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TWO–WEIGHT CACCIOPPOLI–TYPE ESTIMATES AND WEAK REVERSE HÖLDER INEQUALITIES FOR $\mathscr{A}$–HARMONIC TENSORS

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Abstract. In this paper, we first introduce a new weight–$A_r^{\lambda_3}(\lambda_1,\lambda_2,\Omega)$-weight, and then prove the two-weight Caccioppoli-type estimates and the two-weight weak reverse Hölder inequalities for $\mathscr{A}$-harmonic tensors, which can be regarded as generalizations of the classical results.

1. Introduction

The purpose of this paper is to establish the two-weight Caccioppoli-type estimates and the two-weight weak reverse Hölder inequalities for $\mathscr{A}$-harmonic tensors. $\mathscr{A}$-harmonic tensors are interesting and important generalizations of p-harmonic tensors. In the meantime, p-harmonic tensors are extensions of conjugate harmonic functions and p-harmonic functions, $p>1$. In recent years there have been remarkable advances made in the field of $\mathscr{A}$-harmonic tensors. Many interesting results of $\mathscr{A}$-harmonic tensors and their applications in fields such as potential theory, quasiregular mappings and the theory of elasticity have been found; see [1$\sim$3, 6$\sim$12]. For many purposes, we need to know the integrability of $\mathscr{A}$-harmonic tensors and estimate the integrals for $\mathscr{A}$-harmonic tensors. The integral inequalities we will discuss in this paper can be used to study the integrability of $\mathscr{A}$-harmonic tensors and estimate the integrals for them.

Throughout this paper we always assume Ω is a connected open subset of R^n. We use $e_1,e_2,\ldots,e_n$ to denote the standard unit basis of R^n. Let $\bigwedge^l=\bigwedge^l(\mathrm{R}^n)$ be the linear space of l-vectors, spanned by the exterior products $e_I=e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_l}$, corresponding to all ordered l-tuples $I=(i_1,i_2,\cdots,i_l)$, $1\leqslant i_1<i_2<\cdots<i_l\leqslant n$, $l=0,1,\cdots,n$. The Grassman algebra $\bigwedge=\oplus\bigwedge^l$ is a graded algebra with respect to the exterior products. For $\alpha=\sum\alpha^I e_I\in\bigwedge$ and $\beta=\sum\beta^I e_I\in\bigwedge$, the inner product in $\bigwedge$ is given by $\langle\alpha,\beta\rangle=\sum\alpha^I\beta^I$ with summation over all l-tuples $I=(i_1,i_2,\cdots,i_l)$ and all integers $l=0,1,\cdots,n$. We define the Hodge star operator $*:\bigwedge\to\bigwedge$ by the rule $*1=e_1\wedge e_2\wedge\cdots\wedge e_n$ and $\alpha\wedge*\beta=\beta\wedge*\alpha=\langle\alpha,\beta\rangle(*1)$ for all $\alpha,\beta\in\bigwedge$. The norm of $\alpha\in\bigwedge$ is given by the formula $|\alpha|^2=\langle\alpha,\alpha\rangle=*(\alpha\wedge*\alpha)\in\bigwedge^0=\mathrm{R}$. The Hodge star is an isometric isomorphism on $\bigwedge$ with $*:\bigwedge^l\to\bigwedge^{n-l}$ and $**(-1)^{l(n-l)}:\bigwedge^l\to\bigwedge^l$.

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Let $1 \leqslant p < \infty$. We denote the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p,E,w^\alpha} = \left(\int_E |f(x)|^p w^\alpha dx \right)^{1/p}.$$

As we know, l-forms ω on Ω is a Schwartz distribution on Ω with values in $\bigwedge^l(\mathrm{R}^n)$. $D'(\Omega, \bigwedge^l)$ is used to denote the space of all differential l-forms. We write $L^p(\Omega, \bigwedge^l)$ for the l-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathrm{R})$ for all ordered l-tuples I. Thus $L^p(\Omega, \bigwedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx \right)^{1/p} = \left(\int_\Omega (\sum |\omega_I(x)|^2)^{p/2} dx \right)^{1/p}.$$

Similarly, $W^{1,p}(\Omega, \bigwedge^l)$ are those differential l-forms on Ω whose coefficients are in $W^{1,p}(\Omega, \mathrm{R})$. The notations $W^{1,p}_{loc}(\Omega, R)$ and $W^{1,p}_{loc}(\Omega, \bigwedge^l)$ are self-explanatory. We denote the exterior derivative by $d : D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \cdots, n$. Its formal adjoint operator $d^* : D'(\Omega, \bigwedge^{l+1}) \to D'(\Omega, \bigwedge^l)$ is given by $d^* = (-1)^{nl+1} * d *$ on $D'(\Omega, \bigwedge^{l+1})$, $l = 0, 1, \cdots, n$.

