Journal of
Mathematical
Inequalities
Volume 6, Number 2 (2012), 195–204 doi:10.7153/jmi-06-20

SOME NEW INEQUALITIES FOR AN INTERIOR POINT OF A TRIANGLE

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(*Communicated by S. Segura Gomis*)

Abstract. In this paper we establish three new inequalities involving an arbitrary point of a triangle. Some related conjectures and problems are put forward.

1. Introduction

Let P be an arbitrary point in the plane of triangle ABC and let D, E, F be the feet of the perpendiculars from P to BC, CA, AB, respectively. In [1], the author gave the following identity:

$$\vec{S}_{\triangle PBC}\cdot PA^2+\vec{S}_{\triangle PCA}\cdot PB^2+\vec{S}_{\triangle PAB}\cdot PC^2=4R^2\vec{S}_{\triangle DEF}, \tag{1.1}$$

where R is the circumradius of $\triangle ABC$ and $\vec{S}_{\triangle PBC}$, $\vec{S}_{\triangle PCA}$, $\vec{S}_{\triangle PAB}$, $\vec{S}_{\triangle DEF}$ denote directed areas of $\triangle PBC$, $\triangle PCA$, $\triangle PAB$, $\triangle DEF$. The directed area of a triangle is defined as follows: Given a triangle XYZ, if the orientation around the vertexes X, Y, Z in sequence is counterclockwise, then its directed area $\vec{S}_{\triangle XYZ}$ is positive and $\vec{S}_{\triangle XYZ}=S_{\triangle XYZ}$. If that one is clockwise, then the directed area $\vec{S}_{\triangle XYZ}$ is negative and $\vec{S}_{\triangle XYZ}=-S_{\triangle XYZ}$.

In particular, when P lies inside triangle ABC, identity (1.1) becomes

$$S_aR_1^2+S_bR_2^2+S_cR_3^2=4R^2S_p, \tag{1.2}$$

where $R_1=PA$, $R_2=PB$, $R_3=PC$ and S_a, S_b, S_c denote the areas of the $\triangle PBC$, $\triangle PCA$, $\triangle PAB$ respectively, and S_p is the area of the pedal triangle DEF.

It is well known that the following inequality holds between the area S of the triangle ABC and the area S_p of the pedal triangle DEF:

$$S_p\leqslant\frac{1}{4}S, \tag{1.3}$$

with equality if and only if P is the circumcenter of the triangle ABC (see Figure 1). Therefore, it follows from (1.2) that (see Figure 2)

$$S_aR_1^2+S_bR_2^2+S_cR_3^2\leqslant SR^2. \tag{1.4}$$

This inequality inspires the author to find the similar conclusion:

Mathematics subject classification (2010): 51M16.
Keywords and phrases: triangle, interior point, Erdös-Mordell inequality, Euler inequality, conjecture.

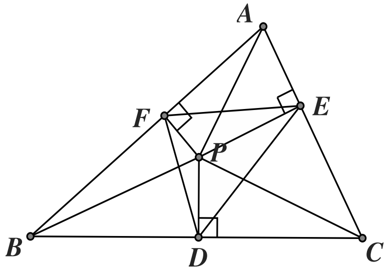

Figure 1

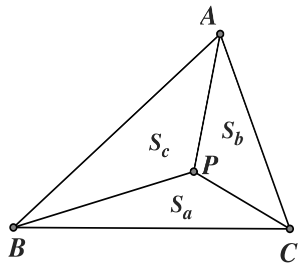

Figure 2

THEOREM 1.1. *Let P be an arbitrary interior point of the triangle ABC. Then*

$$S_aR_1^3 + S_bR_2^3 + S_cR_3^3 \leqslant SR^3, \tag{1.5}$$

with equality if and only if P is the circumcenter of the triangle ABC.

If P coincides with the centroid of $\triangle ABC$, then $S_a = S_b = S_c = \frac{1}{3}S$, $R_1 = \frac{2}{3}m_a$, $R_2 = \frac{2}{3}m_b$, $R_3 = \frac{2}{3}m_c$ (m_a, m_b, m_c are the three medians of $\triangle ABC$) and it follows from (1.5) that

$$m_a^3 + m_b^3 + m_c^3 \leqslant \frac{81}{8}R^3, \tag{1.6}$$

which was conjectured by Ji Chen in [2].

