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A PRIORI BOUNDS FOR ELLIPTIC OPERATORS IN WEIGHTED SOBOLEV SPACES

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Abstract. This paper is concerning with the study of a class of weight functions and their properties. As an application, we prove some a priori bounds for a class of uniformly elliptic second order linear differential operators in weighted Sobolev spaces.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$ (not necessarily bounded), $n \geqslant 3$. Assign in Ω the uniformly elliptic second order linear differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a. \tag{1.1}$$

The aim of this paper is to investigate about a new class of weight functions (introduced in [15]) and to obtain some a priori estimates for the operator L in weighted Sobolev spaces.

In particular, we are interested in the study of the functions $m : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y\in\Omega \\ |x-y|<d}} \frac{m(x)}{m(y)} < +\infty, \tag{1.2}$$

with $d \in \mathbb{R}_+$. Typical examples of such functions are:

$$m(x) = e^{t|x|}, \quad m(x) = (1+|x|^2)^t, \quad x \in \Omega, t \in \mathbb{R}.$$

Then we study the multiplication operator

$$u \to gu \tag{1.3}$$

defined in a weighted Sobolev space and which takes values in a weighted Lebesgue space. We give conditions on g and Ω so that the operator defined by (1.3) is bounded and other ones in order to get its compactness.

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As an application, we obtain some a priori estimates for the operator L. We recall that when Ω is bounded, the problem of determining a priori bounds has been investigated by several authors under various hypotheses on the leading coefficients. It is worth to mention the results proved in [10], [7], [8], [16], [17], where the coefficients a_{ij} are required to be discontinuous. If the open set Ω is unbounded, a priori bounds are established in [12], [2] with analogous assumptions to those required in [10], while in [6], [3], [4], under similar hypotheses asked in [7], [8], the above estimates are obtained. In this paper, we extend some results of [7], [8] to a weighted case.

Actually, assuming that the coefficients a_{ij} are locally VMO and "close" at infinity to certain functions e_{ij} of class VMO, and supposing that the lower – order coefficients verify suitable regularity hypotheses and have a certain behaviour at the infinity, we get the following a priori bound:

$$||u||_{W_s^{2,p}(\Omega)} \leqslant c\left(||Lu||_{L_s^p(\Omega)} + ||u||_{L^p(\Omega_1)} \right) \quad \forall\, u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega),$$

where $s \in \mathbb{R}, \Omega$ is sufficiently regular, $W_s^{2,p}(\Omega), \overset{\circ}{W}{}_s^{1,p}(\Omega)$ and $L_s^p(\Omega)$ are weighted Sobolev spaces in which the weight functions verify (1.2), $c \in \mathbb{R}_+$ is independent of u, and Ω_1 is a bounded open subset of Ω.

As a consequence of the above estimate we can say that the operator L has closed range and finite – dimensional kernel.

We wish to stress that an analogous estimate has been obtained in [5], in a different situation. Indeed, in [5] the open set Ω has singular boundary and the coefficients of the operator L are singular near a subset of $\partial\Omega$. Hence, in [5] the weight function goes to zero on such subset of $\partial\Omega$ and then also the weighted Sobolev spaces are different with respect to those considered in this paper.

2. Notation and function spaces

Let G be any Lebesgue measurable subset of $\mathbb{R}^n$ and $\Sigma(G)$ the collection of all Lebesgue measurable subsets of G. Let $F \in \Sigma(G)$ and $|F|$ denote the Lebesgue measure of F. Let χ_F be the characteristic function of F and $\mathfrak{D}(F)$ the class of restrictions to F of functions $\zeta \in C_\circ^\infty(\mathbb{R}^n)$ with $\bar{F} \cap \operatorname{supp} \zeta \subseteq F$. If $X(F)$ is a space of functions defined on F, $X_{\text{loc}}(F)$ denotes the class of all functions $g : F \to \mathbb{R}$ such that $\zeta g \in X(F)$ for any $\zeta \in \mathfrak{D}(F)$. Finally, for any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, we put $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$, $B_r = B(0,r)$ and $F(x,r) = F \cap B(x,r)$. We now recall the definitions of the function spaces in which the coefficients of the operator are chosen. Indeed, if Ω has the property

$$|\Omega(x,r)| \geqslant A\, r^n \qquad \forall x \in \Omega, \quad \forall r \in]0,1], \tag{2.1}$$

where A is a positive constant independent of x and r, then we can consider the space $BMO(\Omega,\tau)$ ($\tau \in \mathbb{R}_+$) of functions $g \in L^1_{\text{loc}}(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega,\tau)} = \sup_{\substack{x\in\Omega \\ r\in]0,\tau]}} \fint_{\Omega(x,r)} |g - \fint_{\Omega(x,r)} g| < +\infty,$$