There has been remarkable work in the study of the $\mathscr{A}$-harmonic equation

$$d^* \mathscr{A}(x, d\omega) = 0, \tag{1.1}$$

where $\mathscr{A} : \Omega \times \bigwedge^l(\mathrm{R}^n) \to \bigwedge^l(\mathrm{R}^n)$ satisfies the following conditions:

$$|\mathscr{A}(x,\xi)| \leqslant a|\xi|^{p-1} \qquad \text{and} \qquad \langle \mathscr{A}(x,\xi), \xi \rangle \geqslant |\xi|^p$$

for almost every $x \in \Omega$ and $\xi \in \bigwedge^l(\mathrm{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.1). A solution to (1.1) is an element of the Sobolev space $W^{1,p}_{loc}(\Omega, \bigwedge^{l-1})$ such that

$$\int_\Omega \langle \mathscr{A}(x, d\omega), d\varphi \rangle = 0$$

for all $\varphi \in W^{1,p}_{loc}(\Omega, \bigwedge^{l-1})$ with compact support.

DEFINITION 1.1. We call u an $\mathscr{A}$-harmonic tensor in Ω if u satisfies the $\mathscr{A}$-harmonic equation (1.1) in Ω.

A differential l-form $u \in D'(\Omega, \bigwedge^l)$ is called a closed form if $du = 0$ in Ω. Similarly, a differential $(l+1)$-form $v \in D'(\Omega, \bigwedge^{l+1})$ is called a coclosed form if $d^* v = 0$. A differential form u is called a p-harmonic tensor if

$$d^*(|du|^{p-2} du) = 0 \qquad \text{and} \qquad d^* u = 0,$$

where $1 < p < \infty$. The equation

$$\mathscr{A}(x, du) = d^* v \tag{1.2}$$

is called the conjugate $\mathscr{A}$-harmonic equation. For example, $du = d^* v$ is an analogue of a Cauchy-Riemann system in R^n. Clearly, the $\mathscr{A}$-harmonic equation is not affected by adding a closed form to u and coclosed form to v. Therefore, any type of estimates between u and v must be modulo such forms. Suppose that u is a solution to (1.1) in Ω. Then, at least locally in a ball B, there exists a form $v \in W^{1,q}(B, \bigwedge^{l+1})$, $\frac{1}{p}+\frac{1}{q}=1$, such that (1.2) holds.

DEFINITION 1.2. When u and v satisfy (1.2) in Ω, and $\mathscr{A}^{-1}$ exists in Ω, we call u and v conjugate $\mathscr{A}$-harmonic tensors in Ω.

DEFINITION 1.3. We call u a p-harmonic function if u satisfies the p-harmonic equation

$$div(\nabla u|\nabla u|^{p-2}) = 0$$

with $p>1$. Its conjugate in the plane is a q-harmonic function v, $\frac{1}{p}+\frac{1}{q}=1$, which satisfies

$$\nabla u|\nabla u|^{p-2} = \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\right).$$

Note that if $p=q=2$, we get the usual conjugate harmonic functions.

We write $\mathrm{R} = \mathrm{R}^1$. Balls are denoted by B and σB is the ball with the same center as B and with $diam(\sigma B) = \sigma diam(B)$. The n-dimensional Lebesgue measure of a set $E \subseteq \mathrm{R}^n$ is denoted by $|E|$. We call w a weight if $w \in L^1_{loc}(\mathrm{R}^n)$ and $w>0$ a.e.. Also in general $d\mu = wdx$ where w is a weight. We can find the following result in [7]: Let $Q \subset \mathrm{R}^n$ be a cube or a ball. To each $y \in Q$ there corresponds a linear operator $K_y : C^\infty(Q, \bigwedge^l) \to C^\infty(Q, \bigwedge^{l-1})$ defined by

$$(K_y\omega)(x;\xi_1,\xi_2,\cdots,\xi_{l-1}) = \int_0^1 t^{l-1}\omega(tx+y-ty; x-y, \xi_1,\cdots,\xi_{l-1})dt$$

and the decomposition

$$\omega = d(K_y) + K_y(d\omega).$$

We define another linear operator $T_Q : C^\infty(Q, \bigwedge^l) \to C^\infty(Q, \bigwedge^{l-1})$ by averaging K_y over all points y in Q

$$T_Q\omega = \int_Q \varphi(y) K_y \omega dy,$$

where $\varphi \in C_0^\infty(Q)$ is normalized by $\int_Q \varphi(y)dy = 1$. We define the l-form $\omega_Q \in D'(Q, \bigwedge^l)$ by

$$\omega_Q = |Q|^{-1}\int_Q \omega(y)dy, \text{ if } l=0, \text{ and } \omega_Q = d(T_Q\omega), \text{ if } l = 1,2,\cdots,n,$$

for all $\omega \in L^p(Q, \bigwedge^l), 1 \leqslant p < \infty$.

