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NEW SHARP BOUNDS FOR IDENTRIC MEAN IN TERMS OF LOGARITHMIC MEAN AND ARITHMETIC MEAN

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Abstract. Let $x,y>0$ with $x\neq y$. We give new sharp bounds for identric mean $I=e^{-1}\left(x^x/y^y\right)^{1/(x-y)}$ in terms of logarithmic mean $L=(x-y)/(\ln x-\ln y)$ and arithmetic mean $A=(x+y)/2$:

$$\left(\tfrac{1}{2}L^{p_0}+\tfrac{1}{2}A^{p_0}\right)^{1/p_0}<I<\left(\tfrac{1}{2}L^{\tilde{p}_0}+\tfrac{1}{2}A^{\tilde{p}_0}\right)^{1/\tilde{p}_0},$$

where $p_0=8/5$ and $\tilde{p}_0=(\ln 2)/(1-\ln 2)$ are the best possible constants.

1. Introduction

The logarithmic and identric means of two positive real numbers x and y with $x\neq y$ are defined by

$$L=L(x,y)=\frac{x-y}{\ln x-\ln y}\quad\text{and}\quad I=I(x,y)=e^{-1}\left(\frac{x^x}{y^y}\right)^{1/(x-y)},$$

respectively. The power mean of order r of the positive real numbers x and y is defined by

$$M_r=M_r(x,y)=\left(\frac{x^r+y^r}{2}\right)^{1/r}\text{ if } r\neq 0 \text{ and } M_0=M_0(x,y)=\sqrt{xy}.$$

The main properties of these means are given in [5]. In particular, the function $r\mapsto M_r(x,y)$ $(x\neq y)$ is continuous and strictly increasing on $\mathbb{R}$. As special cases, the arithmetic mean and geometric mean are $A=A(x,y)=M_1(x,y)$ and $G=G(x,y)=M_0(x,y)$ respectively.

There has been many bounds for identric mean in terms of other means. Stolarsky [16] first established that

$$L<I<A. \tag{1.1}$$

A reverse inequality of the the second one of (1.1) was given by Alzer [2]:

$$2e^{-1}A<I.$$

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In [17] the author and Pittenger [9] proved that the inequalities

$$M_{2/3} < I \tag{1.2}$$

and

$$I < M_{\ln 2} \tag{1.3}$$

hold, respectively, where the constants $2/3$ and $\ln 2$ are the best possible. The following reverse inequality of (1.2) is due to Yang [22]:

$$I < \sqrt{8}e^{-1}M_{2/3}. \tag{1.4}$$

In 1990, Sandor [11] gave an improvement of the first inequality of (1.1):

$$I > \frac{L+A}{2}. \tag{1.5}$$

For the bounds for identric mean in terms of arithmetic mean and geometric mean, Neuman and Sándor [7] first showed that

$$\frac{A+G}{2} < I < 4e^{-1}\frac{A+G}{2}. \tag{1.6}$$

In [12], Sándor further proved the inequalities

$$\frac{2A+G}{3} < I < \sqrt{\frac{2A^2+G^2}{3}} \tag{1.7}$$

hold. Alzer and Qiu [3] pointed out that

$$\alpha A + (1-\alpha)G < I < \beta A + (1-\beta)G \tag{1.8}$$

hold if and only if $\alpha \leqslant 2/3$ and $\beta \geqslant 2/e$. While Trif [18] derived that for $p \geqslant 2$ inequalities

$$\alpha A^p + (1-\alpha)G^p < I^p < \beta A^p + (1-\beta)G^p \tag{1.9}$$

hold if and only if $\alpha \leqslant (2/e)^p$ and $\beta \geqslant 2/3$. Recently, Kouba [6] proved that the inequalities

$$\left(\tfrac{2}{3}A^p + \tfrac{1}{3}G^p\right)^{1/p} < I < \left(\tfrac{2}{3}A^q + \tfrac{1}{3}G^q\right)^{1/q} \tag{1.10}$$

hold if and only if $p \leqslant 6/5$ and $q \geqslant (\ln 3 - \ln 2)/(1-\ln 2)$.

