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SOME NEW INEQUALITIES SIMILAR TO HILBERT–TYPE INTEGRAL INEQUALITY WITH A HOMOGENEOUS KERNEL

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Abstract. In this paper, we establish some new inequalities similar to Hilbert-type integral inequality, whose kernel is the homogeneous function and the best constant factors are also derived.

1. Introduction

If $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$ and $f,g \geqslant 0$ satisfy

$$0 < \int_0^\infty f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(x)dx < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\frac{\pi}{p})}\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}, \tag{1}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < pq\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}, \tag{2}$$

where the constant factors $\pi/(\sin\pi/p)$ and pq are the best possible. Inequalities (1) and (2) are called Hardy-Hilbert's inequalities ([1]) and are important in analysis and their applications ([2]). In the recent years a lot of results with generalizations of these type of inequalities were obtained. In 2005, Yang ([3]) has given a new Hilbert-type inequality as follows:

If $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\lambda<1$ and $f,g \geqslant 0$ satisfy

$$0 < \int_0^\infty f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(x)dx < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda}dxdy < k_\lambda(p)\left\{\int_0^\infty x^{p-1-\lambda}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{q-1-\lambda}g^q(x)dx\right\}^{\frac{1}{q}}, \tag{3}$$

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where the constant factor $k_\lambda(p) = B\left(\frac{\lambda}{p}, 1-\lambda\right) + B\left(\frac{\lambda}{q}, 1-\lambda\right)$ is the best possible.

In 2008, W. Zhong ([4]) has given a new Hilbert-type integral inequality with a homogeneous kernel of $-\lambda$-degree as follows:

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $\lambda > 0$, $f, g \geqslant 0$, $\omega(x) = x^{p(1-(\lambda/r))-1}$, $\varpi(y) = y^{q(1-(\lambda/s))-1}$. Further, suppose

(a) $K(x,y) \geqslant 0$ is a measurable homogeneous kernel function of $-\lambda$-degree,

and

(b) the weight coefficient $A_\lambda(s) = \int_0^\infty K(1,u)u^{(\lambda/s)-1}du$ is a positive number depending only on the parameters λ, s. Then one has following inequalities.

If $f \in L^p_\omega(\mathbb{R}_+)$, $g \in L^q_\varpi(\mathbb{R}_+)$, $\|f\|_{p,\omega}$, $\|g\|_{q,\varpi} > 0$, then

$$\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy < A_\lambda(s)\|f\|_{p,\omega}\|g\|_{q,\varpi}, \tag{4}$$

$$\left\{\int_0^\infty y^{(p\lambda/s)-1}\left(\int_0^\infty K(x,y)f(x)dx\right)^p dy\right\}^{1/p} < A_\lambda(s)\|f\|_{p,\omega}, \tag{5}$$

where the constant factor $A_\lambda(s)$ is the best possible in both inequalities (4) and (5).

For more general results, please refer to ([5]), ([6]) and ([7]) where ([5]) provides an unified treatment to Hilbert inequalities with general kernels, while ([6]) and ([7]) deals with the problems of the best possible constants in such inequalities (homogeneous case).

In the recent years, many new inequalities similar to (1), (2) and (3) have been established ([8]–[13]). In 2010, Das and Sahoo ([8]) have given two new inequalities similar to Hardy-Hilbert's inequality (1) as follows:

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, λ, $s, r > 0$, $r + s = \lambda$, $f, g \geqslant 0$ and $F(x) = \int_0^x f(t)dt$, $G(x) = \int_0^x g(t)dt$. If $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then the following two inequalities hold:

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}}{(x+y)^\lambda}F(x)G(y)dxdy < pqB(r,s)\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}, \tag{6}$$

$$\int_0^\infty \left(\int_0^\infty \frac{x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}}}{(x+y)^\lambda}F(x)dx\right)^p dy < [qB(r,s)]^p \int_0^\infty f^p(x)dx, \tag{7}$$

where the constant factors $pqB(r,s)$ and $[qB(r,s)]^p$ are the best possible.

