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ON POWER SUMS OF CONVEX FUNCTIONS IN LOCAL MINIMUM ENERGY PROBLEM

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Abstract. In this paper, new inequalities on the power sums of a convex function are derived and the monotonically decreasing nature of the Riemann sum of a function including a certain strong convexity is shown. It is also shown that a derived inequality has a direct implication in a local minimum energy problem in two-dimensional hexagonal packing.

1. Introduction

For a function f on $(0,1]$, it is known that if f is increasing and either convex or concave, then the upper Riemann sum for the integral of f with n equal subintervals,

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$$

decreases with n. This and several related results were derived by several mathematicians, Kuang [12], Bennett and Jameson [2], and Chen, Qi, Cerone, and Dragomir [4]. Furthermore, a recent detailed study by Abramovich, Barić, Matić, and Pečarić [1] suggests that this fact was known at least since 1975 due to Van de Lune and Van Lint.

Here, we obtain the following generalization under a specific restriction:

THEOREM 1.1. *Let $r \geqslant 1$ be a real number and let $p \leqslant 1$. Let $f:(0,1]\to\mathbb{R}$ belong to the class C^1 and be increasing and concave and $f(1)=\lim_{x\to 1} f'(x)=0$. If either (i) $f'(x)^{1/p}$ is convex with $0<p\leqslant 1$ or (ii) $p\leqslant 0$, then*

$$\frac{1}{r}\sum_{i=1}^{\lfloor r\rfloor}\left(\frac{i}{r}\right)^{p} f\left(\frac{i}{r}\right)$$

decreases with r.

Theorem 1.1 is a result of the following theorem.

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THEOREM 1.2. *Let n be a natural number and let $0 < p \leqslant 1$. Let $f : (0,n] \to [0,\infty)$ be decreasing and $f(n) = 0$. If f is convex, then*

$$\sum_{i=1}^{n-1} i^p \int_i^n f(x)^p dx \leqslant \sum_{i=1}^{n-1} \left(\int_0^i x^p dx \right) f(i)^p$$

with equality if and only if f is linear on $[1,n]$. If f is concave, then the inequality reverses.

Over the next two sections, we prove these two main theorems and give some applications. Our motivation for obtaining Theorem 1.2 stems originally from a requirement to solve a certain local minimum energy problem in two-dimensional hexagonal packing. In the final section, we present an implication of Theorem 1.2 in the problem.

2. Inequality on power sums of convex functions

In order to prove Theorem 1.2, we need to first derive an upper estimate of the integral of the p-th power of a convex function $f \geqslant 0$ over $[i, i+1]$, where $0 < p \leqslant 1$ and f is defined on $[i,n]$ with $f(n) = 0$. We first review the following lemma, obtained in [10, Lemma 4.1], giving a similar estimation of such the type.

LEMMA A. *Let $a < b$ and $0 < p \leqslant 1$. If $f : [a,b] \to [0,\infty)$ is decreasing and convex, then*

$$f(xa + (1-x)b)^p \leqslant x^p f(a)^p + (1-x^p) f(b)^p \tag{2.1}$$

holds for $x \in [0,1]$. Hence, by integrating (2.1) over $[0,1]$,

$$\frac{1}{b-a} \int_a^b f(x)^p dx \leqslant \frac{1}{p+1} f(a)^p + \frac{p}{p+1} f(b)^p. \tag{2.2}$$

Although we may obtain an estimate of the integral of $f(x)^p$ over $[i, i+1]$ by putting $a = i$ and $b = i+1$ in (2.2), the estimate is too weak to use in proving Theorem 1.2. For this reason, in this section, we give better inequalities than those in Lemma A under the additional restriction that $f(n) = 0$.

LEMMA 2.1. *Let $I \subset \mathbb{R}$ be any interval. Let $f : I \to [0,\infty)$ be convex and $g : I \to [0,\infty)$ be concave with $f \geqslant g$. Then, for $0 < p \leqslant 1$, the function $h(x) = [f(x)^p - g(x)^p]^{1/p}$ is convex on I.*

Proof. From the concavity of $g \geqslant 0$, we may assume that $g > 0$ at interior points of I. Let $a, b > 0$ with $a + b = 1$. Since the function $k(x,y) = x^{1-p} y^p$ is concave on $[0,\infty)^2$ for any $p \in [0,1]$,

$$ak(p_1,q_1) + bk(p_2,q_2) \leqslant k(ap_1 + bp_2, aq_1 + bq_2)$$

for $(p_1,q_1),(p_2,q_2) \in [0,\infty)^2$. Put $p_1 = f(x), p_2 = f(y), q_1 = g(x)$, and $q_2 = g(y)$. Then,

$$af(x)^{1-p}g(x)^p + bf(y)^{1-p}g(y)^p \leqslant [af(x)+bf(y)]^{1-p}[ag(x)+bg(y)]^p.$$

Moreover, since f is convex and g is concave,

$$\frac{f(ax+by)^p}{g(ax+by)^p} \leqslant \frac{[af(x)+bf(y)]^p}{[ag(x)+bg(y)]^p} \leqslant \frac{af(x)+bf(y)}{af(x)^{1-p}g(x)^p+bf(y)^{1-p}g(y)^p}. \tag{2.3}$$

Thus, by using the convexity of f, (2.3), and the convexity of $x^{1/p}$, in that order,

$$\begin{aligned}
h(ax+by) &= f(ax+by)\left[1-\frac{g(ax+by)^p}{f(ax+by)^p}\right]^{1/p} \\
&\leqslant [af(x)+bf(y)]\left[1-\frac{g(ax+by)^p}{f(ax+by)^p}\right]^{1/p} \\
&\leqslant [af(x)+bf(y)]\left[1-\frac{af(x)^{1-p}g(x)^p+bf(y)^{1-p}g(y)^p}{af(x)+bf(y)}\right]^{1/p} \\
&\leqslant af(x)\left[1-\frac{g(x)^p}{f(x)^p}\right]^{1/p} + bf(y)\left[1-\frac{g(y)^p}{f(y)^p}\right]^{1/p} = ah(x)+bh(y).
\end{aligned}$$

We note that h is linear if and only if f and g are both linear on I and either $p=1$ or $f=g$. $\square$

THEOREM 2.2. *Let $n \geqslant 2$ be a natural number and take any $i \in \{1,\dots,n-1\}$. Let $0 < p \leqslant 1$. Define a sequence of functions $c_{i,k} : [i,i+1] \to \mathbb{R}$ for $k = i,\dots,n-1$ by $c_{i,i}(x) = (i+1-x)^p$ and recursively for $k = i+1,\dots,n-1$,*

$$c_{i,k}(x) = (k+1-x)^p - \sum_{l=i}^{k-1}(k+1-l)^p c_{i,l}(x).$$

Let $f : [i,n] \to [0,\infty)$ be decreasing and $f(n) = 0$. If f is convex, then

$$f(x)^p \leqslant \sum_{k=i}^{n-1} c_{i,k}(x) f(k)^p \tag{2.4}$$

holds for $x \in [i,i+1]$, where equality holds if and only if either $x = i$, $x = i+1$, or f is linear on $[i,n]$. If f is concave, then inequality (2.4) reverses.

