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# LOGARITHMIC CONVEXITY OF GINI MEANS

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*Abstract.* In the paper, alternative proofs for the monotonicity and logarithmic convexity of Gini means and related functions are presented by using new approaches and techniques.

## 1. Introduction

Recall from [7] that Gini means are defined as

$$G(r,s;x,y)=\begin{cases}\left(\dfrac{x^s+y^s}{x^r+y^r}\right)^{1/(s-r)}, & r\neq s;\\ \exp\left(\dfrac{x^r\ln x+y^r\ln y}{x^r+y^r}\right), & r=s;\end{cases}\tag{1}$$

where $x$ and $y$ are positive variables and $r$ and $s$ are real variables. They are also called sum mean values.

There has been a lot of literature such as [3, 4, 5, 6, 12, 13, 15, 16, 17, 24, 25, 26] devoted to studying inequalities and properties of Gini means.

The aim of this paper is to provide alternative proofs for the monotonicity and logarithmic convexity of Gini means $G(r,s;x,y)$ and to establish some new properties of functions involving Gini means.

Our main results can be stated as the following theorems.

THEOREM 1. *Gini means $G(r,s;x,y)$ are*

1. *increasing with respect to both $r\in(-\infty,\infty)$ and $s\in(-\infty,\infty)$;*
2. *logarithmically convex with respect to both $r$ and $s$ if $(r,s)\in(-\infty,0)\times(-\infty,0)$;*
3. *logarithmically concave with respect to both $r$ and $s$ if $(r,s)\in(0,\infty)\times(0,\infty)$.*

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THEOREM 2. *Let*

$$H_{r,s;x,y}(t) = G(r+t, s+t; x, y) \tag{2}$$

*and*

$$K_{r,s;x,y}(t) = H_{r,s;x,y}(t)H_{r,s;x,y}(-t) \tag{3}$$

*for* $t \in \mathbb{R}$. *Then*

1. *Gini means* $H_{r,s;x,y}(t)$ *are*
   - (a) *increasing on* $(-\infty, \infty)$,
   - (b) *logarithmically convex on* $\left(-\infty, -\frac{r+s}{2}\right)$,
   - (c) *logarithmically concave on* $\left(-\frac{r+s}{2}, \infty\right)$;
2. *The function* $K_{r,s;x,y}(t)$ *is*
   - (a) *increasing on* $(-\infty, 0)$ *and decreasing on* $(0, \infty)$ *for* $r+s>0$,
   - (b) *decreasing on* $(-\infty, 0)$ *and increasing on* $(0, \infty)$ *for* $s+r<0$.

THEOREM 3. *The function* $[H_{r,s;x,y}(t)]^t$ *is logarithmically convex*

1. *on* $\left(-\frac{s+r}{2}, 0\right)$ *for* $s+r>0$,
2. *on* $\left(0, -\frac{s+r}{2}\right)$ *for* $s+r<0$.

In next section, we will prove these theorems systematically. In the third section, we will simply summarize and review recent developments and applications on this topic by several remarks.

## 2. Proofs of theorems

In this section we pay our attentions on proving our three theorems.

*Proof of Theorem* 1. It is easy to see that

$$\ln G(r,s;x,y) = \begin{cases} \dfrac{1}{s-r}\displaystyle\int_r^s \frac{x^u \ln x + y^u \ln y}{x^u + y^u}\,\mathrm{d}u, & r \neq s, \\ \dfrac{x^r \ln x + y^r \ln y}{x^r + y^r}, & r = s. \end{cases} \tag{4}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{x^u \ln x + y^u \ln y}{x^u + y^u}\right] = \frac{x^u y^u (\ln x - \ln y)^2}{(x^u + y^u)^2} > 0 \tag{5}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left[\frac{x^u \ln x + y^u \ln y}{x^u + y^u}\right] = -\frac{x^u y^u (x^u - y^u)(\ln x - \ln y)^3}{(x^u + y^u)^3} \gtreqless 0, \quad u \lesseqgtr 0, \tag{6}$$