At the same time when inequality (1.5) has been proven, we obtain the following two interesting geometric inequalities:

THEOREM 1.2. *Let P be an arbitrary point of triangle ABC with circumradius R and inradius r. Let r_p be the inradius of the pedal triangle of P with respect to triangle ABC. Then*

$$\frac{1}{2r_p} \geqslant \frac{1}{R} + \frac{1}{2r}, \tag{1.7}$$

with equality if and only if triangle ABC is equilateral and P is its center.

THEOREM 1.3. *Let P be an arbitrary interior point of $\triangle ABC$ and let D, E, F denote the feet of the perpendiculars from P to BC, CA, AB respectively. Let r_p be the inradius of the pedal triangle DEF and let $PA = R_1$, $PB = R_2$, $PC = R_3$, $PD = r_1$, $PE = r_2$, $PF = r_3$. Then*

$$R_1 + R_2 + R_3 \geqslant r_1 + r_2 + r_3 + 6r_p, \tag{1.8}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center.

In this note we will prove the above three theorems and propose some related conjectures and problems.

2. Some lemmas

To prove our theorems, we need several lemmas.

LEMMA 2.1. *Let P be an arbitrary point with barycentric coordinates (x,y,z) in the plane of the triangle ABC. Then*

$$(x+y+z)^2PA^2 = (x+y+z)(yc^2+zb^2)-(yza^2+zxb^2+xyc^2), \tag{2.1}$$

where a, b, c are the lengths of the edges BC, CA, AB respectively.

The above formulae is well known (see e.g. [3] P_{278}).

LEMMA 2.2. *For any point P inside triangle ABC, we have*

$$cr_2+br_3 \leqslant aR_1. \tag{2.2}$$

If AO (O is the circumcenter of $\triangle ABC$) cuts BC at X, then the equality if and only if P lies on the segment AX.

Analogously to (2.2) we also have two inequalities. Lemma 2.2 is a simple important proposition and it has various proofs, see [4]–[10]. Next, we give a crucial lemma which is substantially equivalent to Lemma 2.2 .

LEMMA 2.3. *For any point P inside triangle ABC, we have*

$$R_1 \geqslant \frac{R_1^2}{2R}+\frac{2RS_p}{S}, \tag{2.3}$$

with equality as in (2.2).

Proof. Note that $S_a=\frac{1}{2}ar_1$, $S_b=\frac{1}{2}br_2$, $S_c=\frac{1}{2}cr_3$, $S_a+S_b+S_c=S$, applying Lemma 2.1 and 2.2 we have

$$\begin{aligned}
(S_a+S_b+S_c)^2R_1^2 &= (S_a+S_b+S_c)(S_bc^2+S_cb^2)-(S_bS_ca^2+S_cS_ab^2+S_aS_bc^2)\\
&= \frac{1}{2}(br_2c^2+cr_3b^2)S-\frac{1}{4}(bcr_2r_3a^2+car_3r_1b^2+abr_1r_2c^2)\\
&= \frac{1}{2}bc(br_2+cr_3)S-\frac{1}{4}abc(ar_2r_3+br_3r_1+cr_1r_2)\\
&\leqslant \frac{1}{2}abcR_1S-\frac{1}{2}abcR(r_2r_3\sin A+r_3r_1\sin B+r_1r_2\sin C)\\
&= \frac{1}{2}abcR_1S-abcR(S_{\triangle PEF}+S_{\triangle PFD}+S_{\triangle PDE})\\
&= \frac{1}{2}abc(R_1S-2RS_p).
\end{aligned}$$

Then make use of $S_a+S_b+S_c=S$ and $abc=4SR$, we get

$$R_1S-2RS_p \geqslant \frac{SR_1^2}{2R},$$

this yields inequality (2.3). Clearly, the condition of the equality in (2.3) is the same as (2.2). □

LEMMA 2.4. *For any point P inside triangle ABC, we have*

$$S_aR_1+S_bR_2+S_cR_3 \geqslant 4RS_p, \tag{2.4}$$

with equality if and only if P is the circumcenter of the triangle ABC.