Proof. From the definition of $c_{i,k}$, we have $c_{i,i}(x) = c_{i+1,i+1}(x+1)$ for any $x \in [i,i+1]$ and $i = 1,\dots,n-1$. Also, we may derive recursively

$$c_{i,k}(x) = c_{i+1,k+1}(x+1) \tag{2.5}$$

for $x \in [i,i+1]$ and $k = i,\dots,n-1$, where $i = 1,\dots,n-1$.

We prove (2.4) by induction on $n \geqslant 2$ and $1 \leqslant i < n$. First consider the case when $n = 2$ and $i = 1$. Since f is convex and $f(2) = 0$, we get

$$f(x) \leqslant (2-x)f(1) + (x-1)f(2) = (2-x)f(1)$$

for $x \in [1,2]$, where equality holds if and only if either $x = 1$, $x = 2$, or f is linear on $[1,2]$. Thus, we have (2.4) as $f(x)^p \leqslant (2-x)^p f(1)^p = c_{1,1}(x)f(1)^p$.

Next, for a fixed n, suppose that (2.4) holds when $i = j < n-1$. By extending $f = 0$ for $x \geqslant n$, define $g : [j,n] \to [0,\infty)$ as $g(x) = f(x+1)$. Since g is decreasing and convex, by applying the assumption of the induction when $i = j$ to the function g and noting $f(n) = 0$ and (2.5), we have

$$\begin{aligned} f(x+1)^p = g(x)^p &\leqslant \sum_{k=j}^{n-1} c_{j,k}(x) g(k)^p \\ &= \sum_{k=j}^{n-1} c_{j,k}(x) f(k+1)^p = \sum_{k=j+1}^{n-1} c_{j+1,k}(x+1) f(k)^p \end{aligned}$$

for $x \in [j, j+1]$, where equality holds if and only if either $x = j$, $x = j+1$, or f is linear on $[j, j+1]$. This assures (2.4) when $i = j+1$.

Next, for a fixed i, suppose that (2.4) holds when $n = m > i$. In order to consider the case when $n = m+1$, let f be decreasing and convex on $[i, m+1]$ with $f(m+1) = 0$. Then, note that f is linear on $[i, m+1]$ if and only if f is written as $f(x) = (m+1-x)f(m)$. From the definition of $c_{i,k}$, we may write

$$\begin{aligned} \sum_{k=i}^{m} c_{i,k}(x) f(k)^p &= \sum_{k=i}^{m-1} c_{i,k}(x) f(k)^p + \left((m+1-x)^p - \sum_{k=i}^{m-1} (m+1-k)^p c_{i,k}(x) \right) f(m)^p \\ &= \sum_{k=i}^{m-1} c_{i,k}(x) \Big[f(k)^p - (m+1-k)^p f(m)^p \Big] + (m+1-x)^p f(m)^p. \end{aligned} \tag{2.6}$$

Here, $f(x)^p - (m+1-x)^p f(m)^p \geqslant 0$ holds because f is convex and $f(m+1) = 0$. Define $h : [i,m] \to [0,\infty)$ by

$$h(x) = [f(x)^p - (m+1-x)^p f(m)^p]^{1/p}.$$

Then, from Lemma 2.1, h is convex and also decreasing because $h(m) = 0$. Hence, by applying the assumption of the induction when $n = m$ to the function h and using (2.6), we have

$$\begin{aligned} f(x)^p &= h(x)^p + (m+1-x)^p f(m)^p \\ &\leqslant \sum_{k=i}^{m-1} c_{i,k}(x) h(k)^p + (m+1-x)^p f(m)^p = \sum_{k=i}^{m} c_{i,k}(x) f(k)^p \end{aligned}$$

for $x \in [i, i+1]$, where equality holds if and only if either $x = i$, $x = i+1$, or h is linear on $[i,m]$. From Lemma 2.1, h is linear if and only if f is linear on $[i,m]$ and either

$p=1$ or $f(x)=(m+1-x)f(m)$; that is, f is linear on $[i,m+1]$ as remarked above. This assures (2.4) when $n=m+1$. This concludes the proof for the case when f is convex.

Next, we consider the case when f is concave on $[i,n]$. First consider the case when $i\leqslant n-2$. For $x\in[i,n-1]$, since $(n-x)f(n-1)\geqslant f(x)$, let

$$h(x)=[(n-x)^p f(n-1)^p - f(x)^p]^{1/p}.$$

Then, from Lemma 2.1, h is decreasing and convex on $[i,n-1]$ and $h(n-1)=0$. Hence, by applying (2.4) to the function h, for $x\in[i,i+1]$, we obtain

$$\begin{aligned} f(x)^p &\geqslant \sum_{k=i}^{n-2} c_{i,k}(x)f(k)^p + f(n-1)^p\left[(n-x)^p - \sum_{k=i}^{n-2} c_{i,k}(x)(n-k)^p\right] \\ &= \sum_{k=i}^{n-2} c_{i,k}(x)f(k)^p + f(n-1)^p c_{i,n-1}(x) = \sum_{k=i}^{n-1} c_{i,k}(x)f(k)^p. \end{aligned}$$

When $i=n-1$, clearly $f(x)^p\geqslant(i+1-x)^p f(i)^p = c_{i,i}(x)f(i)^p$ for $x\in[i,i+1]$. In both cases, equalities hold if and only if either $x=i$, $x=i+1$, or f is linear. $\square$

EXAMPLE 2.1. Let $0<p\leqslant 1$. Let f be decreasing and convex on $[1,4]$ with $f(4)=0$. Define three functions on $[1,2]$ as

$$\begin{aligned} c_{1,1}(x) &= (2-x)^p, \\ c_{1,2}(x) &= (3-x)^p - 2^p(2-x)^p, \\ c_{1,3}(x) &= (4-x)^p - 3^p(2-x)^p - 2^p\big((3-x)^p - 2^p(2-x)^p\big). \end{aligned}$$

Then, from Theorem 2.2, $f(x)^p\leqslant c_{1,1}(x)f(1)^p + c_{1,2}(x)f(2)^p + c_{1,3}(x)f(3)^p$ for $x\in[1,2]$.

REMARK 2.1. In Theorem 2.2, only when f is concave, the assumption that $f(n)=0$ may be relaxed to just $f(n)\geqslant 0$. This is clear from the latter part of the proof of Theorem 2.2. When f is concave and $f(n)>0$, then equality of the reverse of the inequality in (2.4) holds if and only if either of the following is satisfied:
(i) $i\leqslant n-2$ and either $x=i$, $x=i+1$, or f is linear with $p=1$; or
(ii) $i=n-1$ and $x=i$.

From Remark 2.1, we can study some properties of $c_{i,k}(x)$.