then the integrand in (4) is increasing with respect to $u \in (-\infty, \infty)$, convex with respect to $u \in (-\infty, 0)$, and concave with respect to $u \in (0, \infty)$. Recall from [23, Lemma 1] that if $f(t)$ is differentiable and increasing on an interval $I$ then the integral arithmetic mean of $f(t)$,

$$\phi(r,s)=\begin{cases}\dfrac{1}{s-r}\displaystyle\int_r^s f(t)\,\mathrm{d}t, & r\neq s,\\ f(r), & r=s,\end{cases} \tag{7}$$

is also increasing with $r$ and $s$ on $I$, and that if $f(t)$ is twice differentiable and convex on $I$ then $\phi(r,s)$ is also convex with $r$ and $s$ on $I$. Consequently, Gini means $G(r,s;x,y)$ with respect to both $r$ and $s$ are increasing on $(-\infty,\infty)$, logarithmically convex if $(r,s)\in(-\infty,0)\times(-\infty,0)$, and logarithmically concave if $(r,s)\in(0,\infty)\times(0,\infty)$. The proof of Theorem 1 is complete. □

*First proof of Theorem* 2. Taking the logarithm of $H_{r,s;x,y}(t)$ and differentiating consecutively yield

$$\ln H_{r,s;x,y}(t)=\frac{1}{s-r}\left[\ln\left(x^{s+t}+y^{s+t}\right)-\ln\left(x^{r+t}+y^{r+t}\right)\right],$$

$$\left[\ln H_{r,s;x,y}(t)\right]'=\frac{1}{s-r}\left(\frac{x^{s+t}\ln x+y^{s+t}\ln y}{x^{s+t}+y^{s+t}}-\frac{x^{r+t}\ln x+y^{r+t}\ln y}{x^{r+t}+y^{r+t}}\right), \tag{8}$$

$$\left[\ln H_{r,s;x,y}(t)\right]''=\frac{1}{s-r}\left[\frac{x^{s+t}y^{s+t}(\ln x-\ln y)^2}{(x^{s+t}+y^{s+t})^2}-\frac{x^{r+t}y^{r+t}(\ln x-\ln y)^2}{(x^{r+t}+y^{r+t})^2}\right]. \tag{9}$$

By virtue of (5), it follows that $\left[\ln H_{r,s;x,y}(t)\right]'\geqslant 0$ which means that Gini means $H_{r,s;x,y}(t)$ is increasing on $(-\infty,\infty)$.

With the aid of (6), it may be obtained that the function

$$f_{x,y}(u)=\frac{x^u y^u(\ln x-\ln y)^2}{(x^u+y^u)^2} \tag{10}$$

is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$. Note that the function $f_{x,y}(u)$ is even on $(-\infty,\infty)$.

Let

$$F_{x,y}(t)=f_{x,y}(s+t)-f_{x,y}(r+t). \tag{11}$$

If $s+t>r+t>0$, that is, $t>-r>-s$, since $f_{x,y}(u)$ is decreasing on $(0,\infty)$, then $F_{x,y}(t)\leqslant 0$. Similarly, if $r+t<s+t<0$, i.e., $t<-s<-r$, then $F_{x,y}(t)\geqslant 0$. If $r+t<0<s+t$ and $0<-(r+t)<s+t$, equivalently, $t>-\frac{r+s}{2}$, since $f_{x,y}(u)$ is even on $(-\infty,\infty)$ and decreasing on $(0,\infty)$, then $F_{x,y}(t)\leqslant 0$; Similarly, if $t<-\frac{r+s}{2}$, then $F_{x,y}(t)\geqslant 0$. This implies

$$\left[\ln H_{r,s;x,y}(t)\right]''\gtreqless 0,\quad t\lesseqgtr-\frac{r+s}{2}$$

for all $r,s,x,y$ by a recourse to symmetric properties

$$G(r,s;x,y)=G(s,r;x,y)=G(r,s;y,x).$$

Taking the logarithm on both sides of (3) and differentiating give

$$\left[\ln K_{r,s;x,y}(t)\right]' = \frac{H'_{r,s;x,y}(t)}{H_{r,s;x,y}(t)} - \frac{H'_{r,s;x,y}(-t)}{H_{r,s;x,y}(-t)}, \tag{12}$$

where

$$H'_{r,s;x,y}(-t) = \frac{\mathrm{d}H_{r,s;x,y}(u)}{\mathrm{d}u}\bigg|_{u=-t}.$$