Proof. Since the area of the quadrilateral is less than or equal to the half product of two diagonals, so we have

$$S_b+S_c \leqslant \frac{1}{2}aR_1, \qquad S_c+S_a \leqslant \frac{1}{2}bR_2, \qquad S_a+S_b \leqslant \frac{1}{2}cR_3, \tag{2.5}$$

with equalities if and only if $PA \perp BC$, $PB \perp CA$, $PC \perp AB$ respectively. Adding up these inequalities and using identity $S_a+S_b+S_c=S$, we obtain

$$aR_1+bR_2+cR_3 \geqslant 4S. \tag{2.6}$$

Equality holds if and only if P is the orthocenter of $\triangle ABC$.

Applying inequality (2.6) to the pedal triangle $\triangle DEF$ (see Figure 1), we get

$$EF\cdot r_1+FD\cdot r_2+DE\cdot r_3 \geqslant 4S_p,$$

Observe that $EF=\frac{aR_1}{2R}$, $ar_1=2S_a$, etc., then inequality (2.4) follows at once. According to the equality condition of (2.6), we conclude easily that the equality in (2.4) holds if and only if P is the circumcenter of the triangle ABC. □

REMARK 2.1. Inequality (2.4) can also be proven easily by using Lemma 2.2 and the identity (1.2).

LEMMA 2.5. *Suppose that P is any point in the plane of the triangle ABC. Then*

$$aR_1^2+bR_2^2+cR_3^2 \geqslant abc, \tag{2.7}$$

with equality holds if and only if P is the incenter of $\triangle ABC$.

Inequality (2.7) is given first by M. K. Lamkin (see [3]). The author [11] generalized its equivalent form:

$$R_1^2\sin A+R_2^2\sin B+R_3^2\sin C \geqslant 2S \tag{2.8}$$

to the polygon. I proved the following result: For any polygon $A_1A_2\cdots A_n$ and an arbitrary point P

$$\sum_{i=1}^{n} PA_i^2 \sin A_i \geqslant 2F, \tag{2.9}$$

where F is the area of the polygon. Later, the author further generalized inequality (2.9) into the case involving two arbitrary points P, Q (see [12]):

$$\sum_{i=1}^{n} PA_i \cdot QA_i \sin A_i \geqslant 2F. \tag{2.10}$$

LEMMA 2.6. *For any point P inside triangle ABC, we have*

$$\frac{R_1^2+R_2^2+R_3^2}{r_1+r_2+r_3} \geqslant 2R, \tag{2.11}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center.

Inequality (2.11) was first posed by the author, and it was first proved by Xiao-Guang Chu and Zhen-Gang Xiao [13]. Xue-Zhi Yang gave a simple proof in his book [14, P_{15}]. We introduce a brief sketch of his proof as follows:

Applying the Cosine Law, one gets easily

$$4S^2R_1^2 = b^2c^2(r_2^2+r_3^3)+bcr_2r_3(b^2+c^2-a^2). \tag{2.12}$$

Then we use $abc = 4SR$ and the identity:

$$ar_1+br_2+cr_3 = 2S, \tag{2.13}$$

we obtain

$$4S^2\left(\sum R_1^2 - 2R\sum r_1\right) = \sum\left[b^2c^2(r_2^2+r_3^3)+bcr_2r_3(b^2+c^2-a^2)\right] - abc\sum ar_1\sum r_1. \tag{2.14}$$

where $\sum$ denotes cyclic sums. From this we can obtain the identity:

$$4S^2\left(\sum R_1^2 - 2R\sum r_1\right) = \sum bcr_2r_3(b-c)^2 + \frac{1}{2}\sum a\left[c(r_3+r_1)-b(r_1+r_2)\right]^2, \tag{2.15}$$

which implies inequality (2.11).