PROPOSITION 2.3. *For the sequence of functions $c_{i,k}$ for $k=i,\dots,n-1$ defined in Theorem 2.2, $c_{i,k}(x)\geqslant 0$ holds for $x\in[i,i+1]$, where equality holds if and only if one of the following is satisfied:*
(a) $k=i$ and $x=i+1$;
(b) $k=i+1$ and $x=i$; or

(c) $k \geqslant i+2$ *and either* $x=i$, $x=x+1$, *or* $p=1$.
Moreover, for $x \in [i,i+1]$ *with* $i \leqslant n-2$,

$$\sum_{k=i}^{n-1} c_{i,k}(x) \leqslant 1 \tag{2.7}$$

holds, where equality holds if and only if either $x=i$, $x=i+1$, *or* $p=1$.

Proof. When $k=i$, then $c_{i,i}(x) \geqslant 0$ and (a) is trivial. In fact, $c_{i,i}(x)$ is decreasing and concave with $c_{i,i}(i)=1$ and $c_{i,i}(i+1)=0$. Similarly, $c_{i,i+1}(x)$ is increasing and convex with $c_{i,i+1}(i)=0$ and $c_{i,i+1}(i+1)=1$. For $k=i+1,\dots,n-1$, let $f(x)=(k+1-x)$ for $x \in [i,i+1]$. Since $f(k)=1$, from the definition of $c_{i,k}(x)$ and from Remark 2.1, we have $c_{i,k}(x) \geqslant 0$ with equality conditions (b) and (c). Similarly, by letting $f(x)=1$ in Remark 2.1, we have (2.7). (When $i=n-1$, (2.7) also holds with equality only when $x=i$.) $\square$

From Proposition 2.3, when $p=1$, we have $c_{i,k}(x)=0$ for $k \geqslant i+2$ and inequality (2.4) coincides with the usual upper estimate of a convex function. For $x \in [i,i+1]$, the estimated upper-bound for $f(x)^p$ with (2.4) in Theorem 2.2 relies on all of the values of f at $i,i+1,\dots,n-1$, whereas the estimation with (2.1) in Lemma A relies on the values of f at only two points (i and $i+1$). In the following, we check that (2.4) gives a stronger (better) estimate than (2.1) when $f(n)=0$.

PROPOSITION 2.4. *Let* $f \geqslant 0$ *be decreasing and convex on* $[i,n]$ *and* $f(n)=0$. *Then, the right-hand side of (2.4) is less than the value obtained from the right-hand side of (2.1).*

Proof. From (2.1), if $f \geqslant 0$ is convex and decreasing on $[a,b]$, then, for $0<p\leqslant 1$,

$$f(x)^p \leqslant \left(\frac{b-x}{b-a}\right)^p f(a)^p + \left(1-\left(\frac{b-x}{b-a}\right)^p\right) f(b)^p$$

holds for $a \leqslant x \leqslant b$. Substituting $a=i$ and $b=i+1$, for $x \in [i,i+1]$, we obtain

$$f(x)^p \leqslant (i+1-x)^p f(i)^p + \big(1-(i+1-x)^p\big) f(i+1)^p.$$

Then, since f is decreasing and from $c_{i,k}(x) \geqslant 0$ and (2.7), both from Proposition 2.3,

$$\begin{aligned}
&(i+1-x)^p f(i)^p + \big(1-(i+1-x)^p\big) f(i+1)^p - \sum_{k=i}^{n-1} c_{i,k}(x) f(k)^p \\
&= \big[1-c_{i,i}(x)\big] f(i+1)^p - \sum_{k=i+1}^{n-1} c_{i,k}(x) f(k)^p \geqslant \left[1-\sum_{k=i}^{n-1} c_{i,k}(x)\right] f(i+1)^p \geqslant 0,
\end{aligned}$$

where all equalities hold if and only if either $x=i$, $x=i+1$, $p=1$, $i=n-1$, or $f(x)=0$ on $x \geqslant i+1$. $\square$

COROLLARY 2.5. *Let $n \geqslant 2$ be a natural number and take any $i \in \{1,\ldots,n-1\}$. Let $0 < p \leqslant 1$. Define $c_1 = 1/(p+1)$ and recursively for $k = 2,\ldots,n-i$,*

$$c_k = \int_{k-1}^{k} x^p dx - \sum_{l=1}^{k-1} (k+1-l)^p c_l. \tag{2.8}$$

Let $f : [i,n] \to [0,\infty)$ be decreasing and $f(n) = 0$. If f is convex, then

$$\int_i^{i+1} f(x)^p dx \leqslant \sum_{k=i}^{n-1} c_{k-i+1} f(k)^p \tag{2.9}$$

with equality if and only if f is linear on $[i,n]$. If f is concave, then the inequality reverses.

Proof. From the definition of $c_{i,k}(x)$ in Theorem 2.2 and from (2.8),

$$c_1 = \int_i^{i+1} c_{i,i}(x) dx$$

and recursively for $k = i+1,\ldots,n-1$,

$$c_{k-i+1} = \int_i^{i+1} (k+1-x)^p dx - \sum_{l=i}^{k-1} (k+1-l)^p c_{l-i+1} = \int_i^{i+1} c_{i,k}(x) dx.$$

Hence, the required statement follows by integrating (2.4) over $x \in [i,i+1]$. $\square$

From Proposition 2.3, note that $c_k \geqslant 0$ for $k = 1,\ldots,n-i$ and

$$\sum_{k=1}^{n-i} c_k \leqslant 1,$$

where the latter equality holds if and only if $p = 1$. Also, the equality $c_k = 0$ holds if and only if $p = 1$ and $k \geqslant 3$. From Proposition 2.4, the inequality (2.9) gives a stronger (better) estimate than (2.2) in Lemma A.

EXAMPLE 2.2. As in Example 2.1, let $0 < p \leqslant 1$ and let f be decreasing and convex on $[1,4]$ with $f(4) = 0$. Then,

$$\int_1^2 f(x)^p dx \leqslant \frac{1}{p+1} \left[f(1)^p + (2^p - 1) f(2)^p + (2 \cdot 3^p - 2^p - 4^p) f(3)^p \right]$$

holds with equality if and only if f is linear on $[1,2]$. By letting $c_3 = 2 \cdot 3^p - 2^p - 4^p$, note that $c_3 \geqslant 0$ holds with equality if and only if $p = 1$.

3. Proof of the main Theorems

First, we prove the following theorem as a direct application of Corollary 2.5. Hereafter, we admit the case when $n=1$ as a trivial case.

THEOREM 3.1. *Let n be a natural number and take any $i \in \{1,\dots,n-1\}$. Let $0<p\leqslant 1$. Let $f:[i,n]\to[0,\infty)$ be decreasing and $f(n)=0$. If f is convex, then*

$$\sum_{j=1}^{n-i} j^p \int_{i+j-1}^{i+j} f(x)^p dx \leqslant \sum_{j=1}^{n-i}\left(\int_{j-1}^{j} x^p dx\right) f(i+j-1)^p$$

with equality if and only if f is linear on $[i,n]$. If f is concave, then the inequality reverses.