The logarithmic convexity of $H_{r,s;x,y}(t)$ implies that the function $\frac{H'_{r,s;x,y}}{H_{r,s;x,y}}(t)$ is increasing on $\left(-\infty,-\frac{r+s}{2}\right)$ and decreasing on $\left(-\frac{r+s}{2},0\right)$. A careful computation verifies that

$$\frac{H'_{r,s;x,y}(t)}{H_{r,s;x,y}(t)} = \frac{H'_{r,s;x,y}(-t-(s+r))}{H_{r,s;x,y}(-t-(s+r))}$$

for $t \in (-\infty,\infty)$. Consequently, the function

$$Q(t) = \frac{H'_{r,s;x,y}(t-(s+r)/2)}{H_{r,s;x,y}(t-(s+r)/2)}$$

is increasing on $(-\infty,0)$, decreasing on $(0,\infty)$, and even on $(-\infty,\infty)$. Utilization of the approach applied to the function $f_{x,y}$ defined by (10) yields that $Q(t+(s+r))-Q(t)$ is positive on $\left(-\infty,-\frac{s+r}{2}\right)$ and negative on $\left(-\frac{s+r}{2},\infty\right)$ for $s+r>0$, and that it is negative on $\left(-\infty,-\frac{s+r}{2}\right)$ and positive on $\left(-\frac{s+r}{2},\infty\right)$ for $s+r<0$, which is equivalent to

$$Q\left(t+\frac{s+r}{2}\right) - Q\left(t-\frac{s+r}{2}\right) = \frac{H'_{r,s;x,y}(t)}{H_{r,s;x,y}(t)} - \frac{H'_{r,s;x,y}(t-(s+r))}{H_{r,s;x,y}(t-(s+r))} \tag{13}$$

being positive on $(-\infty,0)$ and negative on $(0,\infty)$ for $s+r>0$, and being negative on $(-\infty,0)$ and positive on $(0,\infty)$ for $s+r<0$. Since

$$K_{r,s;x,y}(t) = \frac{xyH_{r,s;x,y}(t)}{H_{r,s;x,y}(t-(s+r))}, \tag{14}$$

then the function in (13) equals $\left[\ln K_{r,s;x,y}(t)\right]'$, which implies that the function $K_{r,s;x,y}(t)$ is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$ for $s+r>0$, and it is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$ for $s+r<0$. The proof of Theorem 2 is complete. □

*Second proof of Theorem 2.* Basing on the first proof of Theorem 2, the derivatives (8), (9), and (12) can be rewritten as

$$[\ln H_{r,s;x,y}(t)]' = \frac{(x^s y^r - x^r y^s)(\ln x - \ln y)}{s-r} \frac{x^t y^t}{(x^{s+t}+y^{s+t})(x^{r+t}+y^{r+t})},$$

$$[\ln H_{r,s;x,y}(t)]'' = \frac{(x^s y^r - x^r y^s)\left(x^{s+r+2t} - y^{s+r+2t}\right)}{s-r} \times \frac{x^t y^t (\ln x - \ln y)^2}{(x^{s+t}+y^{s+t})^2(x^{r+t}+y^{r+t})^2},$$

$$[\ln K_{r,s;x,y}(t)]' = -\frac{(x^s y^r - x^r y^s)\left(x^{s+r} - y^{s+r}\right)\left(x^{2t}-y^{2t}\right)(\ln x - \ln y)}{s-r} \times \frac{x^t y^t}{(x^t y^s + x^s y^t)(x^t y^r + x^r y^t)(x^{s+t}+y^{s+t})(x^{r+t}+y^{r+t})}.$$

