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A HALF–DISCRETE HILBERT–TYPE INEQUALITY WITH A GENERAL HOMOGENEOUS KERNEL OF DEGREE 0

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Abstract. In this paper we establish a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree 0 including two interval variables. The equivalent forms, the operator expressions, the reverses and some particular cases are also considered.

1. Introduction

Assuming that $p>1$, $1/p+1/q=1$, $f(\geqslant 0)\in L^p(\mathbb{R}_+)$, $g(\geqslant 0)\in L^q(\mathbb{R}_+)$, $||f||_p = \{\int_0^\infty f^p(x)dx\}^{1/p}>0$, $||g||_q>0$, we have the following Hilbert integral inequality (cf. [3]):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy<\frac{\pi}{\sin(\pi/p)}||f||_p||g||_q, \tag{1}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. Note also that inequality (1) is sharp unless $f,g=0$ a.e. on $\mathbb{R}_+$.

For a_m, $b_n\geqslant 0$, $a=\{a_m\}_{m=1}^\infty\in l^p$, $b=\{b_n\}_{n=1}^\infty\in l^q$, $||a||_p=\{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}}>0$, $||b||_q>0$, we have the discrete variant of the above inequality

$$\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{a_mb_n}{m+n}<\frac{\pi}{\sin(\pi/p)}||a||_p||b||_q, \tag{2}$$

with the same best possible constant factor. For $p=q=2$, the above two inequalities reduce respectively to the original form of the Hilbert inequality. Inequalities (1) and (2) are important in analysis and its applications (cf. [12], [17], [19]) and they still represent the field of interest to numerous mathematicians.

In 1998, by introducing an independent parameter $\lambda\in(0,1]$, Yang [13] gave an extension of (1) for $p=q=2$. By generalizing the results from [13], Yang gave in [18] the following best extensions of (1) and (2) concerning the homogeneous kernels: Let $\lambda,\lambda_1,\lambda_2\in\mathbb{R}$, $\lambda_1+\lambda_2=\lambda$ and let $k_\lambda(x,y)$ be the non-negative homogeneous function of degree $-\lambda$, i.e. $k_\lambda(tx,ty)=t^{-\lambda}k_\lambda(x,y)$, $x,y,t>0$. If $k(\lambda_1)=$

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$\int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}dt \in \mathbb{R}_+,\ \phi(x)=x^{p(1-\lambda_1)-1},\ \psi(x)=x^{q(1-\lambda_2)-1},\ f(\geqslant 0)\in L_{p,\phi}(\mathbb{R}_+)=\{f;||f||_{p,\phi}=\{\int_0^\infty \phi(x)f^p(x)dx\}^{1/p}<\infty\}, g(\geqslant 0)\in L_{q,\psi}(\mathbb{R}_+),\ ||f||_{p,\phi},||g||_{q,\psi}>0$, then

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover if the function $k_\lambda(x,y)$ $x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing for $x>0(y>0)$, then for $a_m,\ b_n\geqslant 0,\ a=\{a_m\}_{m=1}^\infty\in l_{p,\phi}=\{a;||a||_{p,\phi}=\{\sum_{m=1}^\infty \phi(m)a_m^p\}^{1/p}<\infty\},\ b=\{b_n\}_{n=1}^\infty\in l_{q,\psi},\ ||a||_{p,\phi},\ ||b||_{q,\psi}>0$, we have the discrete inequality

$$\sum_{m=1}^\infty\sum_{n=1}^\infty k_\lambda(m,n)a_mb_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{4}$$

where the constant factor $k(\lambda_1)$ is still the best possible. Clearly, if $\lambda=1,\ k_\lambda(x,y)=1/(x+y),\ \lambda_1=1/q,\ \lambda_2=1/p$, inequality (3) reduces to (1), while (4) reduces to (2).

There are lots of generalizations of the Hilbert inequality. Some of them include different sets of integration, refinements in some particular cases, extension to multi-dimensional case, settings in some more general function spaces etc. For some recent results in the above mentioned directions, the reader is referred, for example, to papers [1], [2], [4], [6], [7], [8], [11], [14], [16], [21]. Let's mention also that the paper [5] provides an unified treatment of Hilbert-type inequalities with conjugate parameters.

Hardy et al. [3], established a few results on the half-discrete Hilbert-type inequalities with the non-homogeneous kernel (see Theorem 351). But they did not prove that the constant factors, included in the inequalities, are the best possible. However, Yang [15], gave a result by introducing an interval variable and proved that the constant factor is the best possible. Recently, Yang [20] gave the following half-discrete Hilbert inequality with the best possible constant factor $B(\lambda_1,\lambda_2)$ $(\lambda,\lambda_1>0, 0<\lambda_2\leqslant 1,\lambda_1+\lambda_2=\lambda)$:

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx < B(\lambda_1,\lambda_2)||f||_{p,\phi}||a||_{q,\psi}. \tag{5}$$

Here, $B(\cdot,\cdot)$ denotes the usual Beta function.

The main objective of this paper is to establish the half-discrete Hilbert-type inequality with a general homogeneous kernel of degree 0 and the best possible constant factor $k(\alpha)$ as follows: Suppose that $\phi_1(x)=x^{p(1+\alpha)-1},\ \psi_1(x)=x^{q(1-\alpha)-1},\ h(t)\geqslant 0$, $k(\alpha)=\int_0^\infty h(t)t^{\alpha-1}dt\in\mathbb{R}_+$, and $x^{-\alpha}\sum_{n=1}^\infty h(n/x)n^{\alpha-1}<k(\alpha),\ x\in\mathbb{R}_+$. We are going to establish the inequality

$$\int_0^\infty f(x)\sum_{n=1}^\infty h\left(\frac{n}{x}\right)a_ndx < k(\alpha)||f||_{p,\phi_1}||a||_{q,\psi_1}, \tag{6}$$

which holds under the previous conditions. Moreover, we are going to derive the best possible extension of (6) with two interval variables, the equivalent forms, the operator expressions and the appropriate inequalities with the reversed sign of inequality.

Since $h(mn/mx)=h(n/x)$, our function h will be referred to as the homogeneous kernel of degree 0.

2. Some lemmas

In order to obtain our main results, we need some auxiliary results. We begin with the following lemma.