In 2010, Das and Sahoo ([9]) have also given two more new inequalities similar to Hardy-Hilbert's inequality (2) as follows:

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, λ, $s, r > 0$, $r + s = \lambda$, $f, g \geqslant 0$ and $F(x) = \int_0^x f(t)dt$, $G(x) = \int_0^x g(t)dt$. If $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then the following two inequalities hold:

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}}{\max\{x^\lambda, y^\lambda\}}F(x)G(y)dxdy < \frac{pq\lambda}{rs}\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}, \tag{8}$$

$$\int_0^\infty\left(\int_0^\infty \frac{x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}}}{\max\{x^\lambda,y^\lambda\}}F(x)dx\right)^p dy<\left(\frac{q\lambda}{rs}\right)^p\int_0^\infty f^p(x)dx, \tag{9}$$

where the constant factors $\frac{pq\lambda}{rs}$ and $\left(\frac{q\lambda}{rs}\right)^p$ are the best possible.

In 2010, Sulaiman ([10, Theorem 1]) derived a new integral inequality similar to (3) as follows:

Let $f,g\geqslant 0$, $\frac{1}{p}+\frac{1}{q}=1$, $1<p<\alpha/2$, $1<q<\beta/2$ and $\alpha,\beta>0$. We define $F(x)$ and $G(x)$ as:

$$F(x)=\int_0^x f(t)dt,\quad G(y)=\int_0^y g(t)dt.$$

Then

$$\int_0^\infty\int_0^\infty \frac{x^{\frac{1}{\beta}-1}y^{\frac{1}{\alpha}-1}F^{\frac{1}{\alpha}+\frac{1}{p}}(x)G^{\frac{1}{\beta}+\frac{1}{q}}(y)}{|x-y|^{\frac{2}{\alpha}+\frac{2}{\beta}}}dxdy$$
$$<K_{p,\alpha}^{1/p}K_{q,\beta}^{1/q}\left\{\int_0^\infty f^{\frac{p}{\alpha}+1}(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^{\frac{q}{\beta}+1}(x)dx\right\}^{\frac{1}{q}}, \tag{10}$$

where

$$K_{p,\alpha}=2\left(1+\frac{\alpha}{p}\right)^{\frac{p}{\alpha}+1}B\left(\frac{p}{\alpha},1-\frac{2p}{\alpha}\right).$$

Very recently, Du and Miao ([13]) obtained the following inequality:

Let $f,g\geqslant 0$ and

$$F(x)=\int_0^x f(t)dt,\quad G(y)=\int_0^y g(t)dt.$$

Furthermore assume that $p,q>1,\alpha,\beta,s,t,\mu,\nu>0$ hold

$$\frac{1}{p}+\frac{1}{q}=1, sp>\beta q+1, tq>\alpha p+1,$$

and

$$(\beta+\mu-s)p+1=0,(\alpha+\nu-t)q+1=0.$$

Then we have

$$\int_0^\infty\int_0^\infty \frac{x^\alpha y^\beta}{(x+y)^{s+t}}F^\mu(x)G^\nu(y)dxdy$$
$$<\kappa\left(\frac{p\mu}{p\mu-1}\right)^\mu\left(\frac{q\nu}{q\nu-1}\right)^\nu\left\{\int_0^\infty f^{p\mu}(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^{q\nu}(x)dx\right\}^{\frac{1}{q}}, \tag{11}$$

where

$$\kappa=B^{1/p}(\beta p+1,sp-(\beta p+1))B^{1/q}(\alpha p+1,tp-(\alpha p+1)).$$

In ([10]) and ([13]), authors do not prove whether the constant factors are the best possible or not.

The main objective of this paper is to build some new inequalities similar to Hilbert-type integral inequalities (4) and (5), whose kernel is the homogeneous function with the best constant factors. As applications, some particular results are given.

2. Preliminary lemmas

In this section we shall prove lemmas, which play crucial roles in proving our main results.

LEMMA 2.1. *Let p and q be conjugate parameters with $p>1$, and let $\lambda, s, r>0$ such that $s+r=\lambda$. If $k_\lambda(x,y):\mathbb{R}_+^2\to\mathbb{R}$ is non-negative homogeneous function of degree $-\lambda$, i.e. $k_\lambda(ux,uy)=u^{-\lambda}k_\lambda(x,y)$, then*

$$\omega_\lambda(s,x)=\varpi_\lambda(r,y)=\widetilde{C}_\lambda(s), \tag{12}$$

where

$$\omega_\lambda(s,x):=\int_0^\infty k_\lambda(x,y)y^{s-1}x^r dy,$$
$$\varpi_\lambda(r,y):=\int_0^\infty k_\lambda(x,y)x^{r-1}y^s dx,$$

