Journal of Mathematical Inequalities
Volume 6, Number 3 (2012), 333–341
doi:10.7153/jmi-06-33

ON A BETA FUNCTION INEQUALITY

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(*Communicated by N. Elezović*)

Abstract. In this paper we present a beta function inequality on $(0,1]\times(0,1]$, which improves an inequality of H. Alzer. Moreover several new inequalities for the gamma and psi functions on $(0,1]$ are provided. Some elementary inequalities of two real variables are proved.

1. Introduction

For $x>0$ the classical gamma function Γ and the Ψ function or digamma function are defined as

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt \qquad \text{and} \qquad \Psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}.$$

The derivatives $\Psi', \Psi'', \Psi''', \ldots$ are known as polygamma functions.

Closely related to the gamma function is the beta function which is the real function of two variables defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x>0, y>0.$$

A well known equation connecting the beta and the gamma functions is

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.1}$$

For a proof of (1.1) see, for example, [9] where a good reference for these functions is also given. While in the recent past, several articles have appeared providing various inequalities for gamma and polygamma functions see [2, 3, 6, 7, 11, 12, 15] and the references therein, only few inequalities concerning the beta function can be found in the literature [4, 5, 10, 11, 15]. Among the various kinds of inequalities concerning

Mathematics subject classification (2010): 26D07, 33B15.
Keywords and phrases: beta function, gamma function, psi function, inequalities, convex functions, Jensen's inequality.

Paper JMI-06-33

the beta function we will select a special one which will be considered in detail on $(0,1]\times(0,1]$.

S. Dragomir at al. [11, p. 114, Theorem 3] established the relation

$$B(x,y)\leqslant\frac{1}{xy}\qquad\text{for}\qquad 0<x,y\leqslant 1.$$

Recently H. Alzer [5, p. 738, Theorem 3.1] obtained the following improved results for all $x,y\in(0,1]$

$$\frac{1}{xy}\left(1-\alpha\frac{1-x}{1+x}\frac{1-y}{1+y}\right)\leqslant B(x,y)\leqslant\frac{1}{xy}\left(1-\beta\frac{1-x}{1+x}\frac{1-y}{1+y}\right),\tag{1.2}$$

with the best possible constants $\alpha=2/3\pi^2-4=2.57973\ldots$ and $\beta=1$ respectively. The aim of this paper is to show that the right side of (1.2) can be strengthened. In addition, a different lower bound for$B(x,y)$ for all $x,y\in(0,1]$ is also given. More precisely, we show that the following sequence of inequalities holds for all real numbers $x,y\in(0,1]$

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma^2\left(\frac{x+y}{2}\right)}\leqslant\frac{1}{xy}\frac{x^2+y^2}{2}\leqslant\frac{1}{xy}(x+y-xy)\tag{1.3}$$

and

$$\frac{1}{xy}(x+y-xy)\leqslant B(x,y)\leqslant\frac{1}{xy}\frac{x+y}{1+xy}\leqslant\frac{1}{xy}\left(1-\frac{1-x}{1+x}\frac{1-y}{1+y}\right).\tag{1.4}$$

Inequalities for the ratio

$$T(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma^2\left(\frac{x+y}{2}\right)},\tag{1.5}$$

are sometimes called “Gurland type” [13].

We recall Jensen’s inequality for a real valued convex function f as follows [15, p. 15] :

$$f(\lambda x+(1-\lambda)y)\leqslant\lambda f(x)+(1-\lambda)f(y),\qquad\lambda\in[0,1]\tag{1.6}$$

for $x,y\in(a,b)$. For a concave function the opposite inequality holds. Furthermore, a positive function f on a given (a,b) interval is logarithmically convex (or log-convex) if $x\mapsto\log f(x)$ is convex on (a,b). Since $x\mapsto\log\Gamma(x+1)$ is strictly log-convex on $(0,\infty)$, we find for all $x,y>0\,(x\neq y)$:

$$1<\frac{(x+y)^2}{4xy}<\frac{\Gamma(x)\Gamma(y)}{\Gamma^2\left(\frac{x+y}{2}\right)}.\tag{1.7}$$

In view of (1.7) we could consider (1.3) as one of its counterparts for $x,y\in(0,1]$. It seems curious that the second inequality of (1.4) is a simple consequence of Jensen’s inequality for convex functions, while it is surprising that the first inequality in (1.4) needs the most preparations.

2. Lemmas

In order to establish the main theorem of this paper we need some lemmas, which we present in this section. The lemmas deal with some useful formulas and inequalities concerning the Ψ function and its derivative Ψ'. Furthermore, we present a simple inequality involving the Γ function. In the first lemma we collect some useful formulas which can be found in [1, Chapter 6].