LEMMA 1. *Suppose* $\alpha \in \mathbb{R}$, *and let* $u(x)$, $x \in (b,c)$, $-\infty \leqslant b < c \leqslant \infty$ *and* $v(y)$, $y \in [n_0,\infty)$, $n_0 \in \mathbb{N}$, *be strictly increasing differentiable functions such that* $u(b^+) = 0$, $v(n_0) > 0$, $u(c^-) = v(\infty) = \infty$. *Further, let* $h(t) \geqslant 0$, $t \in \mathbb{R}_+$ *be a finite measurable function, and let* $\omega(n)$ *and* $\varpi(x)$ *be the weight functions defined by*

$$\omega(n) = [v(n)]^{\alpha} \int_b^c h\left(\frac{v(n)}{u(x)}\right)[u(x)]^{-\alpha-1}u'(x)dx,\ n \geqslant n_0 (n \in \mathbb{N}), \tag{7}$$

$$\varpi(x) = [u(x)]^{-\alpha} \sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right)[v(n)]^{\alpha-1}v'(n),\ x \in (b,c). \tag{8}$$

Then,

$$\omega(n) = k(\alpha) := \int_0^{\infty} h(t)t^{\alpha-1}dt. \tag{9}$$

Moreover, let $f(x,y) := [u(x)]^{-\alpha}h(v(y)/u(x))[v(y)]^{\alpha-1}v'(y)$ *equipped with one of the following conditions:*

Condition (i) $v(y)$, $y \in [n_0-1,\infty)$ *is strictly increasing with* $v(n_0-1) \geqslant 0$ *and for every fixed* $x \in (b,c)$, $f(x,y)$ *is strictly decreasing on the interval* (n_0-1,∞);

Condition (ii) $v(y)$, $y \in [n_0-\frac{1}{2},\infty)$ *is strictly increasing with* $v(n_0-1/2) \geqslant 0$ *and for every fixed* $x \in (b,c)$, $f(x,y)$ *is decreasing and strictly convex on the interval* $(n_0-1/2,\infty)$;

Condition (iii) there exists a constant $\beta \geqslant 0$ *such that* $v(y)$, $y \in [n_0-\beta,\infty)$ *is strictly increasing with* $v(n_0-\beta) \geqslant 0$, *and for every fixed* $x \in (b,c)$, $f(x,y)$ *is differentiable function satisfying*

$$R(x) := \int_{n_0-\beta}^{n_0} f(x,y)dy - \frac{1}{2}f(x,n_0) - \int_{n_0}^{\infty} \rho(y)f_y'(x,y)dy > 0,$$

where $\rho(y) = y - [y] - 1/2$ *is the Bernoulli function of first order.*

If $k(\alpha) \in \mathbb{R}_+$ *and one of the above conditions is fulfilled, then*

$$\varpi(x) < k(\alpha),\ x \in (b,c). \tag{10}$$

Proof. If we apply the substitution $t = v(n)/u(x)$ to relation (7), we easily get (9) after an easy calculation.

Further, if the condition (i) is fulfilled, then we have

$$\begin{aligned}\varpi(x) &= \sum_{n=n_0}^{\infty} f(x,n) < [u(x)]^{-\alpha}\int_{n_0-1}^{\infty} h\left(\frac{v(y)}{u(x)}\right)[v(y)]^{\alpha-1}v'(y)dy \\ &\overset{t=v(y)/u(x)}{=} \int_{\frac{v(n_0-1)}{u(x)}}^{\infty} h(t)t^{\alpha-1}dt \leqslant \int_0^{\infty} h(t)t^{\alpha-1}dt = k(\alpha).\end{aligned}$$

Moreover, if the condition (ii) is satisfied, then by Hadamard's inequality (cf. [10]), we have

$$\begin{aligned}\varpi(x) &= \sum_{n=n_0}^{\infty} f(x,n) < \int_{n_0-\frac{1}{2}}^{\infty} f(x,y)dy \\ &\overset{t=v(y)/u(x)}{=} \int_{\frac{v(n_0-1/2)}{u(x)}}^{\infty} h(t)t^{\alpha-1}dt \leqslant \int_0^{\infty} h(t)t^{\alpha-1}dt = k(\alpha).\end{aligned}$$

Finally, condition (iii) together with the Euler-Maclaurin summation formula (cf. [18]) yields

$$\begin{aligned}\varpi(x)' &= \sum_{n=n_0}^{\infty} f(x,n) = \int_{n_0}^{\infty} f(x,y)dy + \frac{1}{2}f(x,n_0) + \int_{n_0}^{\infty} \rho(y)f_y'(x,y)dy \\ &= \int_{n_0-\beta}^{\infty} f(x,y)dy - R(x) = \int_{\frac{v(n_0-\beta)}{u(x)}}^{\infty} h(t)t^{\alpha-1}dt - R(x) \\ &\leqslant k(\alpha) - R(x) < k(\alpha).\end{aligned}$$

This completes the proof of the lemma. □

LEMMA 2. *Suppose that the parameters* α, $k(\alpha)$ *and the functions* $u(x)$, $v(y)$, $h(t)$, $\varpi(x)$ *are defined as in the statement of Lemma 1. Further, let* $1/p + 1/q = 1$, $p > 0$, $p \neq 1$, $a_n \geqslant 0, n \geqslant n_0$ $(n \in \mathbb{N})$. *If* $f(x)$ *is non-negative measurable function on the interval* (b,c), *then*

(i) *if* $p > 1$, *then the following two inequalities hold:*

$$\begin{aligned}J_1 &:= \left\{\sum_{n=n_0}^{\infty} \frac{v'(n)}{[v(n)]^{1-p\alpha}} \left[\int_b^c h\left(\frac{v(n)}{u(x)}\right) f(x)dx\right]^p\right\}^{\frac{1}{p}} \\ &\leqslant [k(\alpha)]^{\frac{1}{q}} \left\{\int_b^c \varpi(x) \frac{[u(x)]^{p(1+\alpha)-1}}{[u'(x)]^{p-1}} f^p(x)dx\right\}^{\frac{1}{p}}, \qquad (11)\end{aligned}$$

$$\begin{aligned}L_1 &:= \left\{\int_b^c \frac{[\varpi(x)]^{1-q}u'(x)}{[u(x)]^{1+q\alpha}} \left[\sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right) a_n\right]^q dx\right\}^{\frac{1}{q}} \\ &\leqslant \left\{k(\alpha) \sum_{n=n_0}^{\infty} \frac{[v(n)]^{q(1-\alpha)-1}}{[v'(n)]^{q-1}} a_n^q\right\}^{\frac{1}{q}}; \qquad (12)\end{aligned}$$

(ii) *if* $0 < p < 1$, *the reverse inequalities in* (11) *and* (12) *are valid.*

Proof. (i) By using the well known Hölder's inequality (cf. [10]) and relation (9), it follows

$$\left[\int_b^c h\left(\frac{v(n)}{u(x)}\right) f(x)dx\right]^p$$
$$= \left\{\int_b^c h\left(\frac{v(n)}{u(x)}\right)\left[\frac{[u(x)]^{(1+\alpha)/q}}{[v(n)]^{(1-\alpha)/p}}\frac{[v'(n)]^{1/p}}{[u'(x)]^{1/q}}f(x)\right]\left[\frac{[v(n)]^{(1-\alpha)/p}}{[u(x)]^{(1+\alpha)/q}}\frac{[u'(x)]^{1/q}}{[v'(n)]^{1/p}}\right]dx\right\}^p$$
$$\leqslant \int_b^c h\left(\frac{v(n)}{u(x)}\right)\frac{[u(x)]^{(1+\alpha)(p-1)}v'(n)}{[v(n)]^{1-\alpha}[u'(x)]^{p-1}}f^p(x)dx\left\{\int_b^c h\left(\frac{v(n)}{u(x)}\right)\frac{[v(n)]^{(1-\alpha)(q-1)}u'(x)}{[u(x)]^{1+\alpha}[v'(n)]^{q-1}}dx\right\}^{p-1}$$
$$= \left\{\frac{\omega(n)[v(n)]^{q(1-\alpha)-1}}{[v'(n)]^{q-1}}\right\}^{p-1}\int_b^c h\left(\frac{v(n)}{u(x)}\right)\frac{[u(x)]^{(1+\alpha)(p-1)}v'(n)f^p(x)}{[v(n)]^{1-\alpha}[u'(x)]^{p-1}}dx$$
$$= \frac{[k(\alpha)]^{p-1}}{[v(n)]^{p\alpha-1}v'(n)}\int_b^c h\left(\frac{v(n)}{u(x)}\right)\frac{[u(x)]^{(1+\alpha)(p-1)}v'(n)}{[v(n)]^{1-\alpha}[u'(x)]^{p-1}}f^p(x)dx.$$