and

$$\widetilde{C}_\lambda(s):=\int_0^\infty k_\lambda(1,u)u^{s-1}du.$$

Proof. Setting $u=\frac{y}{x}$, we find

$$\omega_\lambda(s,x)=\int_0^\infty k_\lambda(1,u)u^{s-1}du=\widetilde{C}_\lambda(s),$$

and for $y>0$ letting $x=\frac{y}{u}$, it is easy to find that

$$\varpi_\lambda(r,y)=\int_0^\infty k_\lambda\left(\frac{y}{u},y\right)y^s\frac{y^{r-1}}{u^{r-1}}\frac{y}{u^2}du=\int_0^\infty k_\lambda(1,u)u^{s-1}du=\widetilde{C}_\lambda(s),$$

equation (12) is valid. This completes the lemma. $\square$

LEMMA 2.2. (Hardy's inequality, cf. [1]) *If $p>1$, $f\geqslant 0$ and $F(x)=\int_0^x f(t)dt$, then*

$$\int_0^\infty\left(\frac{F(x)}{x}\right)^p dx<\left(\frac{p}{p-1}\right)^p\int_0^\infty f^p(x)dx, \tag{13}$$

unless $f\equiv 0$. The constant is the best possible.

LEMMA 2.3. *Let $p>\frac{1}{\beta}$, $0<\beta\leqslant 1$, $n>\frac{1}{\beta p-1}$ for $x\geqslant 1$, then*

$$(x^{\frac{\beta p-(1+(1/n))}{\beta p}}-1)^\beta\geqslant x^{\frac{\beta p-(1+(1/n))}{p}}-1. \tag{14}$$

Proof. For $x \geqslant 1$, we set

$$H(x) = (x^{\frac{\beta p-(1+(1/n))}{\beta p}} - 1)^{\beta} - x^{\frac{\beta p-(1+(1/n))}{p}} + 1.$$

Simple computations yield for $x > 1$

$$H'(x) = \frac{\beta p - (1+(1/n))}{p} x^{\frac{(\beta-1)p-(1+(1/n))}{p}} \left((1 - x^{\frac{1+(1/n)-\beta p}{\beta p}})^{\beta-1} - 1 \right) > 0.$$

f is increasing function on $(1,\infty)$ and continuous on $[1,\infty)$. In particular, we have $f(x) \geqslant f(1) = 0$, which gives the desired inequality. □

3. Main results

THEOREM 3.1. *Let p and q be conjugate parameters with $p > \frac{1}{\alpha}$, $q > \frac{1}{\beta}$, $0 < \alpha, \beta \leqslant 1$, and let $\lambda, s, r > 0$ such that $s + r = \lambda$, $k_{\lambda}(x,y)$ is non-negative homogeneous function of degree $-\lambda$ in $\mathbb{R}^2_+$. Assume $F(x) := \int_0^x f(t)dt$, $G(y) := \int_0^y g(t)dt$.*

If

$$0 < \widetilde{C}_{\lambda}(s) < \infty, \quad 0 < \int_0^{\infty} k_{\lambda}(1,u) u^{s-\frac{1}{p}-\beta} du < \infty, \quad 0 < \int_0^{\infty} k_{\lambda}(1,u) u^{r-\frac{1}{q}-\alpha} du < \infty$$

and $f, g \geqslant 0$ satisfy

$$0 < \int_0^{\infty} f^{\alpha p}(x) dx < \infty, \quad 0 < \int_0^{\infty} g^{\beta q}(x) dx < \infty,$$

then the following two inequalities hold:

$$\int_0^{\infty}\int_0^{\infty} k_{\lambda}(x,y) x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}-\beta} F^{\alpha}(x) G^{\beta}(y) dx dy < C_{\lambda}(\alpha,\beta,s,p,q) \left\{ \int_0^{\infty} f^{\alpha p}(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^{\infty} g^{\beta q}(x) dx \right\}^{\frac{1}{q}} \tag{15}$$

and

$$\int_0^{\infty} \left(\int_0^{\infty} k_{\lambda}(x,y) x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}} F^{\alpha}(x) dx \right)^{p} dy < \widetilde{C}^{p}_{\lambda}(s) \left(\frac{\alpha p}{\alpha p - 1} \right)^{\alpha p} \int_0^{\infty} f^{\alpha p}(x) dx, \tag{16}$$

where the constant factors $C_{\lambda}(\alpha,\beta,s,p,q) = \widetilde{C}_{\lambda}(s) \left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha} \left(\frac{\beta q}{\beta q-1}\right)^{\beta}$ and $\widetilde{C}^{p}_{\lambda}(s) \left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}$ are the best possible.