with

$$\fint_{\Omega(x,r)} g = |\Omega(x,r)|^{-1} \int_{\Omega(x,r)} g.$$

If $g \in BMO(\Omega) = BMO(\Omega, \tau_A)$, and

$$\tau_A = \sup \left\{ \tau \in \mathbb{R}_+ \ : \ \sup_{\substack{x\in\Omega \\ r\in]0,\tau]}} \frac{r^n}{|\Omega(x,r)|} \leqslant \frac{1}{A} \right\},$$

we say that $g \in VMO(\Omega)$ if $[g]_{BMO(\Omega,\tau)} \to 0$ for $\tau \to 0^+$. A function

$$\eta[g] :]0,1] \longrightarrow \mathbb{R}_+$$

is called a *modulus of continuity* of g in $VMO(\Omega)$ if

$$[g]_{BMO(\Omega,\tau)} \leqslant \eta[g](\tau) \ \forall \tau \in]0,1], \quad \lim_{\tau\to 0^+} \eta[g](\tau) = 0.$$

For $t \in [1,+\infty[$ and $\lambda \in [0,n[$, $M^{t,\lambda}(\Omega)$ denotes the set of all functions g in $L^t_{loc}(\bar{\Omega})$ endowed with the following norm:

$$||g||_{M^{t,\lambda}(\Omega)} = \sup_{\substack{r\in]0,1] \\ x\in\Omega}} r^{-\lambda/t} ||g||_{L^t(\Omega(x,r))} < +\infty. \tag{2.2}$$

Then we define $\tilde{M}^{t,\lambda}(\Omega)$ as the closure of $L^\infty(\Omega)$ in $M^{t,\lambda}(\Omega)$ and $M^{t,\lambda}_\circ(\Omega)$ as the closure of $C^\infty_\circ(\Omega)$ in $M^{t,\lambda}(\Omega)$. In particular, we put $M^t(\Omega) = M^{t,0}(\Omega)$, $\tilde{M}^t(\Omega) = \tilde{M}^{t,0}(\Omega)$ and $M^t_\circ(\Omega) = M^{t,0}_\circ(\Omega)$. Recall that for a function $g \in M^{t,\lambda}(\Omega)$ the following characterization holds:

$$g \in \tilde{M}^{t,\lambda}(\Omega) \iff \lim_{\tau\to 0^+} p_g(\tau) = 0 \tag{2.3}$$

where

$$p_g(\tau) = \sup_{\substack{E\in\Sigma(\Omega) \\ \sup_{x\in\Omega} |E(x,1)|\leqslant\tau}} ||\chi_E g||_{M^{t,\lambda}(\Omega)}, \quad \tau \in \mathbb{R}_+.$$

Thus the *modulus of continuity* of $g \in \tilde{M}^{t,\lambda}(\Omega)$ is a function

$$\tilde{\sigma}[g] :]0,1] \longrightarrow \mathbb{R}_+$$

such that

$$p_g(\tau) \leqslant \tilde{\sigma}[g](\tau) \ \forall \tau \in]0,1], \quad \lim_{\tau\to 0^+} \tilde{\sigma}[g](\tau) = 0.$$

Furthermore, if $g \in M^{t,\lambda}(\Omega)$ then

$$g \in M^{t,\lambda}_\circ(\Omega) \iff \lim_{\tau\to 0^+} \left(p_g(\tau) + ||(1-\zeta_{1/\tau})g||_{M^{t,\lambda}(\Omega)} \right) = 0 \tag{2.4}$$

where ζ_r, $r \in \mathbb{R}_+$, is a function in $C^\infty_\circ(\mathbb{R}^n)$ such that

$$0 \leqslant \zeta_r \leqslant 1, \quad \zeta_{r|B_r} = 1, \quad \operatorname{supp} \zeta_r \subset B_{2r}.$$

Thus the *modulus of continuity* of $g \in M_{\circ}^{t,\lambda}(\Omega)$ is a function

$$\sigma_{\circ}[g] :]0,1] \longrightarrow \mathbb{R}_+$$

such that

$$p_g(\tau) + ||(1-\zeta_{1/\tau})g||_{M^{t,\lambda}(\Omega)} \leqslant \sigma_{\circ}[g](\tau) \quad \forall \tau \in]0,1], \quad \lim_{\tau \to 0^+} \sigma_{\circ}[g](\tau) = 0.$$

A more detailed account of properties of the above defined function spaces can be found in [11], [13] and [14].

