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THE OPTIMAL CONVEX COMBINATION BOUNDS OF ARITHMETIC AND HARMONIC MEANS IN TERMS OF POWER MEAN

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Abstract. In this paper, we answer the question: What are the greatest value $p = p(\alpha)$ and least value $q = q(\alpha)$ such that the double inequality $M_p(a,b) \leqslant \alpha A(a,b) + (1-\alpha)H(a,b) \leqslant M_q(a,b)$ holds for any $\alpha \in (0,1)$ and all $a,b > 0$? Here, $M_p(a,b)$, $A(a,b)$, and $H(a,b)$ are the p-th power, arithmetic, and harmonic means of a and b, respectively

1. Introduction

For $p \in \mathbb{R}$ the p-th power mean $M_p(a,b)$ of two positive numbers a and b is defined by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}$$

It is well-known that $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed a and b with $a \neq b$. In the recent past, the power mean has been the subject of intensive research. In particular, many remarkable inequalities for power mean can be found in literature [1-17]. Let $A(a,b) = \frac{a+b}{2}$, $G(a,b) = \sqrt{ab}$, and $H(a,b) = \frac{2ab}{a+b}$ be the arithmetic, geometric, and harmonic means of two positive number a and b, respectively. Then

$$\begin{aligned} \min\{a,b\} \leqslant M_{-1}(a,b) = H(a,b) \leqslant M_0(a,b) = G(a,b) \\ \leqslant M_1(a,b) = A(a,b) \leqslant \max\{a,b\}. \end{aligned}$$

In [18], Alzer and Janous established the following sharp double inequality (see also [19, p. 350]):

$$M_{\frac{\log 2}{\log 3}}(a,b) \leqslant \frac{2}{3}A(a,b) + \frac{1}{3}G(a,b) \leqslant M_{\frac{2}{3}}(a,b)$$

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for all $a,b>0$.

For any $\alpha\in(0,1)$, Janous [20] found the least value q and the greatest value p such that the inequality $M_p(a,b)\leqslant\alpha A(a,b)+(1-\alpha)G(a,b)\leqslant M_q(a,b)$ holds for all $a,b>0$.

In this paper, we answer the question: What are the greatest value $p=p(\alpha)$ and least value $q=q(\alpha)$ such that the double inequality $M_p(a,b)\leqslant\alpha A(a,b)+(1-\alpha)H(a,b)\leqslant M_q(a,b)$ holds for any $\alpha\in(0,1)$ and all $a,b>0$?

2. Main results

LEMMA 2.1. *For* $\alpha\in(0,1)$ *and* $g(t)=(3\alpha-4)t^{2\alpha}-2\alpha t^{2\alpha-1}-\alpha t^{2\alpha-2}+\alpha t^2+2\alpha t+4-3\alpha$ *we have*

(1) If $\frac{1}{2}<\alpha<1$*, then* $g(t)>0$ *whenever* $t\in(1,\infty)$*;*

(2) If $0<\alpha<\frac{1}{2}$*, then* $g(t)<0$ *whenever* $t\in(0,1)$*.*

Proof. Let $h(t)=t^{5-2\alpha}g'''(t)$, then simple computations lead to

$$g(1)=0, \tag{2.1}$$

$$g'(t)=2\alpha(3\alpha-4)t^{2\alpha-1}-2\alpha(2\alpha-1)t^{2\alpha-2}-\alpha(2\alpha-2)t^{2\alpha-3}+2\alpha t+2\alpha,$$

$$g'(1)=0, \tag{2.2}$$

$$\begin{aligned}g''(t)=&\ 2\alpha(2\alpha-1)(3\alpha-4)t^{2\alpha-2}-2\alpha(2\alpha-1)(2\alpha-2)t^{2\alpha-3}\\&-\alpha(2\alpha-2)(2\alpha-3)t^{2\alpha-4}+2\alpha,\end{aligned}$$

$$g''(1)=0, \tag{2.3}$$

$$\begin{aligned}g'''(t)=&\ 2\alpha(2\alpha-1)(2\alpha-2)(3\alpha-4)t^{2\alpha-3}-2\alpha(2\alpha-1)(2\alpha-2)(2\alpha-3)t^{2\alpha-4}\\&-\alpha(2\alpha-2)(2\alpha-3)(2\alpha-4)t^{2\alpha-5},\end{aligned}$$

$$h(1)=g'''(1)=16\alpha(\alpha-1)\left(\alpha-\frac{5}{4}\right), \tag{2.4}$$

$$h(0)=-8\alpha(\alpha-1)\left(\alpha-\frac{3}{2}\right)(\alpha-2), \tag{2.5}$$

$$h'(t)=4\alpha(2\alpha-1)(2\alpha-2)(3\alpha-4)t-2\alpha(2\alpha-1)(2\alpha-2)(2\alpha-3),$$

$$h'(1)=32\alpha(\alpha-1)\left(\alpha-\frac{1}{2}\right)\left(\alpha-\frac{5}{4}\right), \tag{2.6}$$

$$h'(0)=-16\alpha\left(\alpha-\frac{1}{2}\right)(\alpha-1)\left(\alpha-\frac{3}{2}\right) \tag{2.7}$$

and

$$h''(t) = 48\alpha\left(\alpha - \frac{1}{2}\right)(\alpha - 1)\left(\alpha - \frac{4}{3}\right). \tag{2.8}$$