Proof. By Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned}
J &:= \int_0^\infty \int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}-\beta} F^\alpha(x) G^\beta(y)\,dxdy \\
&= \int_0^\infty \int_0^\infty k_\lambda(x,y) (y^{\frac{s-1}{p}} x^{\frac{r}{p}-\alpha} F^\alpha(x))(x^{\frac{r-1}{q}} y^{\frac{s}{q}-\beta} G^\beta(y))\,dxdy \\
&\leqslant \left\{\int_0^\infty \int_0^\infty k_\lambda(x,y) y^{s-1} x^r \left(\frac{F(x)}{x}\right)^{\alpha p} dxdy\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\int_0^\infty \int_0^\infty k_\lambda(x,y) x^{r-1} y^s \left(\frac{G(y)}{y}\right)^{\beta q} dxdy\right\}^{\frac{1}{q}} \\
&= \widetilde{C}_\lambda(s) \left\{\int_0^\infty \left(\frac{F(x)}{x}\right)^{\alpha p} dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty \left(\frac{G(y)}{y}\right)^{\beta q} dy\right\}^{\frac{1}{q}}.
\end{aligned}$$

Then by Hardy's inequality, (15) is valid.

Supposing there exists a positive constant $C < C_\lambda(\alpha,\beta,s,p,q)$, such that (15) is still valid when $C_\lambda(\alpha,\beta,s,p,q)$ is replaced by C and for $n > \frac{1}{\beta p-1}, n \in \mathbb{N}$, setting $\widetilde{f}(x), \widetilde{g}(y)$ as follows:

$$\widetilde{f}(x) = \begin{cases} 0, & \text{for } x \in (0,1) \\ x^{-\frac{1+(1/n)}{\alpha p}}, & \text{for } x \in [1,\infty) \end{cases},$$

$$\widetilde{g}(y) = \begin{cases} 0, & \text{for } y \in (0,1) \\ y^{-\frac{1+(1/n)}{\beta q}}, & \text{for } y \in [1,\infty) \end{cases},$$

then

$$C\left\{\int_0^\infty \widetilde{f}^{\alpha p}(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty \widetilde{g}^{\beta q}(x)dx\right\}^{\frac{1}{q}} = nC, \tag{17}$$

and

$$\widetilde{F}(x) = \begin{cases} 0, & \text{for } x \in (0,1) \\ \frac{\alpha p}{\alpha p-(1+(1/n))}(x^{\frac{\alpha p-(1+(1/n))}{\alpha p}} - 1), & \text{for } x \in [1,\infty) \end{cases},$$

$$\widetilde{G}(y) = \begin{cases} 0, & \text{for } y \in (0,1) \\ \frac{\beta q}{\beta q-(1+(1/n))}(y^{\frac{\beta q-(1+(1/n))}{\beta q}} - 1), & \text{for } y \in [1,\infty) \end{cases}.$$

Denote $\phi(n) = \left(\frac{\alpha p}{\alpha p-(1+(1/n))}\right)^\alpha \left(\frac{\beta q}{\beta q-(1+(1/n))}\right)^\beta$. Then $\phi(n) \to \left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta$, as $n \to \infty$ and for $x,y \geqslant 1$, by Lemma 2.3, we have

$$\begin{aligned}
\widetilde{F}^\alpha(x)\widetilde{G}^\beta(y) &= \phi(n)(x^{\frac{\alpha p-(1+(1/n))}{\alpha p}} - 1)^\alpha (y^{\frac{\beta q-(1+(1/n))}{\beta q}} - 1)^\beta \\
&\geqslant \phi(n)(x^{\frac{\alpha p-(1+(1/n))}{p}} - 1)(y^{\frac{\beta q-(1+(1/n))}{q}} - 1) \\
&> \phi(n)(x^{\frac{\alpha p-(1+(1/n))}{p}} y^{\frac{\beta q-(1+(1/n))}{q}} - x^{\frac{\alpha p-(1+(1/n))}{p}} - y^{\frac{\beta q-(1+(1/n))}{q}}).
\end{aligned}$$