On the other hand, by Lebesgue term by term integration theorem (cf. [9]), we have

$$J_1 \leqslant [k(\alpha)]^{\frac{1}{q}}\left\{\sum_{n=n_0}^{\infty}\int_b^c h\left(\frac{v(n)}{u(x)}\right)\frac{[u(x)]^{(1+\alpha)(p-1)}v'(n)f^p(x)}{[v(n)]^{1-\alpha}[u'(x)]^{p-1}}dx\right\}^{\frac{1}{p}}$$
$$= [k(\alpha)]^{\frac{1}{q}}\left\{\int_b^c \sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right)\frac{[u(x)]^{(1+\alpha)(p-1)}v'(n)f^p(x)}{[v(n)]^{1-\alpha}[u'(x)]^{p-1}}dx\right\}^{\frac{1}{p}}$$
$$= [k(\alpha)]^{\frac{1}{q}}\left\{\int_b^c \varpi(x)\frac{[u(x)]^{p(1+\alpha)-1}}{[u'(x)]^{p-1}}f^p(x)dx\right\}^{\frac{1}{p}},$$

and (11) holds. Yet another use of Hölder's inequality yields inequality

$$\left[\sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right)a_n\right]^q = \left\{\sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right)\left[\frac{[u(x)]^{(1+\alpha)/q}}{[v(n)]^{(1-\alpha)/p}}\frac{[v'(n)]^{1/p}}{[u'(x)]^{1/q}}\right]\right.$$
$$\left.\times\left[\frac{[v(n)]^{(1-\alpha)/p}}{[u(x)]^{(1+\alpha)/q}}\frac{[u'(x)]^{1/q}}{[v'(n)]^{1/p}}a_n\right]\right\}^q$$
$$\leqslant \left\{\sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right)\frac{[u(x)]^{(1+\alpha)(p-1)}}{[v(n)]^{1-\alpha}}\frac{v'(n)}{[u'(x)]^{p-1}}\right\}^{q-1}$$
$$\times\sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right)\frac{[v(n)]^{(1-\alpha)(q-1)}}{[u(x)]^{1+\alpha}}\frac{u'(x)}{[v'(n)]^{q-1}}a_n^q$$
$$= \frac{[u(x)]^{1+q\alpha}}{[\varpi(x)]^{1-q}u'(x)}\sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right)\frac{u'(x)}{[u(x)]^{1+\alpha}}\frac{[v(n)]^{(q-1)(1-\alpha)}}{[v'(n)]^{q-1}}a_n^q,$$

while Lebesgue term by term integration theorem provides

$$L_1 \leqslant \left\{ \int_b^c \sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right) \frac{u'(x)}{[u(x)]^{1+\alpha}} \frac{[v(n)]^{(q-1)(1-\alpha)}}{[v'(n)]^{q-1}} a_n^q dx \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=n_0}^{\infty} \left[[v(n)]^{\alpha} \int_b^c h\left(\frac{v(n)}{u(x)}\right) \frac{u'(x)dx}{[u(x)]^{1+\alpha}} \right] \frac{[v(n)]^{q(1-\alpha)-1}}{[v'(n)]^{q-1}} a_n^q \right\}^{\frac{1}{q}}$$

$$= \left\{ \sum_{n=n_0}^{\infty} \omega(n) \frac{[v(n)]^{q(1-\alpha)-1}}{[v'(n)]^{q-1}} a_n^q \right\}^{\frac{1}{q}},$$

and then in view of (9), inequality (12) follows.

(ii) By the reverse Hölder's inequality (cf. [10]) and taking into account that $q<0$, we establish the reverses of (11) and (12) in the same way. □

3. Main results

Now we are ready to establish our main results. For that sake we introduce the functions

$$\Phi(x) = \frac{[u(x)]^{p(1+\alpha)-1}}{[u'(x)]^{p-1}}, \quad x \in (b,c), \quad \text{and} \quad \Psi(n) = \frac{[v(n)]^{q(1-\alpha)-1}}{[v'(n)]^{q-1}}, \quad n \geqslant n_0, \quad n \in \mathbb{N},$$

wherefrom we get

$$[\Phi(x)]^{1-q} = \frac{u'(x)}{[u(x)]^{1+q\alpha}}, \quad \text{and} \quad [\Psi(n)]^{1-p} = \frac{v'(n)}{[v(n)]^{1-p\alpha}}.$$

As before, we deal with the non-negative functions and sequences, hence, such types of conditions will go without saying.

THEOREM 1. *Suppose that the assumptions of Lemma 1 are fulfilled, let* $k(\alpha) \in \mathbb{R}_+$, *and let* p *and* q *be conjugate parameters with* $p>1$. *If* $f \in L_{p,\Phi}(b,c), a = \{a_n\}_{n=n_0}^{\infty} \in l_{q,\Psi}$, $||f||_{p,\Phi}$, $||a||_{q,\Psi} > 0$, *then the following inequalities hold and are equivalent:*

$$I := \sum_{n=n_0}^{\infty} a_n \int_b^c h\left(\frac{v(n)}{u(x)}\right) f(x)dx = \int_b^c f(x) \sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right) a_n dx$$
$$< k(\alpha)||f||_{p,\Phi}||a||_{q,\Psi}, \tag{13}$$

$$J := \left\{ \sum_{n=n_0}^{\infty} [\Psi(n)]^{1-p} \left[\int_b^c h\left(\frac{v(n)}{u(x)}\right) f(x)dx \right]^p \right\}^{\frac{1}{p}} < k(\alpha)||f||_{p,\Phi}, \tag{14}$$

$$L := \left\{ \int_b^c [\Phi(x)]^{1-q} \left[\sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right) a_n \right]^q dx \right\}^{\frac{1}{q}} < k(\alpha)||a||_{q,\Psi}. \tag{15}$$

Moreover, if $v'(y)/v(y), y \geqslant n_0$ *is decreasing and there exist the constants* $\delta > \alpha$ *and* $L > 0,$ *such that* $h(t) \leqslant L/t^{\delta}, t \in [v(n_0), \infty),$ *then the constant factor* $k(\alpha)$ *is the best possible in the above inequalities.*

Proof. The proof consist of the two parts. In the first part we prove the above inequalities together with their equivalence. Note also that, by the Lebesgue term by term integration theorem, there are two expressions for I in (13).