Other inequalities involving the identric mean can be found in the literature [7], [10], [12], [13], [14], [15], [21], [22], [23], [24], [25], [26]

The main aim of this paper is to present the sharp bounds for identric mean I in terms of p-order power means of logarithmic mean L and arithmetic mean A, that is, determine the best $p > 1$ such that

$$I > \left(\frac{L^p + A^p}{2}\right)^{1/p} \tag{1.11}$$

and its reverse inequality hold for all $x, y > 0$ with $x \neq y$.

THEOREM 1. *For all* $x,y>0$ *with* $x\neq y$, *the inequality (1.11) holds if and only if* $p\leqslant p_0=8/5$. *Moreover, we have*

$$\left(\tfrac{1}{2}L^{p_0}+\tfrac{1}{2}A^{p_0}\right)^{1/p_0}<I<c_0\left(\tfrac{1}{2}L^{p_0}+\tfrac{1}{2}A^{p_0}\right)^{1/p_0}, \tag{1.12}$$

where $c_0=2^{13/8}e^{-1}=1.13470...$ *is the best possible constant.*

THEOREM 2. *For all* $x,y>0$ *with* $x\neq y$, *the inequality (1.11) is reversed if and only if* $p\geqslant \tilde{p}_0=(\ln 2)/(1-\ln 2)$. *Moreover, we have*

$$\tilde{c}_0\left(L^{\tilde{p}_0}+A^{\tilde{p}_0}\right)^{1/\tilde{p}_0}<I<\left(L^{\tilde{p}_0}+A^{\tilde{p}_0}\right)^{1/\tilde{p}_0}, \tag{1.13}$$

where $\tilde{c}_0\approx 0.97634$ *is the best possible constant.*

2. Lemmas

To prove our main results, we need the following lemmas.

LEMMA 1. ([19], [1]) *Let* $f,g:[a,b]\mapsto\mathbb{R}$ *be two continuous functions which are differentiable on* (a,b). *Further, let* $g'\neq 0$ *on* (a,b). *If* f'/g' *is increasing (or decreasing) on* (a,b), *then so are the functions*

$$x\mapsto\frac{f(x)-f(b)}{g(x)-g(b)}\quad\text{and}\quad x\mapsto\frac{f(x)-f(a)}{g(x)-g(a)}.$$

LEMMA 2. ([4]) *Let* a_n *and* b_n $(n=0,1,2,...)$ *be real numbers and let the power series* $A(t)=\sum_{n=1}^{\infty}a_nt^n$ *and* $B(t)=\sum_{n=1}^{\infty}b_nt^n$ *be convergent for* $|t|<R$. *If* $b_n>0$ *for* $n=0,1,2,...$, *and* a_n/b_n *is strictly increasing (or decreasing) for* $n=0,1,2,...$, *then the function* $t\mapsto A(t)/B(t)$ *is strictly increasing (or decreasing) on* $(0,R)$.

LEMMA 3. *Let* $M(x,y)$ *be a homogeneous mean of positive arguments* x *and* y. *Then*

$$M(x,y)=\sqrt{xy}M\left(e^t,e^{-t}\right), \tag{2.1}$$

where $t=\frac{1}{2}\ln(x/y)$.

The proofs of the following two lemmas are complicated. Some algebraic computations involved in them are preformed with the aid of built-in computer algebra system of *Scientific Workplace Version 5.5*.

LEMMA 4. *Let* (a_n) *and* (b_n) *be the sequences defined by*

$$\begin{aligned} a_n &= -16n^3+22n^2-3n-1 \\ &\quad +3^{2n-2}\left(2n^2-13n+9\right)+2^{2n-1}\left(4n^2+2n-4\right), \end{aligned} \tag{2.2}$$

$$\begin{aligned} b_n &= 16n^4-72n^3+80n^2-35n+1 \\ &\quad +3^{2n-1}(n-3)+2^{2n-3}\left(-4n^3+6n^2+14n+16\right). \end{aligned} \tag{2.3}$$

Then $b_n>0$ *and* a_n/b_n *is strictly increasing for* $n\geqslant 5$.