LEMMA 2.1. *For all x we have*

$$\Gamma(x+1) = x\Gamma(x). \tag{2.1}$$

Further we have

$$\Psi(x) = -\gamma - \frac{1}{x} + x\sum_{k=1}^{\infty} \frac{1}{k(x+k)}, \quad (x \neq 0, -1, -2, \ldots)$$

$$\Psi^{(m)}(x) = (-1)^{m+1} m! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{m+1}}, \tag{2.2}$$

$$(x \neq 0, -1, -2, \ldots \quad m = 1, 2, \ldots)$$

$$\Psi^{(n)}(x+1) = \Psi^{(n)}(x) + (-1)^n \frac{n!}{x^{n+1}}, \quad n \geqslant 0 \tag{2.3}$$

$$\Psi(1/2) = -\gamma - 2\log 2, \quad \Psi(1) = -\gamma, \quad \Psi(2) = 1 - \gamma \tag{2.4}$$

$$\Psi'(1) = \frac{\pi^2}{6}, \quad \Psi'(2) = \frac{\pi^2}{6} - 1 \tag{2.5}$$

where $\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right) = 0.57721\ldots$ *is the Euler-Mascheroni constant.*

The next result is due to H. Alzer [7, p. 3643, Lemma 6].

LEMMA 2.2. *Let*

$$\Phi_t(x) = \frac{\Psi'(x) - \Psi'(x+t)}{[\Psi(x+t) - \Psi(x)]^2}.$$

If $0 < t < 1$, then Φ_t is strictly increasing on $(0,\infty)$. And, if $t > 1$, then Φ_t is strictly decreasing on $(0,\infty)$.

LEMMA 2.3. *For all $x, y \in (0,1]$ we have*

$$-x[\Psi(x+y) - \Psi(x)]^2 + x\left[\Psi'(x+y) - \Psi'(x)\right] + 2[\Psi(x+y) - \Psi(x)] \geqslant 0. \tag{2.6}$$

Proof. We write (2.6) in the form

$$1+\frac{\Psi'(x)-\Psi'(x+y)}{[\Psi(x+y)-\Psi(x)]^2} \leqslant \frac{2}{x[\Psi(x+y)-\Psi(x)]}. \tag{2.7}$$

According to Lemma 2.2 and using (2.3) we can write

$$\frac{\Psi'(x)-\Psi'(x+y)}{[\Psi(x+y)-\Psi(x)]^2} \leqslant \frac{\Psi'(x)-\Psi'(x+1)}{[\Psi(x+1)-\Psi(x)]^2} = 1.$$

In order to prove (2.7) it is sufficient to show that

$$\Psi(x+y)-\Psi(x) < \frac{1}{x},$$

holds, which is clearly the case in view of (2.3) and since Ψ is monoton increasing. □

LEMMA 2.4. *For* $0<t\leqslant 1$, *we have*

$$\frac{2t-t^2}{t^2} \leqslant \frac{\Gamma^2(t)}{\Gamma(2t)}. \tag{2.8}$$

Proof. Let us define the auxiliary function h by

$$h(t) := \frac{2t-t^2}{t^2} - \frac{\Gamma^2(t)}{\Gamma(2t)}.$$

We show that h has a maximum at $t=1$. Since $h(1)=0$, we conclude $h(t)\leqslant 0$, which is equivalent to (2.8). Building the first two derivatives we obtain

$$h'(t) = -2\frac{\Gamma(2t)+t^2\Gamma^2(t)(\Psi(t)-\Psi(2t))}{t^2\Gamma(2t)},$$

and

$$h''(t) = \frac{4\Gamma(2t)-2t^3\Gamma^2(t)h_1(t)}{t^3\Gamma(2t)},$$

where

$$h_1(t) = 2\Psi^2(t)-4\Psi(t)\Psi(2t)+2\Psi^2(2t)+\Psi'(t)-2\Psi'(2t).$$

Using (2.4) in conjunction with (2.5) we obtain $h'(1)=0$ and $h''(1)=\pi^2/3-4=-0.7101\ldots<0$. This means that $h(t)$ has a local maximum at $t=1$ and the lemma follows. A deeper analysis shows that $h(t)$ is strictly increasing on $(0,1]$ and strictly decreasing on $(1,2]$, but we did not really have to use this fact. □

LEMMA 2.5. *For all* $x,y\in(0,1]$, *we have*

$$\Psi'(x+1)-\Psi'(x+y+1) \geqslant \frac{y^2}{(1+xy)^2}.$$

Proof. Using (2.2) we obtain

$$\Psi'(x+1)-\Psi'(x+y+1)=\frac{y^2+2xy+2x+2y+1}{x^2(x+y+1)^2}+\sum_{k=1}^{\infty}\frac{y^2+2xy+2x+2y+1+2k(1+y)}{(x+k)^2(x+y+k+1)^2}$$
$$>\frac{y^2+2xy+2x+2y+1}{x^2(x+y+1)^2}>\frac{y^2}{(1+xy)^2}. \quad \square$$

We also need the following lemma which can be found in [3, p. 382, Theorem 7].