By Lemma 1 $\varpi(x) < k(\alpha),$ so the inequality (14) follows immediately from the relation (11). Now, the inequality (13) follows from (14). Namely, by Hölder's inequality, we have

$$I = \sum_{n=n_0}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_b^c h\left(\frac{v(n)}{u(x)}\right) f(x)dx \right] [\Psi^{\frac{1}{q}}(n) a_n] \leqslant J||a||_{q,\Psi}, \tag{16}$$

so we get (13). Moreover, suppose that the inequality (13) is valid. By considering the sequence

$$a_n = [\Psi(n)]^{1-p} \left[\int_b^c h\left(\frac{v(n)}{u(x)}\right) f(x)dx \right]^{p-1}, n \geqslant n_0, \tag{17}$$

we have $J^{p-1} = ||a||_{q,\Psi}$. Further, the inequality (11) implies that $J < \infty$. If $J = 0,$ then (14) holds trivially. If $J > 0,$ then by (13), we have

$$||a||_{q,\Psi}^q = J^p = I < k(\alpha)||f||_{p,\Phi}||a||_{q,\Psi},$$

i.e.

$$||a||_{q,\Psi}^{q-1} = J < k(\alpha)||f||_{p,\Phi},$$

which means that the inequalities (13) and (14) are equivalent.

The equivalence of (13) and (15) is established in the same way. More precisely, Lemma 1 implies that $[\varpi(x)]^{1-q} > [k(\alpha)]^{1-q},$ hence (15) follows from (12). Now, the Hölder's inequality implies

$$I = \int_b^c [\Phi^{\frac{1}{p}}(x) f(x)] \left[\Phi^{\frac{-1}{p}}(x) \sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right) a_n \right] dx \leqslant ||f||_{p,\Phi} L, \tag{18}$$

that is, we have (13). On the contrary, assuming that (13) is valid and defining

$$f(x) = [\Phi(x)]^{1-q} \left[\sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right) a_n \right]^{q-1}, x \in (b,c),$$

we have $L^{q-1} = ||f||_{p,\Phi}$. Clearly by (12), we find that $L < \infty$. If $L = 0,$ then (15) holds trivially. Moreover, if $L > 0,$ then by (13), we have

$$||f||_{p,\Phi}^p = L^q = I < k(\alpha)||f||_{p,\Phi}||a||_{q,\Psi},$$

i.e.

$$||f||_{p,\Phi}^{p-1} = L < k(\alpha)||a||_{q,\Psi},$$

which yields the equivalence of (13) and (15). Hence, inequalities (13), (14) and (15) are equivalent.

Now, we are going to prove that the constant factor $k(\alpha)$ is the best possible in (13), (14) and (15). Suppose on the contrary that there exists a positive number $k(\leqslant k(\alpha))$, such that (13) is valid if we replace $k(\alpha)$ with k.

By the definition of function u (see the statement of Lemma 1), there exists the unique constant $d \in (b,c)$, satisfying $u(d) = 1$. Let $0 < \varepsilon < p(\delta - \alpha)$. If we substitute the function

$$\widetilde{f}(x) = \begin{cases} 0, & x \in (b,d), \\ [u(x)]^{-\alpha-\frac{\varepsilon}{p}-1}u'(x), & x \in [d,c), \end{cases}$$

and the sequence $\widetilde{a}_n = [v(n)]^{\alpha-\frac{\varepsilon}{q}-1}v'(n), n \geqslant n_0$, in (13) with the smaller constant k, we get the inequality

$$\begin{aligned}
\widetilde{I} &:= \sum_{n=n_0}^{\infty}\int_b^c h\left(\frac{v(n)}{u(x)}\right)\widetilde{a}_n\widetilde{f}(x)dx < k||\widetilde{f}||_{p,\Phi}||\widetilde{a}||_{q,\Psi} \\
&= k\left\{\int_d^c \frac{u'(x)}{[u(x)]^{\varepsilon+1}}dx\right\}^{\frac{1}{p}}\left\{\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \sum_{n=n_0+1}^{\infty}\frac{v'(n)}{[v(n)]^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\
&< k(\frac{1}{\varepsilon})^{\frac{1}{p}}\left\{\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \int_{n_0}^{\infty}\frac{v'(y)}{[v(y)]^{\varepsilon+1}}dy\right\}^{\frac{1}{q}} \\
&= \frac{k}{\varepsilon}\left\{\varepsilon\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + [v(n_0)]^{-\varepsilon}\right\}^{\frac{1}{q}}.
\end{aligned} \tag{19}$$

The left hand side of the previous inequality can be estimated in the following way:

$$\begin{aligned}
\widetilde{I} &= \sum_{n=n_0}^{\infty}[v(n)]^{\alpha-\frac{\varepsilon}{q}-1}v'(n)\int_d^c h\left(\frac{v(n)}{u(x)}\right)[u(x)]^{-\alpha-\frac{\varepsilon}{p}-1}u'(x)dx \\
&\overset{t=v(n)/u(x)}{=} \sum_{n=n_0}^{\infty}[v(n)]^{-\varepsilon-1}v'(n)\int_0^{v(n)} h(t)t^{\alpha+\frac{\varepsilon}{p}-1}dt \\
&= k(\alpha+\frac{\varepsilon}{p})\sum_{n=n_0}^{\infty}[v(n)]^{-\varepsilon-1}v'(n) - A(\varepsilon) \\
&> k(\alpha+\frac{\varepsilon}{p})\int_{n_0}^{\infty}[v(y)]^{-\varepsilon-1}v'(y)dy - A(\varepsilon) \\
&= \frac{1}{\varepsilon}k(\alpha+\frac{\varepsilon}{p})[v(n_0)]^{-\varepsilon} - A(\varepsilon), \\
A(\varepsilon) &:= \sum_{n=n_0}^{\infty}[v(n)]^{-\varepsilon-1}v'(n)\int_{v(n)}^{\infty}h(t)t^{\alpha+\frac{\varepsilon}{p}-1}dt.
\end{aligned} \tag{20}$$