Proof. We first prove that $b_n > 0$ for $n \geqslant 5$. Note that

$$2^{3-2n} b_n > \left(-4n^3 + 6n^2 + 14n + 16\right) + 9\left(\tfrac{3}{2}\right)^{2n-3}(n-3).$$

Using binomial expansion we have

$$\begin{aligned}\left(\tfrac{3}{2}\right)^{2n-3} > {} & 1 + (2n-3)\tfrac{1}{2} + \tfrac{(2n-3)(2n-4)}{2}\tfrac{1}{4} + \tfrac{(2n-3)(2n-4)(2n-5)}{6}\tfrac{1}{8} \\ & + \tfrac{(2n-3)(2n-4)(2n-5)(2n-6)}{24}\tfrac{1}{16},\end{aligned}$$

which yields

$$\begin{aligned} 2^{3-2n} b_n > {} & \left(-4n^3 + 6n^2 + 14n + 16\right) + 9(n-3)\Big(1 + (2n-3)\tfrac{1}{2} + \tfrac{(2n-3)(2n-4)}{2}\tfrac{1}{4} \\ & + \tfrac{(2n-3)(2n-4)(2n-5)}{6}\tfrac{1}{8} + \tfrac{(2n-3)(2n-4)(2n-5)(2n-6)}{24}\tfrac{1}{16}\Big) \\ = {} & \tfrac{3}{8}n^5 - 3n^4 + \tfrac{265}{32}n^3 - \tfrac{153}{8}n^2 + \tfrac{1141}{32}n - \tfrac{41}{16} \\ = {} & \tfrac{3}{8}(n-5)^5 + \tfrac{51}{8}(n-5)^4 + \tfrac{1345}{32}(n-5)^3 + \tfrac{3963}{32}(n-5)^2 + \tfrac{1099}{8}(n-5) + \tfrac{237}{8} \\ > {} & 0. \end{aligned}$$

Hence, $b_n > 0$ for $n \geqslant 5$.

Now we show that a_n/b_n is strictly increasing for $n \geqslant 5$. Since $b_n > 0$, it suffices to show that

$$a_{n+1}b_n - a_n b_{n+1} > 0.$$

Straightforward computation and arrangement yield

$$a_{n+1}b_n - a_n b_{n+1} = c_9 9^{2n} + c_6 6^{2n} + c_4 4^{2n} + c_3 3^{2n} + c_2 2^{2n} + c_1,$$

where

$$\begin{aligned} c_9 &= \tfrac{2}{3}n^2 - \tfrac{10}{3}n + 8, \\ c_6 &= -\tfrac{5}{9}n^5 + \tfrac{34}{9}n^4 - \tfrac{341}{36}n^3 + \tfrac{143}{36}n^2 - \tfrac{217}{18}n - 36, \\ c_4 &= 4n^4 + 12n^3 + 9n^2 + 45n + 40, \\ c_3 &= \tfrac{256}{9}n^6 - \tfrac{2368}{9}n^5 + 800n^4 - \tfrac{7600}{9}n^3 + \tfrac{2924}{9}n^2 + \tfrac{596}{9}n, \\ c_2 &= 72n^6 - 259n^5 - 488n^4 + \tfrac{4469}{4}n^3 - \tfrac{3207}{4}n^2 - \tfrac{239}{2}n + 4, \\ c_1 &= 256n^6 + 64n^5 - 544n^4 + 496n^3 + 478n^2 - 134n - 8. \end{aligned}$$

For $n = 5$, we have

$$a_{n+1}b_n - a_n b_{n+1} = 603\,187\,200 > 0.$$

Hence, it is enough to check that $a_{n+1}b_n - a_n b_{n+1} > 0$ for $n \geqslant 6$.

Clearly, $c_4 > 0$ and

$$c_1 = n^4\left(256n^2 - 544\right) + 64n^5 + \left(496n^3 - 8\right) + n(478n - 134) > 0.$$

We have for $n \geqslant 5$,

$$\begin{aligned} c_3 &= \tfrac{256}{9}(n-5)^6 + \tfrac{5312}{9}(n-5)^5 + \tfrac{44\,000}{9}(n-5)^4 + \tfrac{184\,400}{9}(n-5)^3 \\ &\quad + 45\,436\,(n-5)^2 + \tfrac{459\,836}{9}(n-5) + 25\,120 \\ &> 0, \\ c_2 &= 72\,(n-5)^6 + 1901\,(n-5)^5 + 20\,037\,(n-5)^4 + \tfrac{426\,429}{4}(n-5)^3 \\ &\quad + 294\,007\,(n-5)^2 + \tfrac{1489\,127}{4}(n-5) + 129\,644 \\ &> 0. \end{aligned}$$