(1) If $\frac{1}{2} < \alpha < 1$ and $t \in (1, \infty)$, then (2.4), (2.6) and (2.8) lead to

$$h(1) > 0, \tag{2.9}$$

$$h'(1) > 0 \tag{2.10}$$

and

$$h''(t) > 0. \tag{2.11}$$

From (2.1)–(2.3) and (2.9)–(2.11) we clearly see that $g(t) > 0$.

(2) If $0 < \alpha < \frac{1}{2}$ and $t \in (0,1)$, then from (2.4)–(2.8) we know that

$$h(1) > 0, \tag{2.12}$$

$$h(0) > 0, \tag{2.13}$$

$$h'(1) < 0, \tag{2.14}$$

$$h'(0) > 0 \tag{2.15}$$

and $h'(t)$ is strictly decreasing in $(0,1)$.

From (2.14) and (2.15) together with the monotonicity of $h'(t)$ in $(0,1)$ we clearly see that there exists $\lambda_1 \in (0,1)$, such that $h'(t) > 0$ for $t \in (0, \lambda_1)$ and $h'(t) < 0$ for $t \in (\lambda_1, 1)$, hence $h(t)$ is strictly increasing in $(0, \lambda_1)$ and strictly decreasing in $(\lambda_1, 1)$. From (2.12), (2.13) and the monotonicity of $h(t)$ we know that $h(t) > 0$ for $t \in (0,1)$. Now $g(t) < 0$ follows from (2.1)–(2.3) and $h(t) > 0$ for $t \in (0,1)$.

THEOREM 2.1. *Inequality* $\alpha A(a,b) + (1-\alpha)H(a,b) \geqslant M_{2\alpha-1}(a,b)$ *holds for any* $\alpha \in (0,1)$ *and all* $a,b > 0$, *with equality if and only if* $a = b$, *and the constant* $2\alpha - 1$ *cannot be improved.*

Proof. If $a = b$, then we clearly see that $\alpha A(a,b) + (1-\alpha)H(a,b) = M_{2\alpha-1}(a,b) = a$.

If $a \neq b$, then we divide the proof into three cases.

Case 1. If $\alpha = \frac{1}{2}$, then simple computation leads to

$$\begin{aligned}\frac{1}{2}A(a,b) + \frac{1}{2}H(a,b) - M_0(a,b) &= \frac{a+b}{4} + \frac{ab}{a+b} - \sqrt{ab}\\ &= \frac{(\sqrt{a}-\sqrt{b})^4}{4(a+b)} > 0.\end{aligned}$$

Case 2. If $\frac{1}{2} < \alpha < 1$, then we assume that $a > b > 0$ and $t = \frac{a}{b} > 1$. Elementary computation yields

$$\begin{aligned}&\alpha A(a,b) + (1-\alpha)H(a,b) - M_{2\alpha-1}(a,b)\\ &= b\left[\frac{\alpha(t^2+1) + (4-2\alpha)t}{2(1+t)} - \left(\frac{1+t^{2\alpha-1}}{2}\right)^{\frac{1}{2\alpha-1}}\right].\end{aligned} \tag{2.16}$$

Let

$$f(t) = \log[\alpha(t^2+1)+(4-2\alpha)t] - \log(1+t) - \frac{1}{2\alpha-1}\log(1+t^{2\alpha-1}) + \frac{\log 2}{2\alpha-1} - \log 2, \tag{2.17}$$

then

$$f'(t) = \frac{(3\alpha-4)t^{2\alpha} - 2\alpha t^{2\alpha-1} - \alpha t^{2\alpha-2} + \alpha t^2 + 2\alpha t + 4 - 3\alpha}{(t+1)(t^{2\alpha-1}+1)[\alpha(t^2+1)+(4-2\alpha)t]} \tag{2.18}$$

and

$$f(1) = 0. \tag{2.19}$$

From (2.16)–(2.19) and Lemma 2.1(1) we know that $\alpha A(a,b)+(1-\alpha)H(a,b) > M_{2\alpha-1}(a,b)$.