Proof. Define $c_k \geqslant 0$ for $k=1,\dots,n-i$ by Corollary 2.5. Then, from (2.9),

$$\int_{i+j-1}^{i+j} f(x)^p dx \leqslant \sum_{k=i+j-1}^{n-1} c_{k-i-j+2} f(k)^p = \sum_{k=j}^{n-i} c_{k-j+1} f(k+i-1)^p.$$

Hence, by multiplying both sides with j^p, then summing over $j=1,\dots,n-i$, and using (2.8), we obtain

$$\begin{aligned}
\sum_{j=1}^{n-i} j^p \int_{i+j-1}^{i+j} f(x)^p dx &\leqslant \sum_{j=1}^{n-i} j^p \sum_{k=j}^{n-i} c_{k-j+1} f(k+i-1)^p \\
&= \sum_{j=1}^{n-i}\left(\sum_{k=1}^{j} k^p\cdot c_{j-k+1}\right) f(j+i-1)^p \\
&= \sum_{j=1}^{n-i}\left(\sum_{k=1}^{j} (j-k+1)^p\cdot c_k\right) f(j+i-1)^p \\
&= \sum_{j=1}^{n-i}\left(\int_{j-1}^{j} x^p dx\right) f(j+i-1)^p,
\end{aligned}$$

where equality holds if and only if f is linear on $[i,n]$. □

Let us now prove Theorem 1.2.

Proof. From Theorem 3.1,

$$\begin{aligned}
\sum_{i=1}^{n-1} i^p \int_i^n f(x)^p dx &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} j^p \int_{i+j-1}^{i+j} f(x)^p dx \\
&\leqslant \sum_{i=1}^{n-1}\sum_{j=1}^{n-i}\left(\int_{j-1}^{j} x^p dx\right) f(i+j-1)^p \\
&= \sum_{i=1}^{n-1}\left(\int_0^i x^p dx\right) f(i)^p,
\end{aligned}$$

where equality holds if and only if f is linear on $[1,n]$. □

REMARK 3.1. By substituting $i=1$ in Theorem 3.1, we also obtain the following two identical inequalities. (Each sides of these two inequalities are equal to each other.)

$$\sum_{i=1}^{n-1} i^p \int_i^{i+1} f(x)^p dx \leqslant \sum_{i=1}^{n-1} \left(\int_{i-1}^{i} x^p dx\right) f(i)^p,$$

$$\sum_{i=1}^{n-1} (i^p-(i-1)^p) \int_i^{n} f(x)^p dx \leqslant \sum_{i=1}^{n-1} \left(\int_{0}^{i} x^p dx\right) (f(i)^p - f(i+1)^p).$$

We may also rewrite the expression in Theorem 1.2 as

$$\sum_{i=1}^{n-1} \left(\sum_{j=1}^{i} j^p\right) \int_i^{i+1} f(x)^p dx \leqslant \sum_{i=1}^{n-1} \left(\int_{i-1}^{i} x^p dx\right) \sum_{j=i}^{n-1} f(j)^p.$$

We obtain the following proposition as a simple application of Theorem 1.2.

PROPOSITION 3.2. *Let n be a natural number. Let $0<p\leqslant 1$. If $q\geqslant 1$, then,*

$$\frac{1}{pq+1}\sum_{i=1}^{n-1} i^p (n-i)^{pq+1} \leqslant \frac{1}{p+1}\sum_{i=1}^{n-1} i^{p+1}(n-i)^{pq}$$

with equality if and only if $q=1$. If $0<q\leqslant 1$, then the inequality reverses. The statement also holds when the terms i and $n-i$ are swapped on any side of the inequality.

Proof. By defining $f(x)=(n-x)^q$ in Theorem 1.2, we obtain the required result. □

We are now in a position to prove Theorem 1.1.

Proof. Extend $f=f'=0$ on $x\geqslant 1$. Let

$$V_r(f) = \frac{1}{r}\sum_{i=1}^{\lfloor r\rfloor} \left(\frac{i}{r}\right)^p f\left(\frac{i}{r}\right).$$

Since $f(1)=\lim_{x\to 1} f'(x)=0$, the expression $V_r(f)$ is differentiable with respect to r, as

$$\begin{aligned} {V_r}'(f) &= -\frac{1}{r^2}\sum_{i=1}^{\lfloor r\rfloor}\left(\frac{i}{r}\right)^p f\left(\frac{i}{r}\right) + \frac{1}{r}\sum_{i=1}^{\lfloor r\rfloor}\left(\frac{-i}{r^2}\right)\left[p\left(\frac{i}{r}\right)^{p-1} f\left(\frac{i}{r}\right) + \left(\frac{i}{r}\right)^p f'\left(\frac{i}{r}\right)\right] \\ &= -\frac{1}{r^2}\sum_{i=1}^{\lfloor r\rfloor}\left[(p+1)\left(\frac{i}{r}\right)^p f\left(\frac{i}{r}\right) + \left(\frac{i}{r}\right)^{p+1} f'\left(\frac{i}{r}\right)\right]. \end{aligned}$$

Let $n=\lfloor r\rfloor+1$. First, consider case (i): let $0<p\leqslant 1$ and $f'(x)^{1/p}$ be convex, where $f'\geqslant 0$ follows from the given conditions of f. Define $g:(0,n]\to[0,\infty)$ by $g(x)=$

$f'(x/r)^{1/p}$ for $x \in (0,n]$. Then, since $g(x)$ is decreasing and convex on $(0,n]$ with $g(n)=0$, $f(n/r)=0$, and $0<p\leqslant 1$, from Theorem 1.2, we have

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{n-1}\left(\int_0^i x^p dx\right) f'\left(\frac{i}{r}\right) - \sum_{i=1}^{n-1} i^p \int_i^n f'\left(\frac{x}{r}\right) dx \\ &= \frac{1}{p+1} r^{p+1} \sum_{i=1}^{\lfloor r \rfloor}\left[\left(\frac{i}{r}\right)^{p+1} f'\left(\frac{i}{r}\right) + (p+1)\left(\frac{i}{r}\right)^p f\left(\frac{i}{r}\right)\right] \end{aligned} \tag{3.1}$$

with equality if and only if $f'(x)^p$ is linear and r is a natural number. Thus, $V_f{}'(r)\leqslant 0$. Next, consider case (ii): let $p\leqslant 0$. In this case, it is sufficient to check only the case when $p=0$ because the function $g(x)=x^p f(x)$ is concave and $g(x)=\lim_{x\to 1} g'(x)=0$ for all $p\leqslant 0$. Let $p=0$. Then, since f' is decreasing, we have

$$V_r{}'(f) = -\frac{1}{r^2}\sum_{i=1}^{\lfloor r \rfloor}\left[f\left(\frac{i}{r}\right)+\frac{i}{r}f'\left(\frac{i}{r}\right)\right] = \frac{1}{r^2}\sum_{i=1}^{\lfloor r \rfloor}\left[\int_{i/r}^1 f'(x)dx - \sum_{j=i}^{\lfloor r \rfloor} f'\left(\frac{j}{r}\right)\right] \leqslant 0.$$

Thus, when $p\leqslant 0$, we have $V_r{}'(f)\leqslant 0$ with equality if and only if $f(x)$ is linear, r is a natural number, and $p=0$. $\square$

By using Theorem 1.1, we may give a partial solution to an open question posed by Bennett and Jameson [2, Proposition 13 and related Remark]. They showed that

$$\frac{1}{n^{p+2}}\sum_{i=1}^{n-1} i^p (n-i)$$

increases with n when $0<p\leqslant 1$ and also $p\leqslant 0$. The case when $p>1$ was left as the open question. Here, we show that the expression also increases with n when $p\geqslant 2$.