3. The proofs of the Theorems

3.1. The proof of Theorem 1.1

Proof. We multiply both sides of inequality (2.3) by S_aR_1, then

$$\frac{S_aR_1^3}{2R} + \frac{2RS_pS_aR_1}{S} \leqslant S_aR_1^2.$$

Analogously, we have

$$\frac{S_bR_2^3}{2R}+\frac{2RS_pS_bR_2}{S}\leqslant S_bR_2^2,\qquad \frac{S_cR_3^3}{2R}+\frac{2RS_pS_cR_3}{S}\leqslant S_cR_3^2.$$

By adding up three inequalities and then using identity (1.2) one has

$$\frac{S_aR_1^3+S_bR_2^3+S_cR_3^3}{2R}+\frac{2RS_p}{S}(S_aR_1+S_bR_2+S_cR_3)\leqslant 4R^2S_p.$$

So, it follows from Lemma 2.4 that

$$\frac{S_aR_1^3+S_bR_2^3+S_cR_3^3}{2R}+\frac{8R^2S_p^2}{S}\leqslant 4R^2S_p,$$

Namely

$$\begin{aligned}S_aR_1^3+S_bR_2^3+S_cR_3^3&\leqslant 8R^3\left(S_p-\frac{2S_p^2}{S}\right)\\&=SR^3-\frac{(S-4S_p)^2R^3}{S}\\&\leqslant SR^3.\end{aligned}$$

This completes the proof of (1.5). According to the conditions of equality (2.3) and (1.3), we conclude that the equality in (1.5) occurs if and only if P is the circumcenter of the triangle ABC. □

REMARK 3.1. By applying the inequality of Theorem 1.1 and the weighted power mean inequality, we can get the following generalization of inequality (1.5):

$$S_aR_1^k+S_bR_2^k+S_cR_3^k\leqslant SR^k, \tag{3.1}$$

where $0<k\leqslant 3$. In addition, by using Radon inequality, we can prove that if $k<0$ then (3.1) holds inversely.

3.2. The proof of Theorem 1.2

Proof. From Lemma 2.3 and Lemma 2.5, we have

$$\begin{aligned}aR_1+bR_2+cR_3&\geqslant\frac{1}{2R}(aR_1^2+bR_2^2+cR_3^2)+\frac{2RS_p}{S}(a+b+c)\\&\geqslant\frac{abc}{2R}+\frac{2RS_p}{S}(a+b+c).\end{aligned}$$

Since $abc=4SR, a+b+c=2s$, it follows that

$$aR_1+bR_2+cR_3\geqslant 2S+\frac{4R}{r}S_p. \tag{3.2}$$

Equality occurs if and only if the point P coincide with the circumcenter and the incenter of $\triangle ABC$. This means that $\triangle ABC$ is equilateral and P is its center.

From (3.2) we have

$$\frac{aR_1+bR_2+cR_3}{8RS_p} \geqslant \frac{S}{4RS_p}+\frac{1}{2r}.$$

As

$$r_p = \frac{4RS_p}{aR_1+bR_2+cR_3}, \tag{3.3}$$

we get

$$\frac{1}{2r_p} \geqslant \frac{S}{4RS_p}+\frac{1}{2r}. \tag{3.4}$$

Hence, the inequality (1.7) follows immediately from (3.4) and (1.3). □

3.3. The proof of Theorem 1.3

Proof. First, by adding up the inequality of Lemma 2.3 and its analogues we get

$$R_1+R_2+R_3 \geqslant \frac{R_1^2+R_2^2+R_3^3}{2R}+\frac{6RS_p}{S}. \tag{3.5}$$

Form inequality (2.6) and identity (24), we know (2.6) is equivalent to

$$\frac{S_p}{S} \geqslant \frac{r_p}{R}, \tag{3.6}$$

with equality as in (2.6). Therefore, the inequality (1.8) of Theorem 1.3 follows immediately from (3.5), (3.6) and (2.11). Clearly equality holds in (1.8) if and only if $\triangle ABC$ is equilateral and P is its center. □

REMARK 3.2. From inequality (3.5) and the following identity (we omit its proof)

$$\frac{R_1^2+R_2^2+R_3^2}{2R}+\frac{6RS_p}{S} = r_1\left(\frac{b}{c}+\frac{c}{b}\right)+r_2\left(\frac{c}{a}+\frac{a}{c}\right)+r_3\left(\frac{a}{b}+\frac{b}{a}\right), \tag{3.7}$$

we get

$$R_1+R_2+R_3 \geqslant r_1\left(\frac{b}{c}+\frac{c}{b}\right)+r_2\left(\frac{c}{a}+\frac{a}{c}\right)+r_3\left(\frac{a}{b}+\frac{b}{a}\right). \tag{3.8}$$

Further, we have the following famous Erdös-Mordell inequality (see [5]–[10]):

$$R_1+R_2+R_3 \geqslant 2(r_1+r_2+r_3). \tag{3.9}$$

Recently, the author gave this new proof in [15].