Case 3. If $0<\alpha<\frac{1}{2}$, then we assume that $0<a<b$ and $t=\frac{a}{b}\in(0,1)$. From (2.16)–(2.19) and Lemma 2.1(2) we clearly see that $\alpha A(a,b)+(1-\alpha)H(a,b) > M_{2\alpha-1}(a,b)$.

Next, we prove that $M_{2\alpha-1}(a,b)$ is the best possible lower power mean bound for the sum $\alpha A(a,b)+(1-\alpha)H(a,b)$.

Let $p>2\alpha-1$, $t>0$ and

$$g(t) = \alpha A(1,t)+(1-\alpha)H(1,t) - M_p(1,t). \tag{2.20}$$

Then simple computations lead to

$$g(1) = 0, \tag{2.21}$$

$$g'(1) = 0 \tag{2.22}$$

and

$$g''(1) = -\frac{p-(2\alpha-1)}{4} < 0. \tag{2.23}$$

Inequality (2.23) and the continuity of $g''(t)$ imply that there exists $\delta>0$ such that

$$g''(t) < 0 \tag{2.24}$$

for all $t\in[1,1+\delta)$.

From (2.24) we know that $g'(t)$ is strictly decreasing in $[1,1+\delta]$, then (2.22) leads to the conclusion that

$$g'(t) < 0 \tag{2.25}$$

for all $t \in (1, 1+\delta]$.

Therefore, $\alpha A(1,t) + (1-\alpha)H(1,t) < M_p(1,t)$ for $t \in (1, 1+\delta]$ follows from (2.20) and (2.21) together with (2.25).

LEMMA 2.2. If $\alpha \in (0,1)$, then $(1-2\alpha)\log\alpha + (2\alpha-2)\log 2 < 0$.

Proof. Let $f(\alpha) = (1-2\alpha)\log\alpha + (2\alpha-2)\log 2$, then

$$f(1) = 0, \tag{2.26}$$

$$f'(\alpha) = -2\log\alpha + \frac{1}{\alpha} + 2\log 2 - 2, \tag{2.27}$$

$$f'(1) = 2\log 2 - 1 > 0 \tag{2.28}$$

and

$$f''(\alpha) = -\frac{2}{\alpha} - \frac{1}{\alpha^2} < 0. \tag{2.29}$$

From (2.26)–(2.29) we clearly see that $f(\alpha) < 0$ for $\alpha \in (0,1)$.

LEMMA 2.3. If $\alpha \in (0,1)$, $p = \frac{\log 2}{\log 2 - \log\alpha}$ and

$$g(t) = (3\alpha-4)t^{p+1} - 2\alpha t^p - \alpha t^{p-1} + \alpha t^2 + 2\alpha t + 4 - 3\alpha,$$

then there exists $t_0 \in (0,1)$, such that $g(t) < 0$ for $t \in (0, t_0)$ and $g(t) > 0$ for $t \in (t_0, 1)$.

Proof. From $0 < p < 1$ and Lemma 2.2 together with elementary computations we have

$$g(1) = 0, \tag{2.30}$$

$$\lim_{t\to 0^+} g(t) = -\infty, \tag{2.31}$$

$$g'(t) = (3\alpha-4)(p+1)t^p - 2\alpha p t^{p-1} - \alpha(p-1)t^{p-2} + 2\alpha t + 2\alpha,$$

$$g'(1) = \frac{4}{\log 2 - \log\alpha}[(1-2\alpha)\log\alpha + (2\alpha-2)\log 2] < 0, \tag{2.32}$$

$$\lim_{t\to 0^+} g'(t) = +\infty, \tag{2.33}$$

$$g''(t) = (3\alpha-4)(p+1)pt^{p-1} - 2\alpha p(p-1)t^{p-2} - \alpha(p-1)(p-2)t^{p-3} + 2\alpha,$$

$$\begin{aligned} g''(1) &= -4p(p+1-2\alpha) \\ &= \frac{4p}{\log 2 - \log\alpha}[(1-2\alpha)\log\alpha + (2\alpha-2)\log 2] < 0 \end{aligned} \tag{2.34}$$

and

$$g'''(t) = t^{p-4}[(3\alpha-4)(p+1)p(p-1)t^2 - 2\alpha p(p-1)(p-2)t - \alpha(p-1)(p-2)(p-3)].$$

Since $(3\alpha-4)(p+1)p(p-1) > 0$ and $\Delta = [2\alpha p(p-1)(p-2)]^2 + 4\alpha(3\alpha-4)(p+1)p(p-1)^2(p-2)(p-3) = 4\alpha p(p-1)^2(p-2)[4(1-\alpha)p(2-p) + 9(1-\alpha) +$