Theorem 2 is thus proved. □

*Second proof of the monotonicity of $K_{r,s;x,y}(t)$.* Due to homogeneity in $x$ and $y$, we can assume that $y = 1$. Then we have

$$\begin{aligned} K_{r,s;x,y}(t) &= \left(\frac{x^{r+t}+1}{x^{s+t}+1} \cdot \frac{x^{r-t}+1}{x^{s-t}+1}\right)^{1/(r-s)} \\ &= \left[\frac{x^{2r}+x^r(x^t+x^{-t})+1}{x^{2s}+x^s(x^t+x^{-t})+1}\right]^{1/(r-s)} \\ &= \left\{\frac{x^{2r}+x^{r-s}[x^{2s}+x^s(x^t+x^{-t})+1]+1-x^{r+s}-x^{r-s}}{x^{2s}+x^s(x^t+x^{-t})+1}\right\}^{1/(r-s)} \\ &= \left[x^{r-s}+\frac{(x^{r+s}-1)(x^{r-s}-1)}{x^{2s}+x^s(x^t+x^{-t})+1}\right]^{1/(r-s)}. \end{aligned}$$

The denominator in square brackets increases for $t > 0$, and the numerator is positive for $(r+s)(r-s) > 0$ and negative otherwise, so the expression decreases if and only if $r+s > 0$.

The property follows from the log-convexity of $H$ and its symmetry with respect to $-\frac{r+s}{2}$, as showed in [27, Theorem 2.3] and [31, Theorem 7]. □

*Proof of Theorem 3.* A direct calculation yields

$$\left[t \ln H_{r,s;x,y}(t)\right]'' = 2\left[\ln H_{r,s;x,y}(t)\right]' + t\left[\ln H_{r,s;x,y}(t)\right]''. \tag{15}$$

By Theorem 2, it follows that $\left[\ln H_{r,s;x,y}(t)\right]' > 0$ on $(-\infty,\infty)$, $\left[\ln H_{r,s;x,y}(t)\right]'' > 0$ on $\left(-\infty,-\frac{s+r}{2}\right)$ and $\left[\ln H_{r,s;x,y}(t)\right]'' < 0$ on $\left(-\frac{s+r}{2},\infty\right)$. Therefore, if $s+r<0$ then $\left[t\ln H_{r,s;x,y}(t)\right]'' > 0$, and so $t \ln H_{r,s;x,y}(t)$ is convex on $\left(0,-\frac{s+r}{2}\right)$; if $s+r>0$ then $\left[t\ln H_{r,s;x,y}(t)\right]'' > 0$, and so $t \ln H_{r,s;x,y}(t)$ is convex on $\left(-\frac{s+r}{2},0\right)$. The proof of Theorem 3 is complete. □

## 3. Remarks

After proving our three theorems, we give several remarks on them for simply summarizing and reviewing recent developments and applications on this topic.

REMARK 1. In this paper we provide from new viewpoints alternative proofs for the monotonicity of Gini means $G(r,s;x,y)$, although other proofs have been supplied in [7, 28], to the best of our knowledge.

REMARK 2. The log-convexity of some functions which are more general than Gini means has been demonstrated in, for example, [28, 29, 30, 31, 32]. This can also be deduced from Schur-convexity of Gini means, which was proved in [25] by using the conclusion that for functions of the form

$$\frac{g(x)-g(y)}{x-y} \tag{16}$$

Schur-convexity and convexity are equivalent to each other, as showed in [14, pp. 273–274], see also [18, p. 46, Section 4.3.1].