4. Several conjectures and problems

Euler inequality in the triangle is well known, it states that

$$R \geqslant 2r. \tag{4.1}$$

From this we consider the stronger inequality of Theorem 1.2. After being checked by the computer, we pose the following stronger conjecture:

CONJECTURE 1. *For any arbitrary interior point P, we have*

$$\frac{1}{2r_p} \geqslant \frac{1}{\sqrt{2Rr}} + \frac{1}{2r}. \tag{4.2}$$

From (1.7) and the arithmetic-geometric mean inequality, it is easy to prove:

$$8r_p \leqslant R + 2r. \tag{4.3}$$

For this inequality, we have the following unsolved problem:

PROBLEM 1. *Find the maximum value k such that the inequality*

$$2(k+2)r_p \leqslant R + kr \tag{4.4}$$

is valid for arbitrary interior point P of $\triangle ABC$.

REMARK 4.1. From Euler inequality (4.1) we see that the inequality which takes the maximum value k is the strongest in all inequalities whose type is as (4.4). With the help of the computer, the author finds the maximum value k is about $7.88\cdots$.

Next, we denote the circumradius of the pedal triangle DEF by R_p, note that $R_p \geqslant 2r_p$, we first suppose

$$R_p + 6r_p \leqslant R + 2r, \tag{4.5}$$

which is stronger than the inequality (4.3). Further the following with one parameter conjecture is posed:

CONJECTURE 2. *If real number k satisfies $1.8 \leqslant k \leqslant 7.8$, then the inequality*

$$R_p + 2kr_p \leqslant R + (k-1)r \tag{4.6}$$

holds for an arbitrary interior point P of $\triangle ABC$.

Also, we can put forward the following problem:

PROBLEM 2. *Find the maximum and the minimum value of k such that the inequality (4.6) holds for an arbitrary interior point P of $\triangle ABC$.*

When $k=2$, inequality (4.6) becomes

$$R_p + 4r_p \leqslant R + r. \tag{4.7}$$

This inequality has not yet been proved. The author thinks it has the following exponential generalization:

CONJECTURE 3. *If* $k \geqslant \frac{3}{4}$ *is a real number, then the following inequality*

$$R_p^k + (4r_p)^k \leqslant R^k + r^k \tag{4.8}$$

holds for an arbitrary interior point P *of* $\triangle ABC$.

Another similar difficult conjecture is

CONJECTURE 4. *If* $k \geqslant \frac{1}{2}$ *is a real number, then the following inequality*

$$\frac{1}{R_p^k} + \frac{1}{r_p^k} \geqslant \frac{1}{r^k} + \frac{4^k}{R^k} \tag{4.9}$$

holds for an arbitrary interior point P *of* $\triangle ABC$.

It is possible that the inequality similar to (4.9) holds true for Cevian triangles. So we propose the following dual conjecture:

CONJECTURE 5. *Let* P *be an interior point of* $\triangle ABC$. *Let* LMN *denotes the Cevian triangle of* P *respect to* $\triangle ABC$ *and let* R_q, r_q *denote its circumradius and inradius respectively. If* $k \geqslant \frac{1}{2}$ *is a real number, then the following inequality holds:*

$$\frac{1}{R_q^k} + \frac{1}{r_q^k} \geqslant \frac{1}{r^k} + \frac{4^k}{R^k}. \tag{4.10}$$

Considering the exponential generalization of Theorem 1.3, the following conjecture is brought forward:

CONJECTURE 6. *If* k *is a positive number, then the inequality:*

$$R_1^k + R_2^k + R_3^k - (r_1^k + r_2^k + r_3^k) \geqslant 3 \cdot 2^k(2^k - 1) r_p^k \tag{4.11}$$

holds for an arbitrary interior point P *of* $\triangle ABC$.

Acknowledgements. The author would like to thank the referee for his observations and suggestions.

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(Received May 20, 2011)

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