3. Weight functions

Let Ω be an open subset of $\mathbb{R}^n$, $d \in \mathbb{R}_+$ and $G_d(\Omega)$ the set of all measurable functions $m : \Omega \to \mathbb{R}_+$ such that

$$\sup_{\substack{x,y\in\Omega \\ |x-y|<d}} \frac{m(x)}{m(y)} < +\infty. \tag{3.1}$$

It is easy to verify that $m \in G_d(\Omega)$ if and only if there exists $\gamma \in \mathbb{R}_+$ such that

$$\gamma^{-1} m(y) \leqslant m(x) \leqslant \gamma m(y) \qquad \forall y \in \Omega, \quad \forall x \in \Omega(y,d), \tag{3.2}$$

where $\gamma \in \mathbb{R}_+$ is independent of x and y.

Hence from (3.2) we get

$$m\,,\ m^{-1} \in L^{\infty}_{\mathrm{loc}}(\bar{\Omega}). \tag{3.3}$$

Let $G(\Omega)$ be the class of weight functions defined as follows:

$$G(\Omega) = \bigcup_{d \in \mathbb{R}_+} G_d(\Omega).$$

Hence, if $m \in G(\Omega)$ then:

$$m^s \in G(\Omega), \ \ \lambda m \in G(\Omega) \qquad \forall s \in \mathbb{R}, \forall \lambda \in \mathbb{R}_+.$$

LEMMA 3.1. *Let* m *be a positive function defined on* Ω. *If* $\log m \in \mathrm{Lip}(\Omega)$ *then* $m \in G(\Omega)$.

Proof. By the hypothesis, there is a constant $L \in \mathbb{R}_+$ such that for each $x, y \in \Omega$

$$|\log m(x) - \log m(y)| \leqslant L|x-y|. \tag{3.4}$$

For $x, y \in \Omega$ such that $|x-y| < d \ (d \in \mathbb{R}_+)$, from (3.4) we have

$$\left|\log \frac{m(x)}{m(y)}\right| \leqslant Ld \qquad \forall y \in \Omega, \quad \forall x \in \Omega(y,d),$$

and then the claimed implication. □

Examples of functions in $G(\Omega)$ are:

$$m(x) = e^{t|x|}, \quad m(x) = (1+|x|^2)^t, \quad x \in \Omega, t \in \mathbb{R}.$$

LEMMA 3.2. *If $m \in G(\Omega)$ and Ω has the cone property, then there exists a function $\sigma \in G(\Omega) \cap C^\infty(\bar{\Omega})$ such that*

$$c_1 m(x) \leqslant \sigma(x) \leqslant c_2 m(x) \qquad \forall x \in \Omega, \tag{3.5}$$

$$\sup_{x\in\Omega} \frac{|\partial^\alpha \sigma(x)|}{\sigma(x)} < +\infty \qquad \forall \alpha \in \mathbb{N}_0^n, \tag{3.6}$$

where $c_1, c_2 \in \mathbb{R}_+$ are dependent only on n, Ω, m.

Proof. Since $m \in G(\Omega)$ then there exists a positive number d such that $m \in G_d(\Omega)$. Assume $g \in C_\circ^\infty(\mathbb{R}^n)$ such that

$$g \geqslant 0, \ \ g_{|B_{\frac{1}{2}}} = 1, \ \ \operatorname{supp} g \subset B_1$$

and

$$\sigma : x \in \Omega \longrightarrow \int_\Omega m(y)\, g\Big(\frac{x-y}{d}\Big)\, dy.$$

Since

$$\sigma(x) = \int_{\Omega(x,d)} m(y)\, g\Big(\frac{x-y}{d}\Big)\, dy \qquad \forall x \in \Omega,$$

using (3.2), it follows (3.5). Thus $\sigma \in G_d(\Omega)$.

Again by (3.2), for all $\alpha \in \mathbb{N}_0^n$ and $x \in \Omega$, we have:

$$|\partial^\alpha \sigma(x)| \leqslant \gamma m(x) d^{-|\alpha|} \int_{\Omega(x,d)} \Big| g^{(|\alpha|)}\Big(\frac{x-y}{d}\Big)\Big|\, dy \leqslant c_3 m(x),$$

where c_3 depends on n, Ω, m, α, and then (3.6) follows. □

LEMMA 3.3. *If Ω has the property that there are $r_0 \in \mathbb{R}_+$ and $x_0 \in \Omega \backslash B_{r_0}$ such that for every $x \in \Omega \backslash B_{r_0}$ $\overline{xx_0} \subset \Omega$, then for any $m \in G(\Omega)$ and for every $x \in \Omega$,*

$$c_0^{-1} e^{-c|x|} \leqslant m(x) \leqslant c_0 e^{c|x|},$$

where c and c_0 depend only on n, Ω and m.