Now, if $h(t) \leqslant L/t^{\delta}$, $\delta > \alpha$, $t \in [v(n_0), \infty)$, we find

$$\begin{aligned}
0 < A(\varepsilon) &\leqslant L \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) \int_{v(n)}^{\infty} t^{\alpha-\delta+\frac{\varepsilon}{p}-1} dt \\
&= \frac{L}{\delta-\alpha-\frac{\varepsilon}{p}} \sum_{n=n_0}^{\infty} [v(n)]^{\alpha-\delta-\frac{\varepsilon}{q}-1} v'(n) \\
&= \frac{L}{\delta-\alpha-\frac{\varepsilon}{p}} \left[\frac{v'(n_0)}{[v(n_0)]^{\delta-\alpha+\frac{\varepsilon}{q}+1}} + \sum_{n=n_0+1}^{\infty} \frac{v'(n)}{[v(n)]^{\delta-\alpha+\frac{\varepsilon}{q}+1}} \right] \\
&\leqslant \frac{L}{\delta-\alpha-\frac{\varepsilon}{p}} \left[\frac{v'(n_0)}{[v(n_0)]^{\delta-\alpha+\frac{\varepsilon}{q}+1}} + \int_{n_0}^{\infty} \frac{v'(y)}{[v(y)]^{\delta-\alpha+\frac{\varepsilon}{q}+1}} dy \right] \\
&= \frac{L}{\delta-\alpha-\frac{\varepsilon}{p}} \left[\frac{v'(n_0)}{[v(n_0)]^{\delta-\alpha+\frac{\varepsilon}{q}+1}} + \frac{[v(n_0)]^{\alpha-\delta-\frac{\varepsilon}{q}}}{\delta-\alpha+\frac{\varepsilon}{q}} \right] < \infty,
\end{aligned}$$

that is, $A(\varepsilon) = O(1)(\varepsilon \to 0^+)$. Thus, the relations (19) and (20) imply the inequality

$$k(\alpha+\frac{\varepsilon}{p})[v(n_0)]^{-\varepsilon} - \varepsilon O(1) < k \left\{ \varepsilon \frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + [v(n_0)]^{-\varepsilon} \right\}^{\frac{1}{q}}. \tag{21}$$

The well-known Fatou Lemma (cf. [9]) yields $k(\alpha) \leqslant \underline{\lim}_{\varepsilon\to 0^+} k(\alpha+\frac{\varepsilon}{p})$, thus by (21), it follows $k(\alpha) \leqslant k(\varepsilon \to 0^+)$, i.e. $k(\alpha)$ is the best value for inequality (13).

Due to the equivalence, it follows easily that $k(\alpha)$ is also the best possible constant factor in (14) and (15). Namely, if we suppose that $k(\alpha)$ is not the best possible constant in (14) and (15), then the relations (16) and (18) imply that $k(\alpha)$ is not the best possible constant factor in (13), which contradicts with the previously proved facts. □

Inequalities (14) and (15) enable us to define some interesting operators between some particular function spaces. Due to Theorem 1, we shall be able to determine the norm of such operators.

REMARK 1. (i) Define a half-discrete Hilbert's operator $T : L_{p,\Phi}(b,c) \to l_{p,\Psi^{1-p}}$ in the following way: For $f \in L_{p,\Phi}(b,c)$, we define $Tf \in l_{p,\Psi^{1-p}}$, as

$$Tf(n) = \int_b^c h\left(\frac{v(n)}{u(x)}\right) f(x)dx, n \geqslant n_0.$$

Then by (14), it follows $||Tf||_{p,\Psi^{1-p}} \leqslant k(\alpha)||f||_{p,\Phi}$, i.e. T is the bounded operator with $||T|| \leqslant k(\alpha)$. Since the constant factor in (14) is the best possible, we have $||T|| = k(\alpha)$.

(ii) Define a half-discrete Hilbert's operator $\widetilde{T} : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(b,c)$ as follows: For $a \in l_{q,\Psi}$, we define $\widetilde{T}a \in L_{q,\Phi^{1-q}}(b,c)$, in the following way:

$$\widetilde{T}a(x) = \sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right) a_n, x \in (b,c).$$

Due to (15), it follows $||\widetilde{T}a||_{q,\Phi^{1-q}} \leqslant k(\alpha)||a||_{q,\Psi}$, that is, $\widetilde{T}$ is bounded operator, $||\widetilde{T}|| \leqslant k(\alpha)$. Since the inequality (15) includes the best possible constant factor, we have $||\widetilde{T}|| = k(\alpha)$.

The following result considers the setting with conjugate parameters p and q, where $0 < p < 1$. In such a way we shall obtain the inequalities related with (13), (14) and (15), but with the reversed sign of inequality.

THEOREM 2. *Suppose that the assumptions of Lemma 1 are fulfilled and let* $k(\alpha) \in \mathbb{R}_+$, $k(\alpha)(1-\theta(x)) < \varpi(x) < k(\alpha)$, $\widetilde{\Phi}(x) = (1-\theta(x))\Phi(x)$, *where* $x \in (b,c)$ *and* $\theta(x) \in (0,1)$. *If* p *and* q *are conjugate parameters with* $0 < p < 1$, $f \in L_{p,\widetilde{\Phi}}(b,c), a = \{a_n\}_{n=n_0}^{\infty} \in l_{q,\Psi}$, $0 < ||f||_{p,\widetilde{\Phi}} < \infty$, $0 < ||a||_{q,\Psi} < \infty$, *then the following inequalities hold and are equivalent:*

$$I := \sum_{n=n_0}^{\infty} \int_b^c h\left(\frac{v(n)}{u(x)}\right) a_n f(x)dx = \int_b^c \sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right) a_n f(x)dx$$
$$> k(\alpha)||f||_{p,\widetilde{\Phi}}||a||_{q,\Psi}, \tag{22}$$

$$J := \left\{ \sum_{n=n_0}^{\infty} [\Psi(n)]^{1-p} \left[\int_b^c h\left(\frac{v(n)}{u(x)}\right) f(x)dx \right]^p \right\}^{\frac{1}{p}} > k(\alpha)||f||_{p,\widetilde{\Phi}}, \tag{23}$$

$$\widetilde{L} := \left\{ \int_b^c [\widetilde{\Phi}(x)]^{1-q} \left[\sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right) a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\alpha)||a||_{q,\Psi}. \tag{24}$$

Moreover, if $v'(y)/v(y)$, $y \geqslant n_0$, *is decreasing and there exist constants* η, $\delta_0 > 0$, *satisfying* $\theta(x) = O(1/[u(x)]^{\eta})$, $x \in [d,c)$ *and* $k(\alpha+\delta_0) \in \mathbb{R}_+$, *then the constant factor* $k(\alpha)$ *is the best possible in the above inequalities.*

Proof. Taking into account the relation (9), the reverse inequality in (11), equipped with the relation $\varpi(x) > k(\alpha)(1-\theta(x))$, yields (23) immediately. Moreover, the reverse Hölder's inequality yields

$$I = \sum_{n=n_0}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_b^c h\left(\frac{v(n)}{u(x)}\right) f(x)dx \right] [\Psi^{\frac{1}{q}}(n)a_n] \geqslant J||a||_{q,\Psi}, \tag{25}$$

hence (22) holds, since (23) is valid. On the other hand, suppose that (22) holds. Clearly, $J^{p-1} = ||a||_{q,\Psi}$, where the sequence a_n is defined by (17). Due to the reverse inequality in (11), we find that $J > 0$. Besides, if $J = \infty$, then (23) holds trivially. Finally, if $J < \infty$, then by (22), we have

$$||a||_{q,\Psi}^q = J^p = I > k(\alpha)||f||_{p,\widetilde{\Phi}}||a||_{q,\Psi}, \quad \text{i.e.} \quad ||a||_{q,\Psi}^{q-1} = J > k(\alpha)||f||_{p,\widetilde{\Phi}},$$

that is, the relations (22) and (23) are equivalent.