2. Local $A_r^{\lambda_3}(\lambda_1,\lambda_2,\Omega)$-weighted Caccioppoli-type estimates

DEFINITION 2.1. We say the weight $(w_1(x),w_2(x))$ satisfies the $A_r^{\lambda_3}(\lambda_1,\lambda_2,\Omega)$ condition for some $r>1$ and $\lambda_1,\lambda_2,\lambda_3>0$, write $(w_1(x),w_2(x))\in A_r^{\lambda_3}(\lambda_1,\lambda_2,\Omega)$, if $w_1(x)>0, w_2(x)>0$, a.e., and

$$\sup_B\left(\frac{1}{|B|}\int_B w_1^{\lambda_1}dx\right)\left(\frac{1}{|B|}\int_B\left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)}dx\right)^{\lambda_3(r-1)}<\infty$$

for any ball $B\subset\Omega$.

If we choose $w_1=w_2=w$ and $\lambda_1=\lambda_2=\lambda_3=1$, we will get the A_r-weight, see [4] and [5] for the basic properties of A_r-weights.

Choosing $w_1=w_2=w$, $\lambda_1=\lambda$ and $\lambda_2=\lambda_3=1$, we will get the $A_r(\lambda,\Omega)$-weights which are introduced in [2].

Choosing $w_1=w_2=w$, $\lambda_1=\lambda_2=1$ and $\lambda_3=\lambda$, we will obtain the $A_r^{\lambda}(\Omega)$-weights which are introduced in [13].

In this paper we will need the following generalized Hölder's inequality.

LEMMA 2.1. *Let $0<\alpha<\infty, 0<\beta<\infty$ and $s^{-1}=\alpha^{-1}+\beta^{-1}$. If f and g are measurable functions on R^n, then*

$$\|fg\|_{s,\Omega}\leqslant\|f\|_{\alpha,\Omega}\cdot\|g\|_{\beta,\Omega}$$

for any $\Omega\subset R^n$.

In [9], C. A. Nolder obtains the following local Caccioppoli-type estimate.

THEOREM A. *Let u be an $\mathscr{A}$-harmonic tensor in Ω and let $\sigma>1$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,B}\leqslant C diam(B)^{-1}\|u-c\|_{s,\sigma B}$$

for all balls or cubes B with $\sigma B\subset\Omega$ and all closed forms c. Here $1<s<\infty$.

The following weak reverse Hölder inequality appears in [9].

THEOREM B. *Let u be an $\mathscr{A}$-harmonic tensor in Ω, $\sigma>1$ and $0<s,t<\infty$. Then there exists a constant C, independent of u, such that*

$$\|u\|_{s,B}\leqslant C|B|^{(t-s)/st}\|u\|_{t,\sigma B}$$

for all balls or cubes B with $\sigma B\subset\Omega$.

We now generalize Theorem A into the following local weighted Caccioppoli-type estimate for $\mathscr{A}$-harmonic tensors.

THEOREM 2.1. *Let $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, be an $\mathscr{A}$-harmonic tensor in a domain $\Omega \subset R^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the $\mathscr{A}$-harmonic equation and $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $\lambda_1, \lambda_2, \lambda_3 > 0$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,B,w_1^{\alpha\lambda_1}} \leqslant C diam(B)^{-1} \|u - c\|_{s,\rho B,w_2^{\alpha\lambda_2\lambda_3}} \tag{2.1}$$

for all balls B with $\rho B \subset \Omega$, all closed forms c and any real number α with $0 < \alpha < 1$.

Note that (2.1) can be written as

$$\left(\int_B |du|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \leqslant C diam(B)^{-1} \left(\int_{\rho B} |u - c|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s}.$$

Proof. Choose $k = s/(1-\alpha)$, then $s < k$ and $1/s = 1/k + (k-s)/sk$, by Hölder's inequality and Theorem A we have

$$\begin{aligned}
\|du\|_{s,B,w_1^{\alpha\lambda_1}} &= \left(\int_B |du|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \\
&= \left(\int_B (|du| w_1^{\alpha\lambda_1/s})^s dx\right)^{1/s} \\
&\leqslant \left(\int_B |du|^k dx\right)^{1/k} \cdot \left(\int_B (w_1^{\alpha\lambda_1/s})^{sk/(k-s)} dx\right)^{(k-s)/sk} \\
&= \|du\|_{k,B} \cdot \left(\int_B w_1^{\lambda_1} dx\right)^{(k-s)/sk} \\
&\leqslant C_1 diam(B)^{-1} \|u - c\|_{k,\sigma B} \cdot \left(\int_B w_1^{\lambda_1} dx\right)^{(k-s)/sk}
\end{aligned} \tag{2.2}$$