Proof. Fix $x \in \Omega$. If $x \in \Omega \backslash B_{r_0}$ then $\overline{xx_0} \subset \Omega$ and by Lagrange's theorem, using (3.6), we have

$$|\log \sigma(x) - \log \sigma(x_0)| \leqslant c|x - x_0|, \tag{3.7}$$

where $c \in \mathbb{R}_+$ depends on n, Ω, m. So, by an easy computation via (3.2), we have the result. Otherwise, if $x \in \Omega \cap B_{r_0}$, the result is obtained by (3.3). □

If $m \in G(\Omega)$, $k \in \mathbb{N}_0$, $1 \leqslant p < +\infty$ and $s \in \mathbb{R}$, let $W^{k,p}_s(\Omega)$ be the space of distributions u on Ω such that $m^s \partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leqslant k$, equipped with the norm

$$\|u\|_{W^{k,p}_s(\Omega)} = \sum_{|\alpha| \leqslant k} \|m^s \partial^\alpha u\|_{L^p(\Omega)}. \tag{3.8}$$

Moreover, denote by $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ the closure of $C^\infty_\circ(\Omega)$ in $W^{k,p}_s(\Omega)$ and put $W^{0,p}_s(\Omega) = L^p_s(\Omega)$.

From (3.6), by induction, we can deduce the following property of the function σ defined in Lemma 3.2:

$$\sup_{x \in \Omega} \frac{|\partial^\alpha \sigma^s(x)|}{\sigma^s(x)} < +\infty \qquad \forall \alpha \in \mathbb{N}^n_0, \qquad \forall s \in \mathbb{R}. \tag{3.9}$$

Now, by (3.9), we can easily deduce the following.

LEMMA 3.4. *Let $k \in \mathbb{N}_0$, $1 \leqslant p < +\infty$ and $s \in \mathbb{R}$. If Ω has the cone property, $m \in G(\Omega)$ and σ is the function defined in Lemma 3.2, then the map*

$$u \longrightarrow \sigma^s u$$

defines a topological isomorphism from $W^{k,p}_s(\Omega)$ to $W^{k,p}(\Omega)$ and from $\overset{\circ}{W}{}^{k,p}_s(\Omega)$ to $\overset{\circ}{W}{}^{k,p}(\Omega)$.

A more detailed account of properties of the above defined spaces can be found, for instance, in [15].

4. Some embedding results

Let m be a function of class $G(\Omega)$. We consider the following condition:

(h_0) Ω has the cone property, $p \in]1,+\infty[, s \in \mathbb{R}, k, t$ are numbers such that:

$$k \in \mathbb{N},\ t \geqslant p,\ t \geqslant \frac{n}{k},\ t > p \text{ if } p = \frac{n}{k},\ g \in M^t(\Omega).$$

By Theorem 3.1 of [9] we easily obtain the following.

THEOREM 4.1. *If the assumption (h_0) holds, then for any $u \in W^{k,p}_s(\Omega)$ we have $gu \in L^p_s(\Omega)$ and*

$$||gu||_{L^p_s(\Omega)} \leqslant c\,||g||_{M^t(\Omega)}||u||_{W^{k,p}_s(\Omega)}, \tag{4.1}$$

with c dependent only on Ω, n, k, p and t.

COROLLARY 4.2. *If the assumption (h_0) holds and $g \in \tilde{M}^t(\Omega)$, then for any $\varepsilon \in \mathbb{R}_+$ there exists a constant $c(\varepsilon) \in \mathbb{R}_+$ such that*

$$||gu||_{L^p_s(\Omega)} \leqslant \varepsilon||u||_{W^{k,p}_s(\Omega)} + c(\varepsilon)||u||_{L^p_s(\Omega)} \ \ \forall u \in W^{k,p}_s(\Omega), \tag{4.2}$$

where $c(\varepsilon)$ depends only on $\varepsilon, \Omega, n, k, p, t, \tilde{\sigma}[g]$.

Proof. Fix $\varepsilon > 0$ and let c be the constant in (4.1). Since $g \in \tilde{M}^t(\Omega)$, then there exists $g_\varepsilon \in L^\infty(\Omega)$ such that $||g - g_\varepsilon||_{M^t(\Omega)} < \frac{\varepsilon}{c}$. By Theorem 4.1

$$||gu||_{L^p_s(\Omega)} \leqslant c\,||g - g_\varepsilon||_{M^t(\Omega)}||u||_{W^{k,p}_s(\Omega)} + ||g_\varepsilon||_{L^\infty(\Omega)}||u||_{L^p_s(\Omega)}$$

for any u in $W^{k,p}_s(\Omega)$, and then the result follows. □

COROLLARY 4.3. *If the assumption* (h_0) *holds and* $g \in M^t_\circ(\Omega)$*, then for any* $\varepsilon \in \mathbb{R}_+$ *there exist a constant* $c(\varepsilon) \in \mathbb{R}_+$ *and a bounded open subset* $\Omega_\varepsilon \subset\subset \Omega$ *with the cone property such that*

$$||gu||_{L^p_s(\Omega)} \leqslant \varepsilon||u||_{W^{k,p}_s(\Omega)} + c(\varepsilon)||u||_{L^p(\Omega_\varepsilon)} \quad \forall u \in W^{k,p}_s(\Omega), \tag{4.3}$$

where $c(\varepsilon)$ *and* Ω_ε *depend only on* $\varepsilon, \Omega, n, k, p, m, s, t, \sigma_\circ[g]$.

