

ON A FUNCTION ASSOCIATED WITH THE GENERALIZED EULER CONSTANT FOR AN ARITHMETIC PROGRESSION

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Abstract. The function $\mathfrak{L}(q) = \sum_{p|q} \frac{\log p}{p-1}$ appears in a work by D. H. Lehmer, where he studies a generalization of the Euler constant. In this short note, we obtain lower and upper bounds for $\mathfrak{L}(q)$.

1. Introduction

D. H. Lehmer [1] studied the generalized Euler constant $\gamma(a, q)$ for an arithmetic progression $a \pmod{q}$, defined by

$$\gamma(a, q) = \lim_{x \rightarrow \infty} \left(\sum_{\substack{j \leq x \\ n \equiv a \pmod{q}}} \frac{1}{j} - \frac{1}{q} \log x \right).$$

He proved that

$$\sum_{\substack{1 \leq a \leq q \\ \gcd(a, q) = 1}} \gamma(a, q) = \gamma \frac{\varphi(q)}{q} + \frac{1}{q} \log N(q),$$

for a number theoretic function $N(q)$, where γ is the Euler constant. More precisely, he showed that $N(q)$ is an integer, and its explicit value is given by

$$N(q) = \prod_{p|q} p^{\varphi(q)/(p-1)}.$$

This gives

$$\log N(q) = \varphi(q) \sum_{p|q} \frac{\log p}{p-1} := \varphi(q) \mathfrak{L}(q),$$

say. In this note, we obtain some bounds for the function $\mathfrak{L}(q)$.

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2. Minimax of the function $\mathfrak{L}(q)$

Explicit Minimum. For $x \in (0, \infty) - \{1\}$ we let

$$\ell(x) = \frac{\log x}{x-1},$$

and we put $\ell(1) = 1$. Since ℓ is a decreasing function over $(0, \infty)$, we obtain

$$\min_{1 < q \leq N} \mathfrak{L}(q) = \ell(p^*) \quad \text{where} \quad p^* = \max\{p : p \leq N\}.$$

More precisely, we have

$$\mathfrak{L}(q) \geq \ell(q) \quad \text{for } q \geq 2.$$

Approximate Maximum. Also, we get the maximum value of $\mathfrak{L}(q)$ over $q \in [1, N]$, when we take q to be the largest $\prod_{p \leq x} p$ not larger than N . This happens for

$$x = x(N) := \max\{y : \theta(y) \leq \log N\}, \quad (2.1)$$

where $\theta(x) = \sum_{p \leq x} \log p$. Thus, we obtain

$$\max_{1 < q \leq N} \mathfrak{L}(q) = \sum_{p \leq x(N)} \ell(p).$$

Since $\ell \in C^1(\mathbb{R}^+)$, using Stieljes integral and integrating by parts, we have

$$\begin{aligned} \sum_{p \leq x} \ell(p) &= \int_{2^-}^x \frac{\ell(x)}{\log x} d\theta(x) = \frac{\ell(x)\theta(x)}{\log x} + \int_2^x \theta(t) \frac{d}{dt} \left(\frac{-\ell(t)}{\log t} \right) dt \\ &= \frac{\theta(x)}{x-1} + \int_2^x \frac{\theta(t)}{(t-1)^2} dt = (1 + o(1)) \log x. \end{aligned}$$

This gives $\max_{1 < q \leq N} \mathfrak{L}(q) = (1 + o(1)) \log x(N)$. But, the prime number theorem asserts that $x(N) = (1 + o(1)) \log N$. Thus, we obtain

$$\max_{1 < q \leq N} \mathfrak{L}(q) = (1 + o(1)) \log \log N.$$

Explicit Maximum. Here we give an approach which leads to an explicit upper bound for $\max_{1 < q \leq N} \mathfrak{L}(q)$, as more as has capacity to reprove obtained bound above. To do this, we write

$$\sum_{p \leq x} \ell(p) = \sum_{p \leq x} \frac{\log p}{p} + A - \sum_{p > x} \frac{\log p}{p(p-1)},$$