Then

$$\begin{aligned}
J(n) &= \int_0^\infty \int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}-\beta} \widetilde{F}^\alpha(x) \widetilde{G}^\beta(y) dxdy \\
&> \phi(n) \int_1^\infty \int_1^\infty k_\lambda(x,y) \left(x^{r-\frac{1}{np}-1} y^{s-\frac{1}{nq}-1} - x^{r-\frac{1}{np}-1} y^{s-\frac{1}{p}-\beta} - x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{nq}-1} \right) dxdy \\
&= \phi(n)(I_1 - I_2 - I_3).
\end{aligned}$$

Taking $u = \frac{y}{x}$ and by Fubini's theorem, we obtain

$$\begin{aligned}
I_1 &:= \int_1^\infty \int_1^\infty k_\lambda(x,y) x^{r-\frac{1}{np}-1} y^{s-\frac{1}{nq}-1} dxdy \\
&= \int_1^\infty x^{-1-\frac{1}{n}} \left(\int_1^\infty k_\lambda(x,y) y^{s-\frac{1}{nq}-1} x^{r+\frac{1}{nq}} dy \right) dx \\
&= \int_1^\infty x^{-1-\frac{1}{n}} \left(\int_{1/x}^1 k_\lambda(1,u) u^{s-\frac{1}{nq}-1} du + \int_1^\infty k_\lambda(1,u) u^{s-\frac{1}{nq}-1} du \right) dx \\
&= n \int_1^\infty k_\lambda(1,u) u^{s-\frac{1}{nq}-1} du + \int_1^\infty x^{-1-\frac{1}{n}} dx \int_{1/x}^1 k_\lambda(1,u) u^{s-\frac{1}{nq}-1} du \\
&= n \int_1^\infty k_\lambda(1,u) u^{s-\frac{1}{nq}-1} du + \int_0^1 k_\lambda(1,u) u^{s-\frac{1}{nq}-1} du \int_{1/u}^\infty x^{-1-\frac{1}{n}} dx \\
&= n \left(\int_1^\infty k_\lambda(1,u) u^{s-\frac{1}{nq}-1} du + \int_0^1 k_\lambda(1,u) u^{s+\frac{1}{np}-1} du \right).
\end{aligned}$$

Again taking $u = \frac{y}{x}$, we obtain

$$\begin{aligned}
I_2 &:= \int_1^\infty \int_1^\infty k_\lambda(x,y) x^{r-\frac{1}{np}-1} y^{s-\frac{1}{p}-\beta} dxdy \\
&= \int_1^\infty \int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{np}-1} y^{s-\frac{1}{p}-\beta} dxdy - \int_1^\infty \int_0^1 k_\lambda(x,y) x^{r-\frac{1}{np}-1} y^{s-\frac{1}{p}-\beta} dxdy \\
&< \int_1^\infty x^{-1-\left(\beta-\frac{1}{q}+\frac{1}{np}\right)} dx \int_0^\infty k_\lambda(1,u) u^{s-\frac{1}{p}-\beta} du \\
&= \frac{1}{\beta - \frac{1}{q} + \frac{1}{np}} \int_0^\infty k_\lambda(1,u) u^{s-\frac{1}{p}-\beta} du < \infty.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
I_3 &:= \int_1^\infty \int_1^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{nq}-1} dxdy \\
&< \frac{1}{\alpha - \frac{1}{p} + \frac{1}{nq}} \int_0^\infty k_\lambda(1,u) u^{r-\frac{1}{q}-\alpha} du < \infty.
\end{aligned}$$

Hence by (17), we have

$$\int_1^\infty \phi(n) k_\lambda(1,u) u^{s-\frac{1}{nq}-1} du + \int_0^1 \phi(n) k_\lambda(1,u) u^{s+\frac{1}{np}-1} du - \frac{\phi(n)}{n} \bigcirc(1) < C.$$