Proof. Fix $\varepsilon > 0$ and let c be the constant in (4.1). Since $g \in M^t_\circ(\Omega)$, there exists $g_\varepsilon \in C^\infty_\circ(\Omega)$ such that $||g - g_\varepsilon||_{M^t(\Omega)} < \frac{\varepsilon}{c}$. Let Ω_ε be a bounded open subset of Ω, with the cone property, such that supp $g_\varepsilon \subset \Omega_\varepsilon$, hence by Theorem 4.1 and (3.3), it follows that

$$\begin{aligned} ||gu||_{L^p_s(\Omega)} &\leqslant c\,||g - g_\varepsilon||_{M^t(\Omega)}||u||_{W^{k,p}_s(\Omega)} + ||g_\varepsilon u||_{L^p_s(\Omega_\varepsilon)} \\ &\leqslant \varepsilon||u||_{W^{k,p}_s(\Omega)} + ||g_\varepsilon m^s||_{L^\infty(\Omega_\varepsilon)}||u||_{L^p(\Omega_\varepsilon)} \end{aligned} \tag{4.4}$$

for any u in $W^{k,p}_s(\Omega)$, and then we have the result. □

THEOREM 4.4. *If the assumption* (h_0) *holds and* $g \in M^t_\circ(\Omega)$*, then the operator*

$$u \in W^{k,p}_s(\Omega) \longrightarrow gu \in L^p_s(\Omega) \tag{4.5}$$

is compact.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions which weakly converges to zero in $W^{k,p}_s(\Omega)$. Therefore there exists $b \in \mathbb{R}_+$ such that $||u_n||_{W^{k,p}_s(\Omega)} \leqslant b$ for every $n \in \mathbb{N}$.

For $\varepsilon > 0$, from Corollary 4.3, there exist $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega_\varepsilon \subset\subset \Omega$ with the cone property such that

$$||gu_n||_{L^p_s(\Omega)} \leqslant \frac{\varepsilon}{b}||u_n||_{W^{k,p}_s(\Omega)} + c(\varepsilon)||u_n||_{L^p(\Omega_\varepsilon)} \quad \forall n \in \mathbb{N}. \tag{4.6}$$

Since $W^{k,p}_s(\Omega) \subset W^{k,p}(\Omega_\varepsilon)$, we obtain the result from a well-known compact embedding theorem. □

5. A priori estimates

Assume that Ω is an unbounded open subset of $\mathbb{R}^n, n \geqslant 3$, with the uniform $C^{1,1}$-regularity property, $p \in]1, +\infty[$ and $s \in \mathbb{R}$.

Consider in Ω the differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a, \tag{5.1}$$

with the following conditions on the coefficients:

$$(h_1) \qquad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO_{\text{loc}}(\bar{\Omega}), \;\; i,j = 1, \dots, n, \\ \exists\, \nu > 0 \;\; : \;\; \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geqslant \nu |\xi|^2 \;\; \text{a.e. in } \Omega, \forall \xi \in \mathbb{R}^n, \end{cases}$$

there exist functions e_{ij}, $i,j = 1, \dots, n$, g and $\mu \in \mathbb{R}_+$ such that

$$(h_2) \qquad \begin{cases} e_{ij} = e_{ji} \in L^\infty(\Omega) \cap VMO(\Omega), \;\; i,j = 1, \dots, n, \\ \sum_{i,j=1}^{n} e_{ij} \xi_i \xi_j \geqslant \mu |\xi|^2 \;\; \text{a.e. in } \Omega, \;\; \forall\, \xi \in \mathbb{R}^n, \\ g \in L^\infty(\Omega), \quad \lim_{r \to +\infty} \sum_{i,j=1}^{n} ||e_{ij} - g\, a_{ij}||_{L^\infty(\Omega \setminus B_r)} = 0, \end{cases}$$

$$(h_3) \qquad a_i \in \tilde{M}^{t_1}(\Omega),\; i = 1, \dots, n, \quad a \in \tilde{M}^{t_2}(\Omega),$$

where

$$t_1 \geqslant p, \quad t_1 \geqslant n, \quad t_1 > p \quad \text{if} \;\; p = n,$$
$$t_2 \geqslant p, \quad t_2 \geqslant n/2, \quad t_2 > p \quad \text{if} \;\; p = n/2.$$

Under assumptions (h_1) - (h_3), by Theorem 4.1, the operator $L : W_s^{2,p}(\Omega) \to L_s^p(\Omega)$ is bounded.