LEMMA 2.6. *Let $x>0$ and $s\in(0,1)$ be real numbers. Then we have*

$$\frac{1-s}{x+s+1}+\frac{1-s}{(x+1)(x+s)}<\Psi(x+1)-\Psi(x+s). \tag{2.9}$$

LEMMA 2.7. *Let $0<x<y\leqslant 1$ be real numbers. Then we have*

$$\frac{2x}{x^2+y^2}<\Psi(x+1)-\Psi\left(\frac{x+y}{2}\right).$$

Proof. Applying (2.9) where $s=(y-x)/2>0$ we obtain after a straightforward computation

$$\frac{x^3+4x^2+4-2y^2+2y+6x+2xy-xy^2}{(x+1)(x+y)(x+y+2)}<\Psi(x+1)-\Psi\left(\frac{x+y}{2}\right).$$

On the other hand some computations yield

$$\frac{x^3+4x^2+4-2y^2+2y+6x+2xy-xy^2}{(x+1)(x+y)(x+y+2)}-\frac{2x}{x^2+y^2}$$
$$=\frac{(x-y)\,p(x,y)}{(x+1)(x+y)(x+y+2)(x^2+y^2)}>0,$$

where

$$p(x,y):=x^4+2x^3+x^3y+x^2y^2+xy^3-2y^2-6xy-4y<0$$

for $0<x<y\leqslant 1$. Thus the proof of the lemma is complete. $\square$

3. Main result

Now we give the main results of this paper.

THEOREM. *For all real numbers* $x,y \in (0,1]$, *we have*

$$T(x,y) \leqslant \frac{1}{xy}\frac{x^2+y^2}{2}, \tag{3.1}$$

and

$$\frac{1}{xy}(x+y-xy) \leqslant B(x,y) \leqslant \frac{1}{xy}\frac{x+y}{1+xy} \tag{3.2}$$

with equality if and only if $x=y=1$.

Proof. We begin by proving (3.1). Let function f be defined by

$$f(x,y) := \log\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma^2\left(\frac{x+y}{2}\right)} - \log\left[\frac{x^2+y^2}{2}\right],$$

where we use the notation of (1.5) and the recursion formula (2.1). Because of symmetry we may suppose $0<x\leqslant y\leqslant 1$. Partial differentiation yields

$$\frac{\partial f(x,y)}{\partial x} = \left[\Psi(x+1) - \Psi\left(\frac{x+y}{2}\right)\right] - \frac{2x}{x^2+y^2}.$$

Applying Lemma 2.7 gives $\partial f(x,y)/\partial x > 0$, which implies

$$f(x,y) \leqslant f(y,y) = 0.$$

But $f(x,y) \leqslant 0$ is equivalent to (3.1).

To prove the first inequality of (3.2) let $0<x\leqslant y\leqslant 1$. We wish to investigate the monotonity property of the function g defined by

$$g(x,y) := \frac{1}{xy}(x+y-xy) - B(x,y).$$

Differentiation gives

$$\frac{\partial g(x,y)}{\partial x} = -\frac{1}{x^2} - B(x,y)\left[\Psi(x) - \Psi(x+y)\right].$$

We next claim that $\partial g(x,y)/\partial x \geqslant 0$, thus $g(x,y)$ is increasing in x i.e. $g(x,y) \leqslant g(y,y)$. Making use of Lemma 2.4 yields $g(x,y) \leqslant 0$ and the proof of the left side of (3.2) is complete. The request $\partial g(x,y)/\partial x \geqslant 0$ leads to the inequality

$$\frac{1}{x^2} \leqslant B(x,y)\left[\Psi(x+y) - \Psi(x)\right].$$

Taking

$$g_1(x,y) := -2\log x - \log B(x,y) - \log\left[\Psi(x+y) - \Psi(x)\right],$$

and building the derivative we get after some computations and simplifications

$$\frac{\partial g_1(x,y)}{\partial x} = \frac{-2\left[\Psi(x+y)-\Psi(x)\right]}{x\left[\Psi(x+y)-\Psi(x)\right]} + \frac{x\left[\Psi(x+y)-\Psi(x)\right]^2 + x\left[\Psi'(x)-\Psi'(x+y)\right]}{x\left[\Psi(x+y)-\Psi(x)\right]}.$$

Applying Lemma 2.3 it is clear that $\partial g_1(x,y)/\partial x < 0$, therefore $g_1(x,y)$ is strictly decreasing in x. Since $\lim_{x\to 0} g_1(x,y) = 0$, we infer $g_1(x,y) \leqslant 0$, i.e. $\partial g(x,y)/\partial x \geqslant 0$.