Similarly as in the proof of Theorem 1, it is enough to show the equivalence of the relations (22) and (24). Namely, since $[\varpi(x)]^{1-q} > [k(\alpha)(1-\theta(x))]^{1-q}$, $q<0$, the inequality (24) follows from the reverse inequality in (12). Yet another use of the reverse Hölder's inequality yields

$$I = \int_b^c [\widetilde{\Phi}^{\frac{1}{p}}(x)f(x)]\left[\widetilde{\Phi}^{\frac{-1}{p}}(x)\sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right)a_n\right]dx \geqslant ||f||_{p,\widetilde{\Phi}}\widetilde{L}, \tag{26}$$

i.e. we have (22) due to (24). It remains to prove that inequality (22) implies (24). More precisely, we have that $\widetilde{L}^{q-1} = ||f||_{p,\widetilde{\Phi}}$, where

$$f(x) := [\widetilde{\Phi}(x)]^{1-q}\left[\sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right)a_n\right]^{q-1}, x\in(b,c).$$

Clearly, $\widetilde{L}>0$, due the reverse inequality in (12). If $\widetilde{L}=\infty$, then (24) holds trivially, while for $\widetilde{L}<\infty$ we have (by using (22)),

$$||f||^p_{p,\widetilde{\Phi}} = \widetilde{L}^q = I > k(\alpha)||f||_{p,\widetilde{\Phi}}||a||_{q,\Psi}, \quad \text{i.e.} \quad ||f||^{p-1}_{p,\widetilde{\Phi}} = \widetilde{L} > k(\alpha)||a||_{q,\Psi},$$

that is (24). Hence, we have showed that the inequalities (22), (23) and (24) are mutually equivalent.

In the sequel, we prove that the constant factor $k(\alpha)$ is the best possible in (22), (23) and (24). Suppose on the contrary that there exists a positive number $k(\geqslant k(\alpha))$, such that (22) is valid if we replace $k(\alpha)$ with k.

Let $0<\varepsilon<p\delta_0$. If we substitute the function $\widetilde{f}(x)$ and the sequence $\widetilde{a}_n$ (defined in the proof of Theorem 1), in (22) with the greater constant k, we get the inequality

$$\begin{aligned}
\widetilde{I} &:= \int_b^c \sum_{n=n_0}^{\infty} h\left(\frac{v(n)}{u(x)}\right)\widetilde{a}_n\widetilde{f}(x)dx > k||\widetilde{f}||_{p,\widetilde{\Phi}}||\widetilde{a}||_{q,\Psi} \\
&= k\left\{\int_d^c \left(1-O(\frac{1}{[u(x)]^{\eta}}\right)\frac{u'(x)dx}{[u(x)]^{\varepsilon+1}}\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^{\infty}\frac{v'(n)}{[v(n)]^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\
&= k\{\frac{1}{\varepsilon}-O(1)\}^{\frac{1}{p}}\left\{\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}}+\sum_{n=n_0+1}^{\infty}\frac{v'(n)}{[v(n)]^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\
&> k\{\frac{1}{\varepsilon}-O(1))^{\}\frac{1}{p}}\left\{\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}}+\int_{n_0}^{\infty}\frac{v'(y)}{[v(y)]^{\varepsilon+1}}dy\right\}^{\frac{1}{q}} \\
&= \frac{k}{\varepsilon}\{1-\varepsilon O(1)\}^{\frac{1}{p}}\left\{\varepsilon\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}}+[v(n_0)]^{-\varepsilon}\right\}^{\frac{1}{q}}.
\end{aligned} \tag{27}$$

Note that in this theorem $q<0$, so the notations with norm are formal. On the other hand, the left-hand side of inequality (22) can be estimated in the following way:

$$
\begin{aligned}
\widetilde{I} &= \sum_{n=n_0}^{\infty} [v(n)]^{\alpha-\frac{\varepsilon}{q}-1} v'(n) \int_d^c h\left(\frac{v(n)}{u(x)}\right) [u(x)]^{-\alpha-\frac{\varepsilon}{p}-1} u'(x)dx \\
&\leqslant \sum_{n=n_0}^{\infty} [v(n)]^{\alpha-\frac{\varepsilon}{q}-1} v'(n) \int_b^c h\left(\frac{v(n)}{u(x)}\right) [u(x)]^{-\alpha-\frac{\varepsilon}{p}-1} u'(x)dx \\
&\overset{t=v(n)/u(x)}{=} \sum_{n=n_0}^{\infty} [v(n)]^{-\varepsilon-1} v'(n) \int_0^{\infty} h(t) t^{\alpha+\frac{\varepsilon}{p}-1} dt \\
&\leqslant k(\alpha+\frac{\varepsilon}{p}) \left[\frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + \int_{n_0}^{\infty} [v(y)]^{-\varepsilon-1} v'(y)dy\right] \\
&= \frac{1}{\varepsilon} k(\alpha+\frac{\varepsilon}{p}) \left[\varepsilon \frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + [v(n_0)]^{-\varepsilon}\right]. \qquad (28)
\end{aligned}
$$

Further, since $h(t)t^{\alpha+\frac{\varepsilon}{p}-1} \leqslant h(t)t^{\alpha+\delta_0-1}, t \in [1,\infty)$ and

$$
\int_1^{\infty} h(t)t^{\alpha+\delta_0-1} dt \leqslant k(\alpha+\delta_0) < \infty,
$$

then, by the Lebesgue control convergence theorem (cf. [9]), it follows

$$
\begin{aligned}
k(\alpha+\frac{\varepsilon}{p}) &\leqslant \int_0^1 h(t)t^{\alpha-1} dt + \int_1^{\infty} h(t)t^{\alpha+\frac{\varepsilon}{p}-1} dt \\
&= k(\alpha) + o(1)(\varepsilon \to 0^+).
\end{aligned}
$$

Finally, the relations (27), (28) and the above result yield inequality

$$
\begin{aligned}
&(k(\alpha)+o(1)) \left[\varepsilon \frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + [v(n_0)]^{-\varepsilon}\right] \\
&> k\left\{1-\varepsilon O(1)\right\}^{\frac{1}{p}} \left[\varepsilon \frac{v'(n_0)}{[v(n_0)]^{\varepsilon+1}} + [v(n_0)]^{-\varepsilon}\right]^{\frac{1}{q}},
\end{aligned}
$$

that is, we get $k(\alpha) \geqslant k(\varepsilon \to 0^+)$. Hence $k(\alpha)$ is the best possible constant in (22).

Due to the equivalence, it follows easily that $k(\alpha)$ is also the best possible constant factor in (23) and (24). Namely, if we suppose that $k(\alpha)$ is not the best possible constant in (23) and (24), then the relations (25) and (26) imply that $k(\alpha)$ is not the best possible constant factor in (22), which gives a contradiction. □

4. Some applications

We conclude this paper with few applications of our main results. More precisely, we consider here some examples of homogeneous kernels in particular settings which form the functions $f(x,y)$ which satisfy miscellaneous conditions from Lemma 1. These particular results will be presented in the form (13), while the equivalent forms, as well as the reverse inequalities will here be omitted. Of course, we use the notations as in Lemma 1.