Let

$$L_0 = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

THEOREM 5.1. *Suppose that assumptions $(h_1), (h_2)$ and (h_3) hold. Then there exist r_0, $c \in \mathbb{R}_+$ such that:*

$$||u||_{W_s^{2,p}(\Omega)} \leqslant c\big(||Lu||_{L_s^p(\Omega)} + ||u||_{L_s^p(\Omega)}\big) \quad \forall\, u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega),$$

where c depends only on n, p, t_1, t_2, Ω, ν, μ, $||a_{ij}||_{L^\infty(\Omega)}$, $||e_{ij}||_{L^\infty(\Omega)}$, $||g||_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0} a_{ij}]$, $\eta[e_{ij}]$, $\widetilde{\sigma}[a_i]$, $\widetilde{\sigma}[a]$, m, s, and r_0 depends only on n, p, Ω, μ, $||e_{ij}||_{L^\infty(\Omega)}$, $\eta[e_{ij}]$.

Proof. Let $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)$. By Lemma 3.4 we have that

$$\sigma^s u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega).$$

Then, by Theorem 3.1 of [3], there exist r_0 and $c_0 \in \mathbb{R}_+$ such that

$$||\sigma^s u||_{W^{2,p}(\Omega)} \leqslant c_0 \Big(||L_0(\sigma^s u)||_{L^p(\Omega)} + ||\sigma^s u||_{L^p(\Omega)} \Big), \tag{5.2}$$

where c_0 depends on n, p, Ω, ν, μ, $||a_{ij}||_{L^\infty(\Omega)}$, $||e_{ij}||_{L^\infty(\Omega)}$, $||g||_{L^\infty(\Omega)}, \eta[\zeta_{2r_0} a_{ij}]$, $\eta[e_{ij}]$, and r_0 depends on n, p, Ω, μ, $||e_{ij}||_{L^\infty(\Omega)}$, $\eta[e_{ij}]$. Since

$$\begin{aligned} L_0(\sigma^s u) = \sigma^s Lu - s(s-1)\sigma^{s-2} \sum_{i,j=1}^n a_{ij}\sigma_{x_i}\sigma_{x_j} u - 2s\sigma^{s-1} \sum_{i,j=1}^n a_{ij}\sigma_{x_i} u_{x_j} \\ -s\sigma^{s-1} \sum_{i,j=1}^n a_{ij}\sigma_{x_i x_j} u - \sigma^s \sum_{i=1}^n a_i u_{x_i} - \sigma^s a u, \end{aligned} \tag{5.3}$$

from (5.2) and (5.3) we have

$$\begin{aligned} ||\sigma^s u||_{W^{2,p}(\Omega)} \leqslant c_1 \big(&||\sigma^s Lu||_{L^p(\Omega)} + ||\sigma^s u||_{L^p(\Omega)} \\ &+ \sum_{i,j=1}^n ||\sigma^{s-2}\sigma_{x_i}\sigma_{x_j} u||_{L^p(\Omega)} + \sum_{i,j=1}^n ||\sigma^{s-1}\sigma_{x_i} u_{x_j}||_{L^p(\Omega)} \\ &+ \sum_{i,j=1}^n ||\sigma^{s-1}\sigma_{x_i x_j} u||_{L^p(\Omega)} + \sum_{i=1}^n ||\sigma^s a_i u_{x_i}||_{L^p(\Omega)} + ||\sigma^s a u||_{L^p(\Omega)} \big), \end{aligned} \tag{5.4}$$

where c_1 depends on the same parameters as c_0 and on s.

By Theorem 4.7 of [1], for all $i = 1, \dots, n$ we have:

$$||u_{x_i}||_{L_s^p(\Omega)} \leqslant c_2 \Big(||u_{xx}||_{L_s^p(\Omega)}^{\frac{1}{2}} ||u||_{L_s^p(\Omega)}^{\frac{1}{2}} + ||u||_{L_s^p(\Omega)} \Big), \tag{5.5}$$

where c_2 depends on Ω, m, n, p.