for all balls B with $\sigma B \subset \Omega$, all closed forms c and any real number α with $0 < \alpha < 1$. Since c is a closed form and u is an $\mathscr{A}$-harmonic tensor, then $u - c$ is still an $\mathscr{A}$-harmonic tensor. Taking $m = s/(1 + \alpha\lambda_3(r-1))$, we find $m < s$. Apply Theorem B yields

$$\begin{aligned}
\|u - c\|_{k,\sigma B} &\leqslant C_2 |B|^{(m-k)/mk} \cdot \|u - c\|_{m,\sigma^2 B} \\
&= C_2 |B|^{(m-k)/mk} \cdot \|u - c\|_{m,\rho B}
\end{aligned} \tag{2.3}$$

where $\rho = \sigma^2$. Substituting (2.3) into (2.2), we have

$$\|du\|_{s,B,w_1^{\alpha\lambda_1}} \leqslant C_3 diam(B)^{-1} |B|^{(m-k)/mk} \cdot \|u - c\|_{m,\rho B} \cdot \left(\int_B w_1^{\lambda_1} dx\right)^{(k-s)/sk}. \tag{2.4}$$

Since $1/m = 1/s + (s-m)/sm$, by Hölder's inequality again, we obtain

$$\begin{aligned} \|u-c\|_{m,\rho B} &= \left(\int_{\rho B} |u-c|^m dx\right)^{1/m} \\ &= \left(\int_{\rho B} (|u-c| w_2^{\alpha\lambda_2\lambda_3/s} \cdot w_2^{-\alpha\lambda_2\lambda_3/s})^m dx\right)^{1/m} \\ &\leqslant \left(\int_{\rho B} |u-c|^s w_2^{\alpha\lambda_2\lambda_3} dx\right)^{1/s} \\ &\quad \cdot \left(\int_{\rho B} \left(\left(\frac{1}{w_2}\right)^{\alpha\lambda_2\lambda_3/s}\right)^{sm/(s-m)} dx\right)^{(s-m)/sm} \\ &= \|u-c\|_{s,\rho B, w_2^{\alpha\lambda_2\lambda_3}} \cdot \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{(s-m)/sm} \end{aligned} \tag{2.5}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c.

Combining (2.4) and (2.5), we obtain

$$\begin{aligned} \|du\|_{s,B,w_1^{\alpha\lambda_1}} \leqslant\, & C_3 diam(B)^{-1} |B|^{(m-k)/mk} \cdot \|u-c\|_{s,\rho B, w_2^{\alpha\lambda_2\lambda_3}} \\ & \cdot \left(\int_B w_1^{\lambda_1} dx\right)^{(k-s)/sk} \cdot \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{(s-m)/sm} . \end{aligned} \tag{2.6}$$

Since $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$, we then have

$$\begin{aligned} & \left(\int_B w_1^{\lambda_1} dx\right)^{(k-s)/sk} \cdot \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{(s-m)/sm} \\ \leqslant & \left(\int_{\rho B} w_1^{\lambda_1} dx\right)^{(k-s)/sk} \cdot \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{(s-m)/sm} \\ = & \left(\left(\int_{\rho B} w_1^{\lambda_1} dx\right) \cdot \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{k(s-m)/m(k-s)}\right)^{(k-s)/sk} \\ = & \left(|\rho B|^{1+k(s-m)/m(k-s)} \left(\frac{1}{|\rho B|} \int_{\rho B} w_1^{\lambda_1} dx\right)\right. \\ & \left. \cdot \left(\frac{1}{|\rho B|} \int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\lambda_3(r-1)}\right)^{(k-s)/sk} \\ \leqslant & C_4 |B|^{(k-m)/mk}. \end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.6), we find that

$$\|du\|_{s,B,w_1^{\alpha\lambda_1}} \leqslant C diam(B)^{-1} \|u-c\|_{s,\rho B, w_2^{\alpha\lambda_2\lambda_3}}$$

for all balls B with $\rho B \subset \Omega$, all closed forms c and any real number α with $0 < \alpha < 1$. This ends the proof of Theorem 2.1. □

Note that the parameters $\alpha, \lambda_1, \lambda_2$ and λ_3 in Theorem 2.1 are any real numbers with $0 < \alpha < 1$ and $\lambda_1, \lambda_2, \lambda_3 > 0$. Therefore, we will have different versions of the weighted Caccioppoli-type estimates by choosing $\alpha, \lambda_1, \lambda_2$ and λ_3 to be different values. The following special cases of Theorem 2.1 will be useful in case one meets special weights.

If we choose $\alpha = 1/r$ in Theorem 2.1, we have the following result.