PROPOSITION 3.3. *Let $r\geqslant 1$ be a real number. Let $0<p\leqslant 1$ and $q\geqslant 1$. Then,*

$$\frac{1}{r^{p+pq+2}}\sum_{i=1}^{\lfloor r \rfloor} i^p (r-i)^{pq+1} \tag{3.2}$$

increases with r.

Proof. Let $f(x)=-(1-x)^{pq+1}$. Then, f is increasing and concave with $f(1)=0$. Also, $f'(x)^{1/p}=(pq+1)^{1/p}(1-x)^q$ is convex and $\lim_{x\to 1} f'(x)=0$. Hence, from Theorem 1.1, the expression (3.2) increases with r. From Theorem 1.1, note that the expression (3.2) also increases when $p\leqslant 0$ and $q\leqslant 0$. $\square$

From Proposition 3.3, for $0<p\leqslant 1$ and $q\geqslant 1$, the expression

$$\frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{i}{n}\right)^p\left(1-\frac{i}{n}\right)^{pq+1} = \frac{1}{n}\sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)^p\left(\frac{i}{n}\right)^{pq+1} \tag{3.3}$$

increases with a natural number n. By fixing $p=1$ on the right-hand side of (3.3), we obtain the solution to the question posed by Bennett and Jameson for $p\geqslant 2$. Unfortunately, only the case when $1<p<2$ remains to be established.

4. Relation to local minimum energy problem

We have been interested in energy conditions of a system in which optimal point distributions are assured as the minimum energy state. In [9] and [10], we showed that an energy defined by a certain strong convex function assures optimal point distributions globally in a one-dimensional space and locally in a two-dimensional space. In these studies, we have regarded the optimal point distributions as ideal packing structures. Although general two-dimensional packing structures are unknown [5, D1] and global minimum analysis in two-dimensional space is difficult, we wonder if we may obtain a refinement for the two-dimensional case by studying another local minimum energy condition for the ideal hexagonal packing structures.

A key to this consideration lies in the studies of the Epstein zeta-function on lattices. Given a lattice L in the Euclidian plane, the Epstein zeta-function is defined by

$$\zeta(L,s) = \sum_{\mathbf{x}\in L\setminus\{\mathbf{0}\}} \frac{1}{|\mathbf{x}|^{2s}}$$

for $s>0$. Rankin (1953) [13] proved that $\zeta(\cdot,s)$ has a minimum value at the (equilateral) triangular lattice among all lattices of determinant 1 when $s \geqslant 1.035$. Cassels (1959) [3], Ennola (1964) [7], and Diananda (1964) [6] proved that the same holds for any $s>0$. We also refer to Gruber's related detailed surveys on this topic [8]. From these studies, we may consider a problem of finding a condition in which the energy has a local minimal value at the triangular lattice (as the center points of circles in hexagonal packing) among lattices of determinant 1. Because of the difficulty in treating lattices in two-dimensional space, we will also rely on a simple approximation for defining distances between the origin and triangular lattice points. Under this approximation, we may directly utilize Theorem 1.2.

4.1. Definition

We use similar definitions as in [10].

For a point set $X\subset\mathbb{R}^2$ and $f:(0,\infty)\to\mathbb{R}$, let *the energy (or generally, potential) at a point* $x\in X$ be

$$J(X,x,f) = \sum_{y\in X\setminus\{x\}} f(\|x-y\|),$$

where $\|\cdot\|$ is the Euclidean norm. This definition of the energy is for ease of analysis. When X is a finite set and $f(0)$ is defined, then we may define the energy at $x\in X$ as the average of $f(\|x-y\|)$ for all $y\in X$. The same results as the following study in this section 4 may be obtained also under this definition.

Let $d=2^{1/2}\cdot 3^{-1/4}$, $\mathbf{v}_1=\left(1/2,\sqrt{3}/2\right)$, and $\mathbf{v}_2=\left(1/2,-\sqrt{3}/2\right)$. Let a triangular lattice of determinant 1 as

$$\Lambda = \left\{d(i\mathbf{v}_1+j\mathbf{v}_2) : i\in\mathbb{N}, j=0,\ldots,i-1\right\}.$$

In addition, let Λ^* be the unions of the rotations of Λ around the origin by angles $\pi/3\cdot j$ for $j=0,\ldots,5$. Then, it can be easily checked that $\Lambda^*=\{d(i\mathbf{v}_1+j\mathbf{v}_2) : i,j\in\mathbb{Z}\}\setminus\{\mathbf{0}\}$.

4.2. Analytical condition for local minimum energy

First, we derive analytical energy conditions such that the triangular lattice is the locally optimal configuration among planar lattices of determinant 1, which means that when Λ^* is just continuously deformed into another lattice while keeping the determinant 1, then the energy increases. The imposition of the determinant being 1 is for fixing the distribution ratio of lattice points in all planar lattices.

PROPOSITION 4.1. *Let X be planar lattice points of determinant 1 containing the point $\mathbf{0}$. Let $r > 2^{1/2} \cdot 3^{-1/4}$. Let $f : (0,\infty) \to \mathbb{R}$ belong to the class C^2 and $f(x) = 0$ on $x \geqslant r$. If*

$$\sum_{\mathbf{v}\in\Lambda} \Big[|\mathbf{v}|^2 f''(|\mathbf{v}|) + 3|\mathbf{v}| f'(|\mathbf{v}|) \Big] > 0, \tag{4.1}$$

then the energy $J(X,\mathbf{0},f)$ has a local minimum at $X = \Lambda^* \cup \{\mathbf{0}\}$.

Proof. Every lattice in a plane can be expressed as the square of a positive definite quadratic form

$$xi^2 + 2zij + yj^2$$

for $i,j \in \mathbb{Z}$. Based on the assumption, in order to consider only lattice points of determinant 1, by constraining $xy - z^2 = 1$, let

$$h(x,y) = \sum_{(i,j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} g\big(xi^2 + 2\sqrt{xy-1}\,ij + yj^2\big),$$

where $g(x) = f(x^{1/2})$. Let $L_{(x,y)}$ denote the above defined lattice (containing the origin $\mathbf{0}$) parameterized by $(x,y) \in \mathbb{R}^2$. Then, we have $J(L_{(x,y)},\mathbf{0},f) = h(x,y)$. Since $f = f' = f'' = 0$ for $x \geqslant r$ from the assumption, the function h is twice partially differentiable with respect to both x and y.