Moreover, from Corollary 4.2, for any $\varepsilon \in \mathbb{R}_+$ and $i = 1, \dots, n$ there exist $c_1(\varepsilon)$, $c_2(\varepsilon) \in \mathbb{R}_+$ such that:

$$||a_i u_{x_i}||_{L_s^p(\Omega)} \leqslant \varepsilon ||u||_{W_s^{2,p}(\Omega)} + c_1(\varepsilon) ||u_{x_i}||_{L_s^p(\Omega)}, \tag{5.6}$$

$$||au||_{L_s^p(\Omega)} \leqslant \varepsilon ||u||_{W_s^{2,p}(\Omega)} + c_2(\varepsilon) ||u||_{L_s^p(\Omega)}, \tag{5.7}$$

where $c_1(\varepsilon)$ depends on ε, Ω, n, p, t_1, $\tilde{\sigma}[a_i]$ and $c_2(\varepsilon)$ depends on ε, Ω, n, p, t_2, $\tilde{\sigma}[a]$.

From (5.4)–(5.7), Lemma 3.2 and Lemma 3.4, it follows

$$\begin{aligned} ||u||_{W_s^{2,p}(\Omega)} \leqslant c_3 \big(&||Lu||_{L_s^p(\Omega)} + ||u||_{L_s^p(\Omega)} + \varepsilon ||u||_{W_s^{2,p}(\Omega)} \\ &+ c_3(\varepsilon) (||u_{xx}||_{L_s^p(\Omega)}^{\frac{1}{2}} ||u||_{L_s^p(\Omega)}^{\frac{1}{2}} + ||u||_{L_s^p(\Omega)}) \big), \end{aligned} \tag{5.8}$$

where c_3 depends on the same parameters as c_0 and on s, m, and $c_3(\varepsilon)$ depends on ε, Ω, n, p, t_1, t_2, $\widetilde{\sigma}[a_i]$, $\widetilde{\sigma}[a]$.

For $\varepsilon = \frac{1}{2c_3}$, from (5.8) we have

$$||u||_{W_s^{2,p}(\Omega)} \leqslant c_4\big(||Lu||_{L_s^p(\Omega)} + ||u||_{L_s^p(\Omega)} + ||u_{xx}||^{\frac{1}{2}}_{L_s^p(\Omega)}||u||^{\frac{1}{2}}_{L_s^p(\Omega)}\big), \tag{5.9}$$

where c_4 depends on the same parameters as c_3 and on $t_1, t_2, \widetilde{\sigma}[a_i], \widetilde{\sigma}[a]$.

Using Young's inequality and (5.9), we get the result. □

Add the following assumptions on the coefficients of L and on the weight function:

$$(h_4) \qquad \begin{cases} (e_{ij})_{x_h} \in M_\circ^{t,n-t}(\Omega), \ \text{ with } t \in]2,n], \ i,j,h = 1,\dots,n, \\ a_i \in M_\circ^{t_1}(\Omega), \ i = 1,\dots,n, \\ a = a' + b, a' \in M_\circ^{t_2}(\Omega), b \in L^\infty(\Omega), b_0 = \operatorname{ess\,inf}_\Omega b > 0, \\ g_0 = \operatorname{ess\,inf}_\Omega g > 0, \\ \lim\limits_{|x|\to+\infty} \dfrac{\sigma_x + \sigma_{xx}}{\sigma} = 0, \end{cases}$$

where t_1 and t_2 are defined as in (h_3).

THEOREM 5.2. *Suppose that assumptions $(h_1),(h_2)$ and (h_4) hold. Then there are a real positive number c and a bounded open $\Omega_1 \subset\subset \Omega$ with the cone property such that:*

$$||u||_{W_s^{2,p}(\Omega)} \leqslant c\left(||Lu||_{L_s^p(\Omega)} + ||u||_{L^p(\Omega_1)}\right) \quad \forall\, u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega),$$

where c and Ω_1 are dependent only on n, p, Ω, ν, μ, g_0, b_0, t, t_1, t_2, m, s, $||a_{ij}||_{L^\infty(\Omega)}$, $||e_{ij}||_{L^\infty(\Omega)}$, $||g||_{L^\infty(\Omega)}$, $||b||_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0}a_{ij}]$, $\sigma_0[(e_{ij})_x]$, $\sigma_0[a_i]$, $\sigma_0[a']$.

Proof. Let $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,p}(\Omega)$. By Lemma 3.4 we have that

$$\sigma^s u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega).$$

Applying Theorem 3.3 of [4] to the operator $L_0 + b$, we have that there exist a real number $c_0 \in \mathbb{R}_+$ and an open bounded subset $\Omega_0 \subset \Omega$ with the cone property such that

$$||\sigma^s u||_{W^{2,p}(\Omega)} \leqslant c_0\left(||(L_0+b)(\sigma^s u)||_{L^p(\Omega)} + ||\sigma^s u||_{L^p(\Omega_0)}\right),$$

where c_0 and Ω_0 are dependent on n, p, Ω, ν, μ, g_0, b_0, t, $||a_{ij}||_{L^\infty(\Omega)}$, $||e_{ij}||_{L^\infty(\Omega)}$, $||g||_{L^\infty(\Omega)}$, $||b||_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0}a_{ij}]$, $\sigma_0[(e_{ij})_x]$, and r_0 depends on n, p, Ω, μ, g_0, b_0, t, $||e_{ij}||_{L^\infty(\Omega)}$, $||g||_{L^\infty(\Omega)}$, $||b||_{L^\infty(\Omega)}$, $\sigma_0[(e_{ij})_x]$.