COROLLARY 2.1. *Let $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, be an $\mathscr{A}$-harmonic tensor in a domain $\Omega \subset R^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the $\mathscr{A}$-harmonic equation and $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $\lambda_1, \lambda_2, \lambda_3 > 0$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,B,w_1^{\lambda_1/r}} \leqslant C diam(B)^{-1} \|u - c\|_{s,\rho B,w_2^{\lambda_2\lambda_3/r}}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c.

Choosing $\alpha = 1/s$ in theorem 2.4, we have the following result.

COROLLARY 2.2. *Let $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, be an $\mathscr{A}$-harmonic tensor in a domain $\Omega \subset R^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the $\mathscr{A}$-harmonic equation and $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $\lambda_1, \lambda_2, \lambda_3 > 0$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,B,w_1^{\lambda_1/s}} \leqslant C diam(B)^{-1} \|u - c\|_{s,\rho B,w_2^{\lambda_2\lambda_3/s}}$$

for all balls B with $\rho B \subset \Omega$ and all closed forms c.

If we choose $\lambda_1 = 1$ in Theorem 2.1, we have the following result.

COROLLARY 2.3. *Let $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, be an $\mathscr{A}$-harmonic tensor in a domain $\Omega \subset R^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the $\mathscr{A}$-harmonic equation and $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_2, \Omega)$ for some $r > 1$ and $\lambda_2, \lambda_3 > 0$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,B,w_1^{\alpha}} \leqslant C diam(B)^{-1} \|u - c\|_{s,\rho B,w_2^{\alpha\lambda_2\lambda_3}}$$

for all balls B with $\rho B \subset \Omega$, all closed forms c and any real number α with $0 < \alpha < 1$.

If we choose $\lambda_2 = 1$ in Theorem 2.1, we have the following result.

COROLLARY 2.4. *Let $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, be an $\mathscr{A}$-harmonic tensor in a domain $\Omega \subset R^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the $\mathscr{A}$-harmonic equation and $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \Omega)$ for some $r > 1$ and $\lambda_1, \lambda_3 > 0$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,B,w_1^{\alpha\lambda_1}} \leqslant C diam(B)^{-1} \|u - c\|_{s,\rho B,w_2^{\alpha\lambda_3}}$$

for all balls B with $\rho B \subset \Omega$, all closed forms c and any real number α with $0 < \alpha < 1$.

Choosing $\lambda_3 = 1$ in Theorem 2.1, we have the following result.

COROLLARY 2.5. *Let $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, be an $\mathscr{A}$-harmonic tensor in a domain $\Omega \subset R^n$ and $\rho > 1$. Assume that $1 < s < \infty$ is a fixed exponent associated with the $\mathscr{A}$-harmonic equation and $(w_1, w_2) \in A_r(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $\lambda_1, \lambda_2 > 0$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,B,w_1^{\alpha\lambda_1}} \leqslant C diam(B)^{-1} \|u - c\|_{s,\rho B,w_2^{\alpha\lambda_2}}$$

for all balls B with $\rho B \subset \Omega$, all closed forms c and any real number α with $0 < \alpha < 1$.

3. $A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$-weighted weak reverse Hölder inequality

We now generalize Theorem B into the following weighted form.

THEOREM 3.1. *Let $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, be an $\mathscr{A}$-harmonic tensor in a domain $\Omega \subset R^n$ and $\sigma > 1$. Assume that $1 < s, t < \infty$ and $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ for some $r > 1$ and $\lambda_1, \lambda_2, \lambda_3 > 0$. Then there exists a constant C, independent of u, such that*

$$\left(\int_B |u|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \leqslant C|B|^{(t-s)/st} \left(\int_{\sigma B} |u|^t w_2^{\alpha t\lambda_2\lambda_3/s} dx\right)^{1/t} \tag{3.1}$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

Note that (3.1) can be written as the following symmetric version

$$\left(\frac{1}{|B|}\int_B |u|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \leqslant C \left(\frac{1}{|B|}\int_{\sigma B} |u|^t w_2^{\alpha t\lambda_2\lambda_3/s} dx\right)^{1/t}.$$

The proof of Theorem 3.1 is similar to that of Theorem 2.1. For completion of the paper, we prove Theorem 3.1 as follows.

Proof. Choose $k = s/(1-\alpha)$, then $s < k$ and $1/s = 1/k + (k-s)/sk$, applying the Hölder's inequality yields

$$\begin{aligned}\left(\int_B |u|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} &= \left(\int_B (|u| w_1^{\alpha\lambda_1/s})^s dx\right)^{1/s} \\ &\leqslant \|u\|_{k,B} \left(\int_B (w_1^{\alpha\lambda_1/s})^{sk/(k-s)} dx\right)^{(k-s)/sk} \\ &= \|u\|_{k,B} \left(\int_B w_1^{\lambda_1} dx\right)^{(k-s)/sk}\end{aligned} \tag{3.2}$$