4.1. First example

If $n_0 = 1$, $b = 0$, $c = \infty$, $u(x) = v(x) = x$, $x \in (0, \infty)$, inequality (13) reduces to (6). In particular, if we consider the kernel

$$h(t) = \left(\frac{\min\{1,t\}}{\max\{1,t\}}\right)^{\lambda}, \quad 0 < \lambda \leqslant \frac{1}{2},$$

then for $|\alpha| < \lambda$, we have

$$\begin{aligned} k(\alpha) &= \int_0^{\infty} \left(\frac{\min\{1,t\}}{\max\{1,t\}}\right)^{\lambda} t^{\alpha-1} dt = \int_0^1 t^{\lambda} t^{\alpha-1} dt + \int_1^{\infty} t^{-\lambda} t^{\alpha-1} dt \\ &= \frac{2\lambda}{\lambda^2 - \alpha^2} \in \mathbb{R}_+. \end{aligned}$$

Clearly, for a fixed x, the function

$$f(x,y) = x^{-\alpha} \left(\frac{\min\{x,y\}}{\max\{x,y\}}\right)^{\lambda} y^{\alpha-1} = \begin{cases} x^{\lambda-\alpha} (\frac{1}{y})^{\lambda-\alpha+1}, x \leqslant y \\ x^{-\lambda-\alpha} (\frac{1}{y})^{1-\lambda-\alpha}, x > y \end{cases}$$

is strictly decreasing on interval $(0, \infty)$, i.e. the condition (i) in Lemma 1 is fulfilled. Hence, we have

$$\begin{aligned} \varpi(x) &= x^{-\alpha} \sum_{n=1}^{\infty} \left(\frac{\min\{x,n\}}{\max\{x,n\}}\right)^{\lambda} n^{\alpha-1} < x^{-\alpha} \int_0^{\infty} \left(\frac{\min\{x,y\}}{\max\{x,y\}}\right)^{\lambda} y^{\alpha-1} dy \\ &= \int_0^{\infty} \left(\frac{\min\{1,t\}}{\max\{1,t\}}\right)^{\lambda} t^{\alpha-1} dt = k(\alpha). \end{aligned}$$

Moreover, if we take $\delta = \lambda > \alpha$, it follows that

$$h(t) = \left(\frac{\min\{1,t\}}{\max\{1,t\}}\right)^{\lambda} = t^{-\delta},\ t \in [1, \infty).$$

Hence, the assumptions of Theorem 1 are fulfilled, so we get the following inequality with the best possible constant factor $2\lambda/(\lambda^2 - \alpha^2)$:

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n \int_0^{\infty} \left(\frac{\min\{x,n\}}{\max\{x,n\}}\right)^{\lambda} f(x) dx \\ &< \frac{2\lambda}{\lambda^2 - \alpha^2} \left\{\int_0^{\infty} x^{p(1+\alpha)-1} f^p(x) dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\alpha)-1} a_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{29}$$

4.2. Second example

Let $n_0 = 1$, $b = 0$, $c = \infty$, $u(x) = v(x) = x^{\lambda}$, $x, \lambda \in \mathbb{R}_+$. By introducing the kernel

$$h(t) = \frac{\min\{1,t\}}{1+t},$$

we have

$$\begin{aligned} k(\alpha) &= \int_0^\infty \frac{\min\{1,t\}}{1+t} t^{\alpha-1} dt = \int_0^1 \frac{t^\alpha dt}{1+t} dt + \int_1^\infty \frac{t^{\alpha-1} dt}{1+t} \\ &= \int_0^1 \frac{(t^\alpha + t^{-\alpha})dt}{1+t} = \int_0^1 \sum_{k=1}^\infty (-1)^{k-1}(t^{k-1+\alpha} + t^{k-1-\alpha})dt \\ &= \sum_{k=1}^\infty (-1)^{k-1} \int_0^1 (t^{k-1+\alpha} + t^{k-1-\alpha})dt = 2\sum_{k=1}^\infty \frac{(-1)^{k-1}k}{k^2-\alpha^2} \in \mathbb{R}_+, \end{aligned}$$

where $|\alpha| < 1, \lambda(1+\alpha) \leqslant 1.$ Moreover, since

$$f(x,y) = \frac{\lambda \min\{x^\lambda, y^\lambda\} y^{\lambda\alpha-1}}{x^{\lambda\alpha}(x^\lambda + y^\lambda)} = \begin{cases} \frac{\lambda y^{\lambda(1+\alpha)-1}}{x^{\lambda\alpha}(x^\lambda+y^\lambda)}, y < x \\ \frac{\lambda y^{\lambda\alpha-1}}{x^{\lambda(\alpha-1)}(x^\lambda+y^\lambda)}, y \geqslant x \end{cases}$$

is strictly decreasing for $y \in \mathbb{R}_+$, then the condition (i) from Lemma 1 is fulfilled, i.e. we have $\varpi(x) < k(\alpha)$. Further, for $\delta = 1 > \alpha$, it follows $h(t) = \min\{1,t\}/(1+t) \leqslant t^{-\delta}$, $t \in [1,\infty)$. Hence, in this particular setting, the inequality (13) reads

$$\begin{aligned} &\sum_{n=1}^\infty a_n \int_0^\infty \frac{\min\{x^\lambda, n^\lambda\}}{x^\lambda + n^\lambda} f(x)dx \\ &< \frac{k(\alpha)}{\lambda} \left\{ \int_0^\infty x^{p(1+\lambda\alpha)-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^\infty n^{q(1-\lambda\alpha)-1} a_n^q \right\}^{\frac{1}{q}}, \end{aligned} \tag{30}$$

where the constant factor $k(\alpha)/\lambda$ is the best possible.

4.3. Third example

Consider the following particular case. Let $n_0 = 1$, $b = \beta$, $c = \infty$, $u(x) = v(x) = (x-\beta)^\eta$, $0 < \eta \leqslant 1$, $0 \leqslant \beta \leqslant 1/2$. If we consider the kernel $h(t) = e^{-\gamma t}$, $\gamma > 0$, $0 < \eta\alpha \leqslant 1$, we have

$$k(\alpha) = \int_0^\infty e^{-\gamma t} t^{\alpha-1} dt = \frac{\Gamma(\alpha)}{\gamma^\alpha} \in \mathbb{R}_+,$$

where $\Gamma(\cdot)$ denotes the usual Gamma function. Moreover, for a fixed x, the function

$$f(x,y) = (x-\beta)^{-\eta\alpha} e^{-\gamma(\frac{y-\beta}{x-\beta})^\eta} (y-\beta)^{\eta\alpha-1}$$

is decreasing and strictly convex on interval $(1/2,\infty)$. i.e. the condition (ii) in Lemma 1 is fulfilled. Hence,

$$\begin{aligned} \varpi(x) &= \eta(x-\beta)^{-\eta\alpha} \sum_{n=1}^\infty e^{-\gamma(\frac{n-\beta}{x-\beta})^\eta} (n-\beta)^{\eta\alpha-1} \\ &< \eta(x-\beta)^{-\eta\alpha} \int_{\frac{1}{2}}^\infty e^{-\gamma(\frac{y-\beta}{x-\beta})^\eta} (y-\beta)^{\eta\alpha-1} dy \\ &= \int_{(\frac{\frac{1}{2}-\beta}{x-\beta})^\eta}^\infty e^{-\gamma t} t^{\alpha-1} dt \leqslant \int_0^\infty e^{-\gamma t} t^{\alpha-1} dt = k(\alpha). \end{aligned}$$