Proceeding as in the proof of Theorem 5.1, we have

$$\begin{aligned} ||u||_{W^{2,p}_s(\Omega)} \leqslant c_1\big(||Lu||_{L^p_s(\Omega)} + ||u||_{L^p(\Omega_0)} + \sum_{i,j=1}^n ||\sigma^{s-2}\sigma_{x_i}\sigma_{x_j}u||_{L^p(\Omega)} \\ + \sum_{i,j=1}^n ||\sigma^{s-1}\sigma_{x_i}u_{x_j}||_{L^p(\Omega)} + \sum_{i,j=1}^n ||\sigma^{s-1}\sigma_{x_ix_j}u||_{L^p(\Omega)} \\ + \sum_{i=1}^n ||a_iu_{x_i}||_{L^p_s(\Omega)} + ||a'u||_{L^p_s(\Omega)}\big), \end{aligned} \tag{5.10}$$

where c_1 depends on the same parameters as c_0 and on m, s.

From Corollary 4.3 and (1.6) of [11] it follows that for any $\varepsilon \in \mathbb{R}_+$ and $i, j = 1,\dots,n$ there exist $c_1(\varepsilon), c_2(\varepsilon), c_3(\varepsilon) \in \mathbb{R}_+$ and some bounded open subsets $\Omega_1(\varepsilon) \subset\subset \Omega$, $\Omega_2(\varepsilon) \subset\subset \Omega$, $\Omega_3(\varepsilon) \subset\subset \Omega$ with the cone property such that

$$||\sigma^{s-2}\sigma_{x_i}\sigma_{x_j}u||_{L^p(\Omega)} \leqslant \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_1(\varepsilon)||u||_{L^p(\Omega_1(\varepsilon))}, \tag{5.11}$$

$$||\sigma^{s-1}\sigma_{x_i}u_{x_j}||_{L^p(\Omega)} \leqslant \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_2(\varepsilon)||u_{x_j}||_{L^p(\Omega_2(\varepsilon))}, \tag{5.12}$$

$$||\sigma^{s-1}\sigma_{x_ix_j}u||_{L^p(\Omega)} \leqslant \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_3(\varepsilon)||u||_{L^p(\Omega_3(\varepsilon))}, \tag{5.13}$$

where $c_1(\varepsilon)$, $c_2(\varepsilon)$, $c_3(\varepsilon)$, $\Omega_1(\varepsilon)$, $\Omega_2(\varepsilon)$, $\Omega_3(\varepsilon)$ are dependent on ε, Ω, n, p, m, s.

Using again Corollary 4.3 and Theorem 4.7 of [1] we have that there exist $c_4(\varepsilon)$, $c_5(\varepsilon) \in \mathbb{R}_+$ and bounded open sets $\Omega_4(\varepsilon) \subset\subset \Omega$, $\Omega_5(\varepsilon) \subset\subset \Omega$ with the cone property such that:

$$\begin{aligned} ||a_iu_{x_i}||_{L^p_s(\Omega)} &\leqslant \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_4(\varepsilon)||u_{x_i}||_{L^p(\Omega_4(\varepsilon))} \qquad (5.14) \\ &\leqslant \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_4(\varepsilon)\big(||u_{xx}||^{\frac{1}{2}}_{L^p(\Omega_4(\varepsilon))}||u||^{\frac{1}{2}}_{L^p(\Omega_4(\varepsilon))} + ||u||_{L^p(\Omega_4(\varepsilon))}\big), \\ ||a'u||_{L^p_s(\Omega)} &\leqslant \varepsilon||u||_{W^{2,p}_s(\Omega)} + c_5(\varepsilon)||u||_{L^p(\Omega_5(\varepsilon))}, \qquad (5.15) \end{aligned}$$

where $c_4(\varepsilon)$ and $\Omega_4(\varepsilon)$ depend on ε, Ω, n, p, m, s, t_1, $\sigma_0[a_i]$, and $c_5(\varepsilon)$ and $\Omega_5(\varepsilon)$ depend on ε, Ω, n, p, m, s, t_2, $\sigma_0[a']$.

From (5.10)–(5.15) and Young's inequality we have the result. □

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