for all balls B with $\sigma B \subset \Omega$. Next choose $m = st/(s + \alpha t\lambda_3(r-1))$ then $m < t$. By Theorem B, we obtain

$$\|u\|_{k,B} \leqslant C_1 |B|^{(m-k)/mk} \|u\|_{m,\sigma B}. \tag{3.3}$$

Since $1/m = 1/t + (t-m)/mt$, by the Hölder inequality again, we obtain

$$\begin{aligned}
\|u\|_{m,\sigma B} &= \left(\int_{\sigma B} (|u| w_2^{\alpha\lambda_2\lambda_3/s} \cdot w_2^{-\alpha\lambda_2\lambda_3/s})^m dx\right)^{1/m} \\
&\leqslant \left(\int_{\sigma B} |u|^t w_2^{\alpha t\lambda_2\lambda_3/s} dx\right)^{1/t} \left(\int_{\sigma B} \left(\left(\frac{1}{w_2}\right)^{\alpha\lambda_2\lambda_3/s}\right)^{mt/(t-m)} dx\right)^{(t-m)/mt} \\
&= \left(\int_{\sigma B} |u|^t w_2^{\alpha t\lambda_2\lambda_3/s} dx\right)^{1/t} \left(\int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{(t-m)/mt}.
\end{aligned} \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we arrive at the following estimate

$$\begin{aligned}
\left(\int_B |u|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \leqslant C_1 |B|^{(m-k)/mk} \left(\int_{\sigma B} |u|^t w_2^{\alpha t\lambda_2\lambda_3/s} dx\right)^{1/t} \left(\int_B w_1^{\lambda_1} dx\right)^{(k-s)/sk} \\
\cdot \left(\int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{(t-m)/mt}.
\end{aligned} \tag{3.5}$$

Since $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$, we find that

$$\begin{aligned}
&\left(\int_B w_1^{\lambda_1} dx\right)^{(k-s)/sk} \left(\int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{(t-m)/mt} \\
&\leqslant \left(\int_{\sigma B} w_1^{\lambda_1} dx\right)^{(k-s)/sk} \left(\int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{(t-m)/mt} \\
&= \left(|\sigma B|^{1+sk(t-m)/mt(k-s)} \left(\frac{1}{|\sigma B|} \int_{\sigma B} w_1^{\lambda_1} dx\right)\right. \\
&\quad \left. \cdot \left(\frac{1}{|\sigma B|} \int_{\sigma B} \left(\frac{1}{w_2}\right)^{\lambda_2/(r-1)} dx\right)^{\lambda_3(r-1)}\right)^{(k-s)/sk} \\
&\leqslant C_2 |B|^{\frac{1}{s}-\frac{1}{k}+\frac{1}{m}-\frac{1}{t}}.
\end{aligned} \tag{3.6}$$

Finally, substituting (3.6) into (3.5), we obtain

$$\left(\int_B |u|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \leqslant C|B|^{(t-s)/st} \left(\int_{\sigma B} |u|^t w_2^{\alpha t\lambda_2\lambda_3/s} dx\right)^{1/t}. \qquad \square$$

Similar to Section 2, we also have some corollaries of Theorem 3.1, which will have special use in case one meets special weights. If we choose $\alpha = 1/r$ in Theorem 3.1, we then have the following result.

COROLLARY 3.1. *Let $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, be an $\mathscr{A}$-harmonic tensor in a domain $\Omega \subset R^n$ and $\sigma > 1$. Assume that $1 < s, t < \infty$ and $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$*

for some $r > 1$ *and* $\lambda_1, \lambda_2, \lambda_3 > 0$. *Then there exists a constant C, independent of u , such that*

$$\left(\int_B |u|^s w_1^{\lambda_1/r} dx\right)^{1/s} \leqslant C|B|^{(t-s)/st} \left(\int_{\sigma B} |u|^t w_2^{t\lambda_2\lambda_3/rs} dx\right)^{1/t}$$

for all balls B with $\rho B \subset \Omega$.

If we choose $\alpha = 1/s$ in Theorem 3.1, we then have the following result.

COROLLARY 3.2. *Let* $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, *be an* $\mathscr{A}$*-harmonic tensor in a domain* $\Omega \subset R^n$ *and* $\sigma > 1$. *Assume that* $1 < s, t < \infty$ *and* $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ *for some* $r > 1$ *and* $\lambda_1, \lambda_2, \lambda_3 > 0$. *Then there exists a constant C, independent of u , such that*

$$\left(\int_B |u|^s w_1^{\lambda_1/s} dx\right)^{1/s} \leqslant C|B|^{(t-s)/st} \left(\int_{\sigma B} |u|^t w_2^{t\lambda_2\lambda_3/s^2} dx\right)^{1/t}$$

for all balls B with $\rho B \subset \Omega$.

If we choose $\alpha = 1/t$ in Theorem 3.1, we then have the following version.

