{p,i}(K,K)}{W_i(K)}=n(2\pi)^n.
\end{aligned} \tag{3.22}$$

And

$$\begin{aligned}
&\int_{S^{n-1}}\widehat{h_K^p}(\theta)\frac{\widehat{f_{p,i}(D,\cdot)}(\theta)}{W_i(D)}d\theta\\
&=\frac{(2\pi)^n}{W_i(D)}\int_{S^{n-1}}h_K^p(\theta)f_{p,i}(D,\theta)d\theta\\
&=\frac{(2\pi)^n}{W_i(D)}\int_{S^{n-1}}h_K^p(\theta)dS_{p,i}(D,\theta)\\
&=n(2\pi)^n\frac{W_{p,i}(D,K)}{W_i(D)}
\end{aligned} \tag{3.23}$$

Thus

$$W_{p,i}(D,K) < W_i(D). \tag{3.24}$$

As in the previous lemma, this implies

$$W_i(K) < W_i(D). \qquad \square$$

Taking $i = 0$ to Theorem 1 and Theorem 2, respectively, we obtain that

COROLLARY 3.2. [15] *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, and $p \geqslant 1, p \neq n, p$ is not an even integer. Suppose that the support function h_K is infinitely smooth and the functions $C_p \hat{h_K^p}(\theta) \geqslant 0$ for all $\theta \in S^{n-1}$. If*

$$\Gamma_{-p}K \subseteq \Gamma_{-p}L,$$

then

$$\mathrm{vol}_n(K) \leqslant \mathrm{vol}_n(L).$$

COROLLARY 3.3. [15] *Let K be an origin-symmetric convex body in $\mathbb{R}^n$, and $p \geqslant 1, p \neq n, p$ is not an even integer. If the curvature function $f(K,\cdot)$ is positive on S^{n-1} and $C_p \hat{h_K^p}(\theta)$ is negative on an open subset of S^{n-1}, then there exists an origin-symmetric convex body D so that*

$$\Gamma_{-p}D \subseteq \Gamma_{-p}K,$$

but

$$\mathrm{vol}_n(D) > \mathrm{vol}_n(K).$$

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