We now proceed to prove the right-hand side of (3.2). Let $x \in (0,1]$ and fix y where $0 < y \leqslant 1$. Denote by g_2 the function

$$g_2(x) := \log(1+xy) + \log\left[\Gamma(x+1)\right] + \log\left[\Gamma(y+1)\right] - \log\left[\Gamma(x+y+1)\right].$$

If $g_2(x)$ is a convex function on $[0,1]$, then from Jensen's inequality for convex functions (3.2) immediately follows. First we find $g_2(0) = 0$ and $g_2(1) = 0$. For checking the convexity of g_2 it is sufficient to verify that $g_2''(x) > 0$. Indeed the second derivative of g_2 and Lemma 2.5 lead to

$$g_2''(x) = -\frac{y^2}{(1+xy)^2} + \Psi'(x+1) - \Psi'(x+y+1) > 0.$$

On account of Jensen's inequality (1.6) for the convex function g_2 we have by $a = 0$ and $b = 1$

$$g_2(\lambda a + (1-\lambda)b) \leqslant \lambda g_2(a) + (1-\lambda) g_2(b), \quad \lambda \in [0,1].$$

This clearly forces $g_2(x) \leqslant 0$, hence the proof of the theorem is complete. □

4. Concluding remarks

First we establish the connection between (3.1) and (3.2) as announced in (1.3) and (1.4). Moreover we show that the right side of (3.2) improves the right side of (1.2).

COROLLARY 4.1. *For all real numbers $x,y \in (0,1)$, we have*

$$\frac{x^2+y^2}{2} < \frac{x+y}{2} < 1 - \sqrt{(1-x)(1-y)} < x+y-xy.$$

The proof is trivial, we omit details.

COROLLARY 4.2. *For all real numbers $x,y \in [0,1]$, we have*

$$\frac{x+y}{1+xy} \leqslant \left(\sqrt{x}+\sqrt{y}-\sqrt{xy}\right)^2 \leqslant 1 - \frac{(1-x)(1-y)}{(1+x)(1+y)}. \tag{4.1}$$

Equality holds in (4.1) if and only if $x = y = 0$ or $x = y = 1$.

Proof. We begin with the verification of the first part of (4.1). Let $a,b \in [0,1]$. It now follows that

$$\frac{a^2+b^2}{1+a^2b^2}-(a+b-ab)^2=\frac{(1-a)(b-1)ab}{1+a^2b^2}p(a,b)\leqslant 0,$$

where

$$p(a,b):=2+a^2b^2-ab-a^2b-ab^2.$$

We show that $p(a,b)\geqslant 0$. Clearly,

$$p(a,b)\geqslant 1+a^2b^2-a^2b-ab^2\geqslant 1-b-(1-b)a^2b=(1-b)\left(1-a^2b\right)\geqslant 0.$$

Substituting $a=\sqrt{x}$ and $b=\sqrt{y}$, gives the desired result.

To establish the second part of (4.1) let us write

$$(a+b-ab)^2-\frac{2\left(a^2+b^2\right)}{(1+a^2)(1+b^2)}=\frac{(1-a)(1-b)}{(1+a^2)(1+b^2)}q(a,b)\leqslant 0,$$

where

$$q(a,b):=-\left[(a-b)^2+(1-a)a^2b+(1-b)ab^2+(1-a)a^2b^3\right]$$
$$-\left(a^3+b^3+a^2b^2+a^3b^2\right).$$

The substitutions $a=\sqrt{x}$ and $b=\sqrt{y}$ establish (4.1) and the corollary follows. □

Finally, we mention that neither the left side of (3.1), $\Gamma(x)\Gamma(y)/\Gamma^2((x+y)/2)$ nor the right side $(1/xy)\left(\left(x^2+y^2\right)/2\right)$ are comparable with the left side of (1.2), $1/xy\left(1-\alpha(1-x)/(1+x)(1-y)/(1+y)\right)$ on the whole $(0,1)$ interval. The same is also true for the left side of (3.2), $1/(xy(x+y-xy))$ and the left side of (1.2).

Acknowledgement. The author thanks the referee for the careful reading of the manuscript and for pointing out a mistake.

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(Received August 16, 2010)

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