We make an extreme value analysis for h at $(x,y) = (2\cdot 3^{-1/2}, 2\cdot 3^{-1/2})$, where the lattice conforms with Λ^*, the triangular lattice of determinant 1. For convenience, let

$$q(x,y) = xi^2 + 2\sqrt{xy-1}\,ij + yj^2.$$

Then, the derivatives of h are given by

$$\frac{\partial h}{\partial x} = \sum_{(i,j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \left(i^2 + \frac{yij}{\sqrt{xy-1}} \right) g'\big(q(x,y)\big),$$

$$\frac{\partial h}{\partial y} = \sum_{(i,j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \left(j^2 + \frac{xij}{\sqrt{xy-1}} \right) g'\big(q(x,y)\big),$$

and also second-order derivatives are given by

$$\frac{\partial^2 h}{\partial x^2} = \sum_{(i,j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \left[\left(i^2 + \frac{yij}{\sqrt{xy-1}} \right)^2 g''\big(q(x,y)\big) - \frac{y^2 ij}{2(xy-1)^{\frac{3}{2}}} g'\big(q(x,y)\big) \right],$$

$$\frac{\partial^2 h}{\partial y^2} = \sum_{(i,j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \left[\left(j^2 + \frac{xij}{\sqrt{xy-1}} \right)^2 g''\big(q(x,y)\big) - \frac{x^2 ij}{2(xy-1)^{\frac{3}{2}}} g'\big(q(x,y)\big) \right],$$

and

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x} = \sum_{(i,j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \Bigg[\left(i^2 + \frac{yij}{\sqrt{xy-1}} \right) \left(j^2 + \frac{xij}{\sqrt{xy-1}} \right) g''\big(q(x,y)\big) + \left(\frac{ij}{\sqrt{xy-1}} - \frac{xyij}{2(xy-1)^{\frac{3}{2}}} \right) g'\big(q(x,y)\big) \Bigg].$$

Thus, at $(x,y) = (2\cdot 3^{-1/2}, 2\cdot 3^{-1/2})$, we have

$$\frac{\partial h}{\partial x} = \sum_{(i,j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} (i^2+2ij)\cdot g'\left(\frac{2}{\sqrt{3}}(i^2+ij+j^2) \right),$$

$$\frac{\partial h}{\partial y} = \sum_{(i,j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} (j^2+2ij)\cdot g'\left(\frac{2}{\sqrt{3}}(i^2+ij+j^2) \right).$$

Further, at $(x,y) = (2\cdot 3^{-1/2}, 2\cdot 3^{-1/2})$, we have

$$\frac{\partial^2 h}{\partial x^2} = \sum_{(i,j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \left[(i^2+2ij)^2 g''\left(\frac{2}{\sqrt{3}}(i^2+ij+j^2) \right) - 2\sqrt{3}ij g'\left(\frac{2}{\sqrt{3}}(i^2+ij+j^2) \right) \right],$$

$$\frac{\partial^2 h}{\partial y^2} = \sum_{(i,j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \left[(j^2+2ij)^2 g''\left(\frac{2}{\sqrt{3}}(i^2+ij+j^2) \right) - 2\sqrt{3}ij g'\left(\frac{2}{\sqrt{3}}(i^2+ij+j^2) \right) \right],$$

and

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial^2 h}{\partial y \partial x} = \sum_{(i,j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \Bigg[(i^2+2ij)(j^2+2ij) g''\left(\frac{2}{\sqrt{3}}(i^2+ij+j^2) \right) - \sqrt{3}ij g'\left(\frac{2}{\sqrt{3}}(i^2+ij+j^2) \right) \Bigg].$$

Let $\mathbf{u}_1 = (1/2, \sqrt{3}/2)$, $\mathbf{u}_2 = (1,0)$, and $d = 2^{1/2}\cdot 3^{-1/4}$. From the hexagonal isotropic nature of triangular lattices, for each point $d(p\mathbf{u}_1 + q\mathbf{u}_2)$ on Λ for $p \in \mathbb{N}$ and $q \in \mathbb{N}\cup\{0\}$, we can choose six points of the form $d(i\mathbf{u}_1 + j\mathbf{u}_2)$ on Λ^* that are equidistant from the origin and form the vertices of a regular hexagon. Such pairs of (i,j) with respect to a given (p,q) are given by

$$H_{(p,q)} = \{(p,q), (-q,p+q), (-p-q,p), (-p,-q), (q,-p-q), (p+q,-p)\}.$$

Then, for each $(p,q) \in \mathbb{N} \times (\mathbb{N} \cup \{0\})$, we obtain

$$\begin{aligned}
\sum_{(i,j)\in H_{(p,q)}} ij &= -2(p^2+pq+q^2),\\
\sum_{(i,j)\in H_{(p,q)}} (i^2+2ij) = \sum_{(i,j)\in H_{(p,q)}} (j^2+2ij) &= 0,\\
\sum_{(i,j)\in H_{(p,q)}} (i^2+2ij)^2 = \sum_{(i,j)\in H_{(p,q)}} (j^2+2ij)^2 &= 4(p^2+pq+q^2)^2,\\
\sum_{(i,j)\in H_{(p,q)}} (i^2+2ij)(j^2+2ij) &= 2(p^2+pq+q^2)^2.
\end{aligned}$$

Hence, since $g(x) = f(x^{1/2})$, from the fact that

$$\begin{aligned}
g'(x) &= 1/2 \cdot f'(x^{\frac{1}{2}})/x^{\frac{1}{2}},\\
g''(x) &= 1/(4x) \cdot \left[f''(x^{\frac{1}{2}}) - f'(x^{\frac{1}{2}})/x^{\frac{1}{2}}\right],
\end{aligned}$$

at $(x,y) = (2\cdot 3^{-1/2}, 2\cdot 3^{-1/2})$, we obtain

$$\begin{aligned}
\frac{\partial^2 h}{\partial x^2} &= \sum_{(p,q)\in\mathbb{N}\times(\mathbb{N}\cup\{\mathbf{0}\})} \sum_{(i,j)\in H_{(p,q)}} \left[(i^2+2ij)^2 g''\left(\frac{2}{\sqrt{3}}(p^2+pq+q^2)\right)\right.\\
&\qquad\qquad \left. -2\sqrt{3}ij g'\left(\frac{2}{\sqrt{3}}(p^2+pq+q^2)\right)\right]\\
&= \sum_{\mathbf{v}\in\Lambda}\left[3|\mathbf{v}|^4 g''(|\mathbf{v}|^2) + 6|\mathbf{v}|^2 g'(|\mathbf{v}|^2)\right] = \frac{3}{4}\sum_{\mathbf{v}\in\Lambda}\left[|\mathbf{v}|^2 f''(|\mathbf{v}|) + 3|\mathbf{v}| f'(|\mathbf{v}|)\right].
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} &= 0,\\
\frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 h}{\partial y^2} = 2\frac{\partial^2 h}{\partial x \partial y} = 2\frac{\partial^2 h}{\partial y \partial x} &= \frac{3}{4}\sum_{\mathbf{v}\in\Lambda}\left[|\mathbf{v}|^2 f''(|\mathbf{v}|) + 3|\mathbf{v}| f'(|\mathbf{v}|)\right].
\end{aligned}$$