Then by Fatou lemma, we have

$$\begin{aligned} C_\lambda(\alpha,\beta,s,p,q) &= \left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta \int_0^\infty k_\lambda(1,u)u^{s-1}du \\ &= \int_1^\infty \lim_{n\to\infty} \phi(n)k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du \\ &\quad + \int_0^1 \lim_{n\to\infty} \phi(n)k_\lambda(1,u)u^{s+\frac{1}{np}-1}du - \lim_{n\to\infty}\frac{\phi(n)}{n}\bigcirc(1) \\ &\leqslant \varliminf_{n\to\infty}\left(\int_1^\infty \phi(n)k_\lambda(1,u)u^{s-\frac{1}{nq}-1}du\right. \\ &\quad \left. + \int_0^1 \phi(n)k_\lambda(1,u)u^{s+\frac{1}{np}-1}du - \frac{\phi(n)}{n}\bigcirc(1)\right) < C. \end{aligned}$$

Hence, the constant factor $C = C_\lambda(\alpha,\beta,s,p,q)$ is the best possible.

By Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned} L(y) &:= \int_0^\infty k_\lambda(x,y)x^{r-\frac{1}{q}-\alpha}y^{s-\frac{1}{p}}F^\alpha(x)dx \\ &= \int_0^\infty k_\lambda(x,y)(x^{\frac{r}{p}-\alpha}y^{\frac{s-1}{p}}F^\alpha(x))(x^{\frac{r-1}{q}}y^{\frac{s}{q}})dx \\ &\leqslant \left\{\int_0^\infty k_\lambda(x,y)x^{r-\alpha p}y^{s-1}F^{\alpha p}(x)dx\right\}^{1/p}\left\{\int_0^\infty k_\lambda(x,y)x^{r-1}y^s dx\right\}^{1/q} \\ &= (\widetilde{C}_\lambda(s))^{\frac{1}{q}}\left\{\int_0^\infty k_\lambda(x,y)(x^{r-\alpha p}y^{s-1}F^{\alpha p}(x)dx\right\}^{1/p}. \end{aligned}$$

Hence again applying Lemma 2.1, we have

$$\begin{aligned} \int_0^\infty L^p(y)dy &\leqslant (\widetilde{C}_\lambda(s))^{\frac{p}{q}}\int_0^\infty \left(\int_0^\infty k_\lambda(x,y)x^r y^{s-1}dy\right)x^{-\alpha p}F^{\alpha p}(x)dx \\ &= (\widetilde{C}_\lambda(s))^p \int_0^\infty \left(\frac{F(x)}{x}\right)^{\alpha p} dx. \end{aligned}$$

Then by Hardy's inequality, (16) is valid.

If the constant factor $\widetilde{C}^p_\lambda(s)\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}$ in (16) is not the best possible, then there exists a positive constant K such that $K < \widetilde{C}_\lambda(s)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha$ and (16) still remains valid if $\widetilde{C}^p_\lambda(s)\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}$ is replaced by K^p. Then by Hölder's inequality, (16) and Hardy's inequality, we obtain

$$\begin{aligned} J &= \int_0^\infty \left(\int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}} F^\alpha(x)dx \right) \left(\frac{G(y)}{y} \right)^\beta dy \\ &\leqslant \left\{ \int_0^\infty \left(\int_0^\infty k_\lambda(x,y) x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}} F^\alpha(x)dx \right)^p dy \right\}^{1/p} \left\{ \int_0^\infty \left(\frac{G(y)}{y} \right)^{\beta q} dy \right\}^{1/q} \\ &< \left(\frac{\beta q}{\beta q - 1} \right)^\beta K \left\{ \int_0^\infty f^{\alpha p}(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^{\beta q}(x)dx \right\}^{\frac{1}{q}}, \end{aligned}$$

which results that the constant factor $C_\lambda(\alpha,\beta,s,p,q)$ in (15) is not the best possible. This contradiction shows that the constant factor $\widetilde{C}_\lambda^p(s)\left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}$ in (16) is the best possible. The theorem is proved. □

If $k_\lambda(x,y) = 1/(x+y)^\lambda, 1/\max\{x^\lambda, y^\lambda\}$ or $1/|x-y|^\lambda$ then we obtain the following corollaries correspondingly,