where A is an absolute constant given by

$$A = \sum_p \frac{\log p}{p(p-1)}.$$

We recall that

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + E + O(\exp(-c\sqrt{x}))$$

is valid for absolute constants $c > 0$, and E given by

$$E = -\gamma - \sum_{n=2}^{\infty} \sum_p \frac{\log p}{p^n}.$$

More precisely, it is known [2] that

$$\sum_{p \leq x} \frac{\log p}{p} < \log x + E + \frac{1}{\log x} \quad \text{for } x \geq 32.$$

Thus, we obtain

$$\sum_{p \leq x} \ell(p) < \log x + A + E + \frac{1}{\log x} \quad \text{for } x \geq 32,$$

or, we obtain

$$\max_{1 < q \leq N} \mathcal{L}(q) < \log x(N) + A + E + \frac{1}{\log x(N)} \quad \text{for } x(N) \geq 32.$$

It is known [2] that $t(1 - \frac{1}{\log t}) < \theta(t)$ is valid for $t \geq 41$. Thus, considering the relation (2.1) we have $x(N) < x^*$, in which

$$x^* \left(1 - \frac{1}{\log x^*} \right) = \log N.$$

It is clear that $x^* > \log N$, so we put $x^* = \log N + \varepsilon_N$ for some function $\varepsilon_N > 0$, with the property $\varepsilon_N = o(\log N)$. We obtain

$$1 + \frac{\log N}{\varepsilon_N} - \log \log N = \log \left(1 + \frac{\varepsilon_N}{\log N} \right).$$

Exact solution of this equation is not accessible, but if we put $\varepsilon_N := \frac{\log N}{\log \log N} + r_N$ then the left hand side of it becomes $1 - \frac{r_N(\log \log N)^2}{\log N} + O\left(\frac{r_N^2(\log \log N)^3}{\log^2 N}\right)$, and the right hand side of it becomes $\frac{1}{\log \log N} + \frac{r_N}{\log N} + O\left(\left(\frac{1}{\log \log N} + \frac{r_N}{\log N}\right)^2\right)$. Taking $r_N = \frac{\log N}{(\log \log N)^2}$, both of above terms become $O\left(\frac{1}{\log \log N}\right)$. Thus, we obtain

$$x^* = \log N \left(1 + \frac{1}{\log \log N} + \frac{(1 + o(1))}{(\log \log N)^2} \right).$$

Therefore, for N large enough, we obtain

$$\max_{1 < q \leq N} \mathcal{L}(q) < \log \log N + A + E + O\left(\frac{1}{\log \log N}\right).$$

Here, we note that $A + E \approx -0.58$, and above result asserts that we must have $\max_{1 < q \leq N} \mathcal{L}(q) < \log \log N + C$ for a constant C . Below figure shows that $C = 0.5$ works as well at least for $7 \leq q \leq 250$.

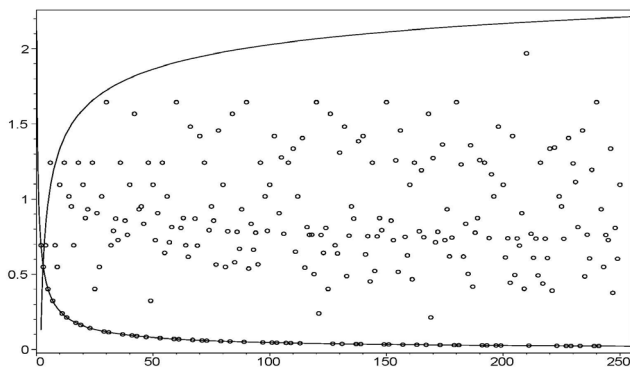


Figure 1: Graphs of points $(q, \mathcal{L}(q))$ for $1 \leq q \leq 250$, the function $\frac{\log x}{x-1}$, and the function $\log \log x + 0.5$.

REFERENCES

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