Besides, for a constant $\delta \in (\alpha,\infty)$, there exists a constant $L > 0$, such that $0 < h(t)t^{\delta} = t^{\delta}e^{-\gamma t} \leqslant L$, $t \in [(1-\beta)^{\eta},\infty)$. This means that the conditions of Theorem 1 are fulfilled, so the inequality (13) reduces to the following inequality which includes the best possible constant factor:

$$\begin{aligned}&\sum_{n=1}^{\infty} a_n \int_{\beta}^{\infty} e^{-\gamma(\frac{n-\beta}{x-\beta})^{\eta}} f(x)dx \\ &< \frac{1}{\eta\gamma^{\alpha}}\Gamma(\alpha)\left\{\int_{\beta}^{\infty}(x-\beta)^{p(1+\eta\alpha)-1}f^{p}(x)dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}(n-\beta)^{q(1-\eta\alpha)-1}a_n^{q}\right\}^{\frac{1}{q}}. \quad (31)\end{aligned}$$

4.4. Fourth example

Setting $n_0 = 1$, $b = 1-\beta = \gamma$, $c = \infty$, $u(x) = v(x) = x-\gamma$, $\gamma \leqslant \frac{1}{4}(3-\sqrt{5}) = 0.19^{+}$, $h(t) = \frac{1}{1+t^2}$, $\alpha = 1$, we have

$$k(1) = \int_0^{\infty} h(t)dt = \int_0^{\infty} \frac{1}{1+t^2}dt = \frac{\pi}{2} \in \mathbb{R}_+,$$

and

$$f(x,y) = \frac{1}{(x-\gamma)[1+(\frac{y-\gamma}{x-\gamma})^2]} = \frac{x-\gamma}{(x-\gamma)^2+(y-\gamma)^2}.$$

Hence $v(y)$, $y \in [\gamma,\infty)$, is strictly increasing with $v(1-\beta) = v(\gamma) = 0$, and for any fixed $x \in (\gamma,\infty)$, $f(x,y)$ is differentiable with

$$\begin{aligned} f'_y(x,y) &= \frac{-2(x-\gamma)(y-\gamma)}{[(x-\gamma)^2+(y-\gamma)^2]^2} \\ &= \frac{-2(x-\gamma)[(y-\gamma)^2+(x-\gamma)^2-(x-\gamma)^2]}{(y-\gamma)[(x-\gamma)^2+(y-\gamma)^2]^2} \\ &= \frac{-2(x-\gamma)}{(y-\gamma)[(x-\gamma)^2+(y-\gamma)^2]} + \frac{2(x-\gamma)^3}{(y-\gamma)[(x-\gamma)^2+(y-\gamma)^2]^2}. \end{aligned}$$

Further, set

$$R(x) := \int_{\gamma}^{1} f(x,y)dy - \frac{1}{2}f(x,1) - \int_1^{\infty} \rho(y) f'_y(x,y)dy. \quad (32)$$

In view of the following improvement of Euler-Maclaurin summation formula (cf. [18], relation (2.2.13))

$$-\frac{1}{8}g(1) < \int_1^{\infty} \rho(y)g(y)dy < 0, \quad (33)$$

where $g^{(i)}(\infty)=0,\ (-1)^i g^{(i)}(y)>0,\ y\in[1,\infty),\ i=0,1,$ by (32), we find that

$$\begin{aligned}
R(x) &= \int_\gamma^1 \frac{(x-\gamma)dy}{(x-\gamma)^2+(y-\gamma)^2} - \frac{x-\gamma}{2[(x-\gamma)^2+(1-\gamma)^2]} \\
&\quad + \int_1^\infty \rho(y)\frac{2(x-\gamma)}{(y-\gamma)[(x-\gamma)^2+(y-\gamma)^2]}dy \\
&\quad - \int_1^\infty \rho(y)\frac{2(x-\gamma)^3}{(y-\gamma)[(x-\gamma)^2+(y-\gamma)^2]^2}dy \\
&> \arctan\frac{1-\gamma}{x-\gamma} - \frac{x-\gamma}{2[(x-\gamma)^2+(1-\gamma)^2]} \\
&\quad -\frac{1}{8}\frac{2(x-\gamma)}{(1-\gamma)[(x-\gamma)^2+(1-\gamma)^2]} \\
&= \arctan\frac{1-\gamma}{x-\gamma} - \frac{A(x-\gamma)}{(x-\gamma)^2+(1-\gamma)^2} := h(x),
\end{aligned} \tag{34}$$

where $A := \frac{1}{2}+\frac{1}{4(1-\gamma)}>0.$ Now, for $x>\gamma,$ we have

$$\begin{aligned}
h'(x) &= \frac{-(1-\gamma)}{(x-\gamma)^2+(1-\gamma)^2} - \frac{A}{(x-\gamma)^2+(1-\gamma)^2} \\
&\quad + \frac{2A[(x-\gamma)^2+(1-\gamma)^2-(1-\gamma)^2]}{[(x-\gamma)^2+(1-\gamma)^2]^2} \\
&= \frac{1}{(x-\gamma)^2+(1-\gamma)^2}[-(1-\gamma-A) - \frac{2A(1-\gamma)^2}{(x-\gamma)^2+(1-\gamma)^2}].
\end{aligned}$$

Since for $1-\gamma \geqslant \frac{1}{4}(1+\sqrt{5}),$

$$1-\gamma-A = \frac{1}{1-\gamma}[(1-\gamma)^2 - \frac{1}{2}(1-\gamma) - \frac{1}{4}] \geqslant 0,$$

it follows that $h'(x)<0,$ that is, $h(x)$ is strictly decreasing on $(\gamma,\infty).$ Then by (34), we have $R(x)>h(x)>h(\infty)=0,\ x\in(\gamma,\infty).$

Hence due to Condition (iii), it follows that $\varpi(x)<k(1)=\frac{\pi}{2},\ x\in(\gamma,\infty).$ Moreover, for a constant $\delta=2>1=\alpha,$ we have

$$h(t)=\frac{1}{1+t^2}\leqslant\frac{1}{t^\delta},\ t\in[1-\gamma,\infty),$$

so by (13), we have the following inequality with the best possible constant factor $\frac{\pi}{2}$:

$$\sum_{n=1}^\infty \int_\gamma^\infty \frac{a_n(x-\gamma)^2 f(x)dx}{(x-\gamma)^2+(n-\gamma)^2} < \frac{\pi}{2}\left\{\int_\gamma^\infty \frac{f^p(x)dx}{(x-\gamma)^{1-2p}}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty \frac{a_n^q}{n-\gamma}\right\}^{\frac{1}{q}}. \tag{35}$$

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