Hence, $\nabla h = \mathbf{0}$ holds at $(x,y) = (2\cdot 3^{-1/2}, 2\cdot 3^{-1/2})$ and the discriminant is positive. Further, from the assumption of (4.1), we have $\frac{\partial^2 h}{\partial x^2} > 0$. These yield that h has a local minimum at $(x,y) = (2\cdot 3^{-1/2}, 2\cdot 3^{-1/2})$. □

4.3. Approximative local minimum energy condition

Let $\{a_i\}$ be the sequence obtained by sorting the value $|\mathbf{v}|$ for all $\mathbf{v} \in \Lambda$ in increasing order. Then, we may consider a generalization of inequality (4.1) as

$$\sum_{i=1}^{\infty}\left[a_i^{m-1} f''(a_i) + m a_i^{m-2} f'(a_i)\right] > 0, \tag{4.2}$$

where $m \geqslant 1$ is a real number. Inequality (4.1) is the case when $m=3$ in (4.2). Moreover, in [10], we have derived a sufficient condition for the energy J at a point such that each triangular lattice point is locally optimal with respect to the energy, which means that a small perturbation of a triangular lattice point increases the energy, as in the case when $m=1$ in (4.2). Although we have derived the actual conditions of f for satisfying (4.2) with $m=1$ in [10], the proof is overly complicated because of the essential difficulties in treating lattice points that are represented by two variables. Here, in order to consider inequality (4.1), we would take a reasonable approximative approach that uses a single variable.

Consider the case that there are k triangular lattice points in a circle of radius $r>1$ centered at the origin. Then, the area of the circle, πr^2, can be approximated by the total area of k identical equilateral triangles of the area $\sqrt{3}/2$. Here, if $r=a_i$, we have $k=6i$. Thus, we have $a_i \approx b_i$, where

$$b_i = 3^{\frac{3}{4}} \cdot \pi^{-\frac{1}{2}} \cdot i^{\frac{1}{2}}.$$

Thus, by considering the substitution a_i with b_i in (4.2), we may consider an approximative condition for the function f so that inequality (4.2) holds. That is, by omitting the constant coefficients of b_i and substituting a_i with b_i in (4.2), we may consider an approximation of inequality (4.2),

$$\sum_{i=1}^{\infty}\left[\left(\frac{i}{r}\right)^{\frac{m-1}{2}} f''\left(\left(\frac{i}{r}\right)^{\frac{1}{2}}\right)+m\left(\frac{i}{r}\right)^{\frac{m}{2}-1} f'\left(\left(\frac{i}{r}\right)^{\frac{1}{2}}\right)\right]>0, \tag{4.3}$$

where the variable i is divided by $r \geqslant 1$ so as to substantially restrict the domain of f to $[0,1]$ for ease of consideration. Inequality (4.3) with $m=3$ is an approximation of inequality (4.1), which we would like to solve.

There may be another consideration for inequality (4.2). In the expression $|\mathbf{v}| = [i^2-ij+j^2]^{1/2}/r$ for $i \in \mathbb{N}$, and $j=0,\dots,i-1$, we may consider rough estimation by fixing $|\mathbf{v}| = i/r$ for all $j=0,\dots,i-1$. Then, as a rough approximation of (4.2), we may also consider the inequality

$$\sum_{i=1}^{\infty}\left[\left(\frac{i}{r}\right)^{m} f''\left(\frac{i}{r}\right)+m\left(\frac{i}{r}\right)^{m-1} f'\left(\frac{i}{r}\right)\right]>0. \tag{4.4}$$

For the three inequalities (4.2), (4.3), and (4.4), the key conditions are the two convexity conditions for $f''(x)^p$ and $(f''(x^{1/2})/x^{1/2})^p$ that indicate the "strength of the convexity" of f, where a smaller p yields stronger convexities. In Table 4.1, we list in advance the required conditions of the values p so that inequalities (4.2), (4.3), and (4.4) are satisfied for each $m=1,2$, and 3. In the table, an asterisk means that strict convexity or concavity is required.

In the following, we prove the conditions given in Table 4.1 except for the results that have been already obtained in [10]. First, we show the following simple lemma for giving a relation between two types of convexities.

Table 4.1: *Requirements for a convex function f for satisfying (4.2), (4.3), and (4.4).*

	convexity of $f''(x)^p$ type	convexity of $(f''(x^{1/2})/x^{1/2})^p$ type
(4.2) with $m=1$	f' is concave ([10, Thm. 5.2])	$p=17.048$ ([10, Thm. 5.2])
(4.2) with $m=2$	unknown	unknown
(4.2) with $m=3$	unknown	unknown
(4.3) with $m=1$	f' is concave	$f'(x^{1/2})$ is concave (Prop. 4.4)
(4.3) with $m=2$	f' is concave*	$f'(x^{1/2})$ is concave (Prop. 4.4)*
(4.3) with $m=3$	$p=2$ (Lemma 4.2)*	$p=2$ (Prop. 4.3)*
(4.4) with $m=1$	f' is concave ([10, Prop. 5.1])	$p=2$ ([10, Prop. 5.1])
(4.4) with $m=2$	$p=1$ (Prop. 4.3)*	unknown
(4.4) with $m=3$	unknown	unknown

LEMMA 4.2. *Let $0<p$ and $0<q<1$. Let $f:(0,\infty)\to[0,\infty)$. If $f(x)^p$ is decreasing and convex, then $(f(x^q)/x^q)^p$ is also decreasing and convex.*

Proof. Let $a,b\geqslant 0$ with $a+b=1$. Since x^q is concave, $ax^q+by^q\leqslant(ax+by)^q$ holds. Thus, since $f(x)^p$ is decreasing and convex, we have

$$f((ax+by)^q)^p\leqslant f(ax^q+by^q)^p\leqslant af(x^q)^p+bf(y^q)^p,$$

from which $f(x^q)^p$ is convex. The fact that $f(x^q)^p$ is decreasing is trivial. In general, for decreasing convex functions g and h defined on $(0,\infty)$, $g(x)h(x)$ is non-negative decreasing and convex. Thus, by letting $g(x)=f(x^q)^p$ and $h(x)=1/x^{pq}$, the function $(f(x^q)/x^q)^p$ is decreasing and convex because of the convexities of g and h. $\square$

Next, we show the conditions required for satisfying (4.4) with $m=2$ and (4.3) with $m=3$ presented in Table 4.1. For the latter case, from Lemma 4.2, the condition given in the right column of the table is the essential condition we need to prove.