REMARK 3. The logarithmic convexity of $H_{r,s;x,y}(t)$ for $(r,s)=(1,0)$ have been proved in [1], see also [31, Theorem 6]. Since the graph of $\ln H_{r,s}$ can be obtained from the graph of $\ln H_{1,0}$ by an affine change of variables, so the logarithmic convexity of $H_{r,s;x,y}(t)$ follows. This is also an alternative proof of the logarithmic convexity of $H_{r,s;x,y}(t)$.

REMARK 4. From [27, Property 1.2 and Theorem 2.1] for $a=x$ and $b=y$, the logarithmic convexity of $H_{r,s;x,y}(t)$ can also be derived. This is the last alternative proof we find as possible as we can.

REMARK 5. Our approaches and techniques in this paper can be used to study similar properties of extended mean values $E(r,s;x,y)$ and other means, see, for example, [2, 8, 11, 19, 20]. It is much worthwhile to mentioning that these have been applied in [9, 10, 22, 33] to establish inequalities for bounding the ratio of two gamma functions in the theory of special functions, see also [18] and plenty of references cited therein.

REMARK 6. This paper is a slightly revised and corrected version of the preprint [21].

## REFERENCES


[1] H. ALZER, *Über Lehmers Mittelwertefamilie*, Elem. Math. **43** (1988), 50–54.

[2] W.-S. CHEUNG AND F. QI, *Logarithmic convexity of the one-parameter mean values*, Taiwanese J. Math. **11**, 1 (2007), 231–237.

[3] P. CZINDER AND ZS. PÁLES, *A general Minkowski-type inequality for two variable Gini means*, Publ. Math. Debrecen **57**, 1-2 (2000), 203–216.

[4] P. CZINDER AND ZS. PÁLES, *Local monotonicity properties of two-variable Gini means and the comparison theorem revisited*, J. Math. Anal. Appl. **301**, 2 (2005), 427–438.

[5] Z. DARÓCZY AND L. LOSONCZI, *Über den Vergleich von Mittelwerten*, Publ. Math. Debrecen **17** (1970), 289–297.

[6] D. FARNSWORTH AND R. ORR, *Gini means*, Amer. Math. Monthly **93**, 8 (1986), 603–607.

[7] C. GINI, *Di una formula compresive delle medie*, Metron **13** (1938), 3–22.

[8] B.-N. GUO AND F. QI, *A simple proof of logarithmic convexity of extended mean values*, Numer. Algorithms **52**, 1 (2009), 89–92;
Available online at `http://dx.doi.org/10.1007/s11075-008-9259-7`.

[9] B.-N. GUO AND F. QI, *An alternative proof of Elezović-Giordano-Pečarić's theorem*, Math. Inequal. Appl. **14**, 1 (2011), 73–78; Available online at `http://dx.doi.org/10.7153/mia-14-06`.

[10] B.-N. GUO AND F. QI, *Properties and applications of a function involving exponential functions*, Commun. Pure Appl. Anal. **8**, 4 (2009), 1231–1249;
Available online at `http://dx.doi.org/10.3934/cpaa.2009.8.1231`.

[11] B.-N. GUO AND F. QI, *The function* $(b^x - a^x)/x$*: Logarithmic convexity and applications to extended mean values*, Filomat **25**, 4 (2011), 63–73;
Available online at `http://dx.doi.org/10.2298/FIL1104063G`.

[12] Z. LIU, *Minkowski's inequality for extended mean values*, Proc. 2nd ISAAC Congress, Vol. 1 (Fukuoka, 1999), 585–592, Int. Soc. Anal. Appl. Comput., 7, Kluwer Acad. Publ., Dordrecht, 2000.

[13] L. LOSONCZI, *Inequalities for integral mean values*, J. Math. Anal. Appl. **61** (1977), 586–606.

[14] M. MERKLE, *Conditions for convexity of a derivative and some applications to the Gamma function*, Aequationes Math. **55** (1998), 273–280.

[15] E. NEUMAN AND ZS. PÁLES, *On comparison of Stolarsky and Gini means*, J. Math. Anal. Appl. **278**, 2 (2003), 274–284.