COROLLARY 3.3. *Let* $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, *be an* $\mathscr{A}$*-harmonic tensor in a domain* $\Omega \subset R^n$ *and* $\sigma > 1$. *Assume that* $1 < s, t < \infty$ *and* $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \lambda_2, \Omega)$ *for some* $r > 1$ *and* $\lambda_1, \lambda_2, \lambda_3 > 0$. *Then there exists a constant C, independent of u, such that*

$$\left(\int_B |u|^s w_1^{\lambda_1/t} dx\right)^{1/s} \leqslant C|B|^{(t-s)/st} \left(\int_{\sigma B} |u|^t w_2^{\lambda_2\lambda_3/s} dx\right)^{1/t}$$

for all balls B with $\rho B \subset \Omega$.

If we choose $\lambda_1 = 1$ in Theorem 3.1, we then have the following result.

COROLLARY 3.4. *Let* $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, *be an* $\mathscr{A}$*-harmonic tensor in a domain* $\Omega \subset R^n$ *and* $\sigma > 1$. *Assume that* $1 < s, t < \infty$ *and* $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_2, \Omega)$ *for some* $r > 1$ *and* $\lambda_2, \lambda_3 > 0$. *Then there exists a constant C, independent of u , such that*

$$\left(\int_B |u|^s w_1^{\alpha} dx\right)^{1/s} \leqslant C|B|^{(t-s)/st} \left(\int_{\sigma B} |u|^t w_2^{\alpha t\lambda_2\lambda_3/s} dx\right)^{1/t}$$

for all balls B with $\rho B \subset \Omega$ *and any real number* α *with* $0 < \alpha < 1$.

If we choose $\lambda_2 = 1$ in Theorem 3.1, we have the following result.

COROLLARY 3.5. *Let* $u \in D'(\Omega, \bigwedge^l), l = 0, 1, \cdots, n$, *be an* $\mathscr{A}$*-harmonic tensor in a domain* $\Omega \subset R^n$ *and* $\sigma > 1$. *Assume that* $1 < s, t < \infty$ *and* $(w_1, w_2) \in A_r^{\lambda_3}(\lambda_1, \Omega)$ *for some* $r > 1$ *and* $\lambda_1, \lambda_3 > 0$. *Then there exists a constant C, independent of u , such that*

$$\left(\int_B |u|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \leqslant C|B|^{(t-s)/st} \left(\int_{\sigma B} |u|^t w_2^{\alpha t\lambda_3/s} dx\right)^{1/t}$$

for all balls B with $\rho B \subset \Omega$ *and any real number* α *with* $0 < \alpha < 1$.

If we choose $\lambda_3 = 1$ in Theorem 3.1, we have the following result.

COROLLARY 3.6. *Let $u \in D'(\Omega, \bigwedge^l)$, $l = 0,1,\cdots,n$, be an $\mathscr{A}$-harmonic tensor in a domain $\Omega \subset R^n$ and $\sigma > 1$. Assume that $1 < s,t < \infty$ and $(w_1,w_2) \in A_r(\lambda_1,\lambda_2,\Omega)$ for some $r > 1$, and $\lambda_1,\lambda_2 > 0$. Then there exists a constant C, independent of u, such that*

$$\left(\int_B |u|^s w_1^{\alpha\lambda_1} dx\right)^{1/s} \leqslant C|B|^{(t-s)/st} \left(\int_{\sigma B} |u|^t w_2^{\alpha t\lambda_2/s} dx\right)^{1/t}$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha < 1$.

4. Global weighted inequalities

We need the following properties of the Whitney covers appearing in [6] to prove the global results.

LEMMA 4.1. *Each Ω has a modified Whitney cover of cubes $\nu = Q_i$ such that*

$$\bigcup_i Q_i = \Omega,$$

$$\sum_{Q \in \nu} \chi_{\sqrt{5/4}Q} \leqslant N\chi_\Omega$$

for all $x \in R^n$ and some $N > 1$, and if $Q_i \bigcap Q_j \neq \phi$, then there exists a cube R (this cube does not need to be a member of ν) in $Q_i \bigcap Q_j$ such that $Q_i \bigcup Q_j \subset NR$. Moreover, if Ω is a δ-John, then there is a distinguished cube $Q_0 \in \nu$ which can be connected with every cube $Q \in \nu$ by a chain of cubes $Q_0, Q_1, \cdots, Q_k = Q$ from ν and such that $Q \subset \sigma Q_i$, $i = 0,1,2,\cdots,k$, for some $\sigma = \sigma(n,\delta)$.

We prove the following $A_r^{\lambda_3}(\lambda_1,\lambda_2,\Omega)$-weighted Caccioppoli-type estimate and weak reverse Hölder inequality for $\mathscr{A}$-harmonic tensors in a bounded domain Ω.