It remains to prove that $c_9 9^{2n} + c_6 6^{2n} > 0$ for $n \geqslant 6$, which is equivalent to

$$c_9\left(\tfrac{9}{6}\right)^{2n} + c_6 = c_9\left(\tfrac{9}{4}\right)^n + c_6 > 0.$$

Using binomial expansion again we get

$$\begin{aligned} \left(\tfrac{9}{4}\right)^n &= \left(1+\tfrac{5}{4}\right)^n \\ &> 1 + n\tfrac{5}{4} + \tfrac{n(n-1)}{2}\left(\tfrac{5}{4}\right)^2 + \tfrac{n(n-1)(n-2)}{6}\left(\tfrac{5}{4}\right)^3 \\ &\quad + \tfrac{n(n-1)(n-2)(n-3)}{24}\left(\tfrac{5}{4}\right)^4 + \tfrac{n(n-1)(n-2)(n-3)(n-4)}{120}\left(\tfrac{5}{4}\right)^5, \end{aligned}$$

which yields

$$\begin{aligned} c_9\left(\tfrac{9}{4}\right)^n + c_6 &> \left(\tfrac{2}{3}n^2 - \tfrac{10}{3}n + 8\right) \times \Big(1 + n\tfrac{5}{4} + \tfrac{n(n-1)}{2}\left(\tfrac{5}{4}\right)^2 + \tfrac{n(n-1)(n-2)}{6}\left(\tfrac{5}{4}\right)^3 \\ &\quad + \tfrac{n(n-1)(n-2)(n-3)}{24}\left(\tfrac{5}{4}\right)^4 + \tfrac{n(n-1)(n-2)(n-3)(n-4)}{120}\left(\tfrac{5}{4}\right)^5\Big) \\ &\quad + \left(-\tfrac{5}{9}n^5 + \tfrac{34}{9}n^4 - \tfrac{341}{36}n^3 + \tfrac{143}{36}n^2 - \tfrac{217}{18}n - 36\right) \\ &= \tfrac{625}{36\,864}n^7 - \tfrac{6875}{36\,864}n^6 + \tfrac{20\,645}{36\,864}n^5 + \tfrac{11\,339}{36\,864}n^4 - \tfrac{50\,207}{18\,432}n^3 \\ &\quad - \tfrac{961}{512}n^2 - \tfrac{463}{72}n - 28. \end{aligned}$$

The above seven degree polynomial can be written as

$$\begin{aligned} &\tfrac{625}{36\,864}(n-6)^7 + \tfrac{19\,375}{36\,864}(n-6)^6 + \tfrac{245\,645}{36\,864}(n-6)^5 + \tfrac{1643\,189}{36\,864}(n-6)^4 \\ &+ \tfrac{3126\,961}{18\,432}(n-6)^3 + \tfrac{377\,105}{1024}(n-6)^2 + \tfrac{1878\,053}{4608}(n-6) + \tfrac{29\,125}{384}, \end{aligned}$$

which is clearly positive for $n \geqslant 6$.

Consequently, $a_{n+1}b_n - a_n b_{n+1} > 0$ for $n \geqslant 5$, and thus a_n/b_n is strictly increasing for $n \geqslant 5$.

This completes the proof. □

LEMMA 5. *Let f_1 be the function defined on $(0,\infty)$ by*

$$f_1(t) = \frac{\ln\left(\left(\sinh^2 t - t^2\right)\cosh t\right) - \ln\left(t\left(t\cosh t - \sinh t\right)\right)}{\ln\left(t\cosh t\right) - \ln\sinh t}. \tag{2.4}$$

Then $f_1(t)$ increases from $8/5$ to ∞ as t increases from 0 to ∞.

Proof. We first prove the function $t \mapsto f_1(t)$ is strictly increasing on $(0,\infty)$. Define

$$f_