[16] E. NEUMAN AND J. SÁNDOR, *Inequalities involving Stolarsky and Gini means*, Math. Pannon. **14**, 1 (2003), 29–44.

[17] ZS. PÁLES, *Comparison of two variable homogeneous means*, General Inequalities 6. Proc. 6th Internat. Conf. Math. Res. Inst. Oberwolfach, Birkhäuser Verlag Basel, 1992, pp. 59–69.

[18] F. QI, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. **2010** (2010), Article ID 493058, 84 pages; Available online at `http://dx.doi.org/10.1155/2010/493058`.

[19] F. QI, P. CERONE, S. S. DRAGOMIR, AND H. M. SRIVASTAVA, *Alternative proofs for monotonic and logarithmically convex properties of one-parameter mean values*, Appl. Math. Comput. **208**, 1 (2009), 129–133; Available online at `http://dx.doi.org/10.1016/j.amc.2008.11.023`.

[20] F. QI AND S.-X. CHEN, *Complete monotonicity of the logarithmic mean*, Math. Inequal. Appl. **10**, 4 (2007), 799–804; Available online at `http://dx.doi.org/10.7153/mia-10-73`.

[21] F. QI AND B.-N. GUO, *Logarithmic convexities of Gini's mean values*,
Available online at `http://arxiv.org/abs/0903.1208`.

[22] F. QI, B.-N. GUO, AND C.-P. CHEN, *The best bounds in Gautschi-Kershaw inequalities*, Math. Inequal. Appl. **9**, 3 (2006), 427–436; Available online at `http://dx.doi.org/10.7153/mia-09-41`.

[23] F. QI, S.-L. XU, AND L. DEBNATH, *A new proof of monotonicity for extended mean values*, Internat. J. Math. Math. Sci. **22**, 2 (1999), 417–421;
Available online at `http://dx.doi.org/10.1155/S0161171299224179`.

[24] J. SÁNDOR, *A note on the Gini means*, Gen. Math. **12**, 4 (2004), 17–21.

[25] J. SÁNDOR, *The Schur-convexity of Stolarsky and Gini means*, Banach J. Math. Anal. **1**, 2 (2007), 212–215.

[26] S. TOADER AND G. TOADER, *Complementaries of Greek means with respect to Gini means*, Int. J. Appl. Math. Stat. **11**, 7 (2007), 187–192.

[27] A. WITKOWSKI, *Convexity of weighted Stolarsky means*, J. Inequal. Pure Appl. Math. **7**, 2 (2006), Art. 73; Available online at `http://www.emis.de/journals/JIPAM/article690.html`.

[28] Z.-H. YANG, *On the homogeneous functions with two parameters and its monotonicity*, J. Inequal. Pure Appl. Math. **6**, 4 (2005), Art. 101;
Available online at `http://www.emis.de/journals/JIPAM/article575.html`.

[29] Z.-H. YANG, *On the logarithmic convexity for two parameter homogeneous functions*, RGMIA Res. Rep. Coll. **8**, 2 (2005), Art. 21; Available online at `http://rgmia.org/v8n2.php`.

[30] Z.-H. YANG, *On the log-convexity of two-parameter homogeneous functions*, Math. Inequal. Appl. **10**, 3 (2007), 499–516; Available online at `http://dx.doi.org/10.7153/mia-10-47`.

[31] Z.-H. YANG, *On the monotonicity and log-convexity for one-parameter homogeneous functions*, RGMIA Res. Rep. Coll. **8**, 2 (2005), Art. 14; Available online at http://rgmia.org/v8n2.php.

[32] Z.-H. YANG, *On refinements and extensions of log-convexity for two-parameter homogeneous functions*, RGMIA Res. Rep. Coll. **8**, 3 (2005), Art. 12;
Available online at http://rgmia.org/v8n3.php.

[33] S.-Q. ZHANG, B.-N. GUO, AND F. QI, *A concise proof for properties of three functions involving the exponential function*, Appl. Math. E-Notes **9** (2009), 177–183.

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