$3] < 0$, hence $g'''(t) > 0$ and $g''(t)$ is strictly increasing in $(0,1)$. From (2.34) and the monotonicity of $g''(t)$ we know that $g''(t) < 0$ for $t \in (0,1)$, hence $g'(t)$ is strictly decreasing in $(0,1)$. Then from (2.32) and (2.33) together with the monotonicity of $g'(t)$ in $(0,1)$ we know that there exists $\lambda \in (0,1)$, such that $g'(t) > 0$ for $t \in (0,\lambda)$ and $g'(t) < 0$ for $t \in (\lambda,1)$, which implies that $g(t)$ is strictly increasing in $(0,\lambda)$ and strictly decreasing in $(\lambda,1)$. Now, from (2.30) and (2.31) together with the monotonicity of $g(t)$ in $(0,1)$ we clearly see that there exists $t_0 \in (0,1)$ such that $g(t) < 0$ for $t \in (0,t_0)$ and $g(t) > 0$ for $t \in (t_0,1)$.

THEOREM 2.2. Inequality $\alpha A(a,b)+(1-\alpha)H(a,b) \leqslant M_{\frac{\log 2}{\log 2-\log\alpha}}(a,b)$ holds for any $\alpha \in (0,1)$ and all $a,b > 0$, with equality if and only if $a=b$, and $M_{\frac{\log 2}{\log 2-\log\alpha}}(a,b)$ is the best possible upper power mean bound for the sum $\alpha A(a,b)+(1-\alpha)H(a,b)$.

Proof. If $a=b$, then we clearly see that

$$\alpha A(a,b)+(1-\alpha)H(a,b) = M_{\frac{\log 2}{\log 2-\log\alpha}}(a,b) = a.$$

If $a \neq b$, then we assume that $b > a > 0$. Let $p = \frac{\log 2}{\log 2-\log\alpha}$ and $t = \frac{a}{b} \in (0,1)$, then

$$\begin{aligned}&\alpha A(a,b)+(1-\alpha)H(a,b)-M_p(a,b)\\ &= b\left[\frac{\alpha(1+t^2)+(4-2\alpha)t}{2(1+t)} - \left(\frac{1+t^p}{2}\right)^{\frac{1}{p}}\right].\end{aligned} \tag{2.35}$$

Let

$$f(t) = \log[\alpha(1+t^2)+(4-2\alpha)t]-\log(1+t)-\frac{1}{p}\log(1+t^p)+\left(\frac{1}{p}-1\right)\log 2, \tag{2.36}$$

then

$$f(1) = f(0) = 0 \tag{2.37}$$

and

$$f'(t) = \frac{(3\alpha-4)t^{p+1}-2\alpha t^p-\alpha t^{p-1}+\alpha t^2+2\alpha t+4-3\alpha}{(t+1)(t^p+1)[\alpha(1+t^2)+(4-2\alpha)t]}. \tag{2.38}$$

Lemma 2.3 and (2.38) imply that there exists $t_0 \in (0,1)$, such that $f(t)$ is strictly decreasing in $(0,t_0)$ and $f(t)$ is strictly increasing in $(t_0,1)$. Then the monotonicity of $f(t)$ in $(0,1)$ and (2.37) imply that $f(t) < 0$ for $t \in (0,1)$. Now from (2.35) and (2.36) we clearly see that $\alpha A(a,b)+(1-\alpha)H(a,b) < M_{\frac{\log 2}{\log 2-\log\alpha}}(a,b)$.

Next, we prove the that $M_{\frac{\log 2}{\log 2-\log\alpha}}(a,b)$ is the best possible upper power mean bound for the sum $\alpha A(a,b)+(1-\alpha)H(a,b)$.

For any $0<\varepsilon<p$, let $0<t<1$, making use of the Taylor expansion we have

$$\begin{aligned}
&\alpha A(1,t)+(1-\alpha)H(1,t)-M_{p-\varepsilon}(1,t)\\
=&\frac{2^{\frac{1}{p-\varepsilon}}[\alpha+(4-2\alpha)t+\alpha t^2]-2(1+t)(1+t^{p-\varepsilon})^{\frac{1}{p-\varepsilon}}}{2^{1+\frac{1}{p-\varepsilon}}(1+t)}\\
=&\frac{(\alpha\cdot 2^{\frac{1}{p-\varepsilon}}-2)+o(t^{p-\varepsilon})}{2^{1+\frac{1}{p-\varepsilon}}(1+t)}. \qquad (2.39)
\end{aligned}$$

It is not difficult to verify that

$$\alpha\cdot 2^{\frac{1}{p-\varepsilon}}>2. \qquad (2.40)$$

From (2.39) and (2.40) we know that for any $0<\varepsilon<p$, there exists $\delta=\delta(\varepsilon,\alpha)>0$, such that $\alpha A(1,t)+(1-\alpha)H(1,t)>M_{p-\varepsilon}(1,t)$ for $t\in(0,\delta)$.

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