THEOREM 4.1. *Let $u \in D'(\Omega, \bigwedge^l), l = 0,1,\cdots,n$, be an $\mathscr{A}$-harmonic tensor in a domain $\Omega \subset R^n$ which has a finite open cover $\nu = \{B_1,\cdots,B_m\}$, where B_i is an open ball, $i = 1,\cdots,m$. Assume that $\rho > 1$ and $1 < s < \infty$ is a fixed exponent associated with the $\mathscr{A}$-harmonic equation and $(w_1,w_2) \in A_r^{\lambda_3}(\lambda_1,\lambda_2,\Omega)$ for some $r > 1$ and $\lambda_1,\lambda_2,\lambda_3 > 0$. Then there exists a constant C, independent of u and du, such that*

$$\|du\|_{s,\Omega,w_1^{\alpha\lambda_1}} \leqslant C diam(\Omega)^{-1} \|u - c\|_{s,\Omega,w_2^{\alpha\lambda_2\lambda_3}}$$

for all closed forms c and any real number α with $0 < \alpha < 1$.

Proof. Let $\nu = \{B_1,\cdots,B_m\}$ be an open cover of the bounded domain $\Omega \subset \mathrm{R}^n$ and $d_i = diam(B_i) > 0$, $i = 1,\cdots,m$. Assume that $d = \min\{d_1,\cdots,d_m\}$. Since Ω is bounded, then there exists a constant C_1 such that $\frac{1}{d} \leqslant \frac{C_1}{diam(\Omega)}$. By Theorem 2.4 and

Lemma 4.1, we obtain

$$
\begin{aligned}
\|du\|_{s,\Omega,w_1^{\alpha\lambda_1}} &= \left(\int_\Omega |du|^s w_1^{\alpha\lambda_1}dx\right)^{1/s} \\
&\leqslant \sum_{Q\in\mathcal{V}} \left(\int_Q |du|^s w_1^{\alpha\lambda_1}dx\right)^{1/s} \\
&\leqslant \sum_{Q\in\mathcal{V}} C_2 diam(Q)^{-1}\left(\int_{\rho Q} |u-c|^s w_2^{\alpha\lambda_2\lambda_3}dx\right)^{1/s} \\
&\leqslant C_3 diam(\Omega)^{-1}\sum_{Q\in\mathcal{V}} \left(\int_{\rho Q} |u-c|^s w_2^{\alpha\lambda_2\lambda_3}dx\right)^{1/s} \\
&\leqslant C_4 diam(\Omega)^{-1}\left(\int_{\Omega} |u-c|^s w_2^{\alpha\lambda_2\lambda_3}dx\right)^{1/s}.
\end{aligned}
$$

The proof of Theorem 4.1 has been completed. □

We now generalize Theorem 3.1 into the following global weighted form.

THEOREM 4.2. *Let $u\in D'(\Omega,\bigwedge^l), l=0,1,\cdots,n,$ be an $\mathscr{A}$-harmonic tensor in a domain $\Omega\subset R^n$ and $\sigma>1$. Assume that $1<s\leqslant t<\infty$ and $(w_1,w_2)\in A_r^{\lambda_3}(\lambda_1,\lambda_2,\Omega)$ for some $r>1$, and $\lambda_1,\lambda_2,\lambda_3>0$. Then there exists a constant C, independent of u, such that*

$$
\left(\int_\Omega |u|^s w_1^{\alpha\lambda_1}dx\right)^{1/s} \leqslant C|\Omega|^{(t-s)/st}\left(\int_\Omega |u|^t w_2^{\alpha t\lambda_2\lambda_3/s}dx\right)^{1/t}
$$

for any real number α with $0<\alpha<1$.

Proof. By Theorem 3.1 and Lemma 4.1, we obtain

$$
\begin{aligned}
\left(\int_\Omega |u|^s w_1^{\alpha\lambda_1}dx\right)^{1/s} &\leqslant \sum_{Q\in\mathcal{V}}\left(\int_Q |u|^s w_1^{\alpha\lambda_1}dx\right)^{1/s} \\
&\leqslant \sum_{Q\in\mathcal{V}} C_1|Q|^{(t-s)/st}\left(\int_{\sigma Q} |u|^t w_2^{\alpha t\lambda_2\lambda_3/s}dx\right)^{1/t} \\
&\leqslant C_1|\Omega|^{(t-s)/st}\sum_{Q\in\mathcal{V}}\left(\int_{\sigma Q} |u|^t w_2^{\alpha t\lambda_2\lambda_3/s}dx\right)^{1/t} \\
&\leqslant C_2|\Omega|^{(t-s)/st}\left(\int_{\Omega} |u|^t w_2^{\alpha t\lambda_2\lambda_3/s}dx\right)^{1/t}.
\end{aligned}
$$

This ends the proof of Theorem 4.2. □

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