PROPOSITION 4.3. *Let $f:(0,\infty)\to[0,\infty)$ belong to the class C^2 and be decreasing and convex and $f=f'=f''=0$ for $x\geqslant 1$. Let $r\geqslant 1$ and $p\geqslant 1$. If $f''(x)^p$ is convex, then*

$$\sum_{i=1}^{\infty}\left[\left(\frac{i}{r}\right)^{\frac{p+1}{p}}f''\left(\frac{i}{r}\right)+\frac{p+1}{p}\left(\frac{i}{r}\right)^{\frac{1}{p}}f'\left(\frac{i}{r}\right)\right]\geqslant 0 \tag{4.5}$$

with equality if and only if $f''(x)^p$ is linear on $(0,1]$ and r is a natural number. Also, if $(f''(x^{1/2})/x^{1/2})^p$ is convex, then

$$\sum_{i=1}^{\infty}\left[\left(\frac{i}{r}\right)^{\frac{1}{p}+\frac{1}{2}}f''\left(\left(\frac{i}{r}\right)^{\frac{1}{2}}\right)+\frac{2(p+1)}{p}\left(\frac{i}{r}\right)^{\frac{1}{p}}f'\left(\left(\frac{i}{r}\right)^{\frac{1}{2}}\right)\right]\geqslant 0 \tag{4.6}$$

with equality if and only if $(f''(x^{1/2})/x^{1/2})^p$ *is linear on* $(0,1]$ *and* r *is a natural number.*

Proof. Suppose that $f''(x)^p$ is convex. Substitute f and p in Theorem 1.1 for f' and $1/p$, respectively, where $f' \leqslant 0$ is certainly increasing and concave from the assumption. Then, (4.5) follows from (3.1). Note that $\lfloor r \rfloor$ in (3.1) may be substituted by ∞ because $f(x) = f'(x) = 0$ on $x \geqslant 1$. Next, suppose that $(f''(x^{1/2})/x^{1/2})^p$ is convex. Let $g''(x) = f''(x^{1/2})/x^{1/2}$. From the assumption, $g''(x)^p$ is convex and decreasing. Thus, from the fact that $g'(x) = 2f'(x^{1/2})$ and using (4.5), we obtain (4.6). □

REMARK 4.1. Proposition 4.3 may be rewritten for the exact forms (4.4) and (4.3). For $p \geqslant 1$, by letting $m = 1/p + 1 \in (1,2]$, if $f''(x)^{1/(m-1)}$ is strictly convex, then (4.4) holds. Also, by letting $m = 2/p + 2 \in (2,4]$, if $(f''(x^{1/2})/x^{1/2})^{2/(m-2)}$ is strictly convex, then (4.3) holds. Note that these restrictions arise in the ranges of m.

Finally, we check the conditions required for satisfying (4.3) with $m = 1$ and $m = 2$ presented in Table 4.1, where the concavity of $f'(x)$ in the left column immediately follows from the concavity of $f'(x^{1/2})$ in the right column.

PROPOSITION 4.4. *Let* $f : (0,\infty) \to \mathbb{R}$ *belong to the class* C^2 *and be decreasing and convex with* $f = f' = f'' = 0$ *for* $x \geqslant 1$. *If* $f'(x^{1/2})$ *is concave, then (4.3) holds for* $m < 2$. *Also, if* $f'(x^{1/2})$ *is strictly concave, then (4.3) holds for* $m = 2$.

Proof. Let $g(x) = f'(x^{1/2})$. Then, g is increasing and concave and $g = g' = 0$ on $x \geqslant 1$. Thus, from the proof of Theorem 1.1, we have $V_r{}'(g) \leqslant 0$ for $p \leqslant 0$ with equality if and only if g is linear, r is a natural number, and $p = 0$. By substituting $m = 2(p+1) \leqslant 2$, the left-hand side of (4.3) equals $-2r^2 V_r{}'(g)$. Hence, the required statements hold. □

4.4. Summary and Examples

From Propositions 4.1 and 4.3, if f belongs to the class C^2 and is decreasing and convex with $f = f' = f'' = 0$ on $x \geqslant 1$ and $(f''(x^{1/2})/x^{1/2})^2$ is strictly convex, then inequality (4.3) with $m = 3$ holds, which is the approximative condition so that the triangular lattice is the locally optimal configuration among planar lattices of determinant 1. This obtained condition for f requires stronger convexity than that given by condition (4.2) for $m = 1$, which was the local minimum energy condition for a local move of a noticed point on triangular lattice points studied in [10].

For giving examples of f satisfying the obtained conditions, it is convenient to normalize f on $[0,1]$ with the function $h : [0,1] \to [0,\infty)$ defined by

$$f(x) = \begin{cases} h(x/r) & x \leqslant r, \\ 0 & x > r \end{cases}$$

for $r > 0$. By utilizing this definition, any "sphere of influence" $r > 0$ may be imposed on the energy without changing the function h. Then, the conditions for h satisfying inequality (4.3) with $m = 3$ are summarized in the following three points.

(T1) $h(x)$ belongs to the class C^2 and is monotone decreasing and convex.
(T2) $h(1) = \lim_{x\to 1} h'(x) = \lim_{x\to 1} h''(x) = 0$.
(T3) $(h''(x^{1/2})/x^{1/2})^p$ is strictly convex, where $p = 2$.

With these conditions T1–T3, 1-D global minimum [9], 2-D local minimum [10], and 2-D approximative local minimum in this study are all satisfied. Thus, T1–T3 is the current best result for assuring stable good point distributions by means of the minimum energy.

EXAMPLE 4.1. Let $p > 1$. Define functions $h(x) = (1-x)^p$ and $h(x) = (2/3 - x + 1/3\cdot x^3)^p$. Then, the strict convexities of $(h''(x^{1/2})/x^{1/2})^2$ are achieved when $p > 2.5$ and $p > 1.25$, respectively. Table 4.2 shows general conditions so that $(h''(x^{1/2})/x^{1/2})^q$ is strictly convex. Hence, for assuring optimal point distributions with respect to the energy, we should require at least $(1-x)^p$ with $p > 2.5$ or $(2/3 - x + 1/3\cdot x^3)^p$ with $p > 1.25$.

Table 4.2: *List of power values of two functions for satisfying required convexities.*

	$(1-x)^p$	$(2/3 - x + 1/3\cdot x^3)^p$
$f'(x^{1/2})$ is strictly concave	$p > 2.0$	$p > 1.00$
$(f''(x^{1/2})/x^{1/2})^2$ is strictly convex	$p > 2.5$	$p > 1.25$
$f''(x^{1/2})/x^{1/2}$ is strictly convex	$p > 3.0$	$p > 1.50$
$(f''(x^{1/2})/x^{1/2})^{2/3}$ is strictly convex	$p > 3.5$	$p > 1.75$
$(f''(x^{1/2})/x^{1/2})^{1/2}$ is strictly convex	$p > 4.0$	$p > 2.00$

T1–T3 is still a necessary condition in two-dimensional space and it may not seem worthwhile to attempt a slight refinement. However, we think that T1–T3 (with a smaller p in T3) is nearly a sufficient condition from a practical standpoint for assuring stable good point distributions in a two-dimensional space [11]. Our study is motivated by a requirement for determining a spatial measure which can assure stable, good point distributions in a plane even if the space is discrete. T1–T3 is a suitable condition for this purpose.

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