COROLLARY 3.2. *Let p and q be conjugate parameters with $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \leqslant 1$, and let $\lambda, s, r > 0$, such that $r+s=\lambda, f, g \geqslant 0$ and $F(x) = \int_0^x f(t)dt, G(y) = \int_0^y g(t)dt$. If $0 < \int_0^\infty f^{\alpha p}(x)dx < \infty$ and $0 < \int_0^\infty g^{\beta q}(x)dx < \infty$, then the following two inequalities hold:*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}-\beta}}{(x+y)^\lambda} F^\alpha(x) G^\beta(y) dxdy \\ &\qquad < B(r,s) \left(\frac{\alpha p}{\alpha p - 1} \right)^\alpha \left(\frac{\beta q}{\beta q - 1} \right)^\beta \left\{ \int_0^\infty f^{\alpha p}(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^{\beta q}(x)dx \right\}^{\frac{1}{q}}, \end{aligned}$$

$$\int_0^\infty \left(\int_0^\infty \frac{x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}}}{(x+y)^\lambda} F^\alpha(x)dx \right)^p dy < [B(r,s)]^p \left(\frac{\alpha p}{\alpha p - 1} \right)^{\alpha p} \int_0^\infty f^{\alpha p}(x)dx,$$

where the constant factors $B(r,s)\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta$ and $[B(r,s)]^p \left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}$ are the best possible.

COROLLARY 3.3. *Let p and q be conjugate parameters with $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \leqslant 1$, and let $\lambda, s, r > 0$, such that $r+s=\lambda, f, g \geqslant 0$ and $F(x) = \int_0^x f(t)dt, G(y) = \int_0^y g(t)dt$. If $0 < \int_0^\infty f^{\alpha p}(x)dx < \infty$ and $0 < \int_0^\infty g^{\beta q}(x)dx < \infty$, then the following two inequalities hold:*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}-\beta}}{\max\{x^\lambda, y^\lambda\}} F^\alpha(x) G^\beta(y) dxdy \\ &\qquad < \frac{\lambda}{rs} \left(\frac{\alpha p}{\alpha p - 1} \right)^\alpha \left(\frac{\beta q}{\beta q - 1} \right)^\beta \left\{ \int_0^\infty f^{\alpha p}(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^{\beta q}(x)dx \right\}^{\frac{1}{q}}, \end{aligned}$$

$$\int_0^\infty \left(\int_0^\infty \frac{x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}}}{\max\{x^\lambda, y^\lambda\}} F^\alpha(x)dx\right)^p dy < \left(\frac{\lambda}{rs}\right)^p \left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p} \int_0^\infty f^{\alpha p}(x)dx,$$

where the constant factors $\frac{\lambda}{rs}\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta$ *and* $\left(\frac{\lambda}{rs}\right)^p \left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}$ *are the best possible.*

COROLLARY 3.4. *Let* p *and* q *be conjugate parameters with* $p > \frac{1}{\alpha}, q > \frac{1}{\beta}, 0 < \alpha, \beta \leqslant 1$, *and let* $0 < \lambda < 1, s, r > 0$, *such that* $r + s = \lambda, f, g \geqslant 0$ *and* $F(x) = \int_0^x f(t)dt, G(y) = \int_0^y g(t)dt$. *If* $0 < \int_0^\infty f^{\alpha p}(x)dx < \infty$ *and* $0 < \int_0^\infty g^{\beta q}(x)dx < \infty$, *then the following two inequalities hold:*

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}-\beta}}{|x-y|^\lambda} F^\alpha(x) G^\beta(y) dxdy$$
$$< (B(s,1-\lambda)+B(r,1-\lambda)) \left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta \left\{\int_0^\infty f^{\alpha p}(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^{\beta q}(x)dx\right\}^{\frac{1}{q}},$$

$$\int_0^\infty \left(\int_0^\infty \frac{x^{r-\frac{1}{q}-\alpha} y^{s-\frac{1}{p}}}{|x-y|^\lambda} F^\alpha(x)dx\right)^p dy$$
$$< (B(s,1-\lambda)+B(r,1-\lambda))^p \left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p} \int_0^\infty f^{\alpha p}(x)dx,$$

where the constant factors $(B(s,1-\lambda)+B(r,1-\lambda))\left(\frac{\alpha p}{\alpha p-1}\right)^\alpha \left(\frac{\beta q}{\beta q-1}\right)^\beta$ *and* $(B(s,1-\lambda)+B(r,1-\lambda))^p \left(\frac{\alpha p}{\alpha p-1}\right)^{\alpha p}$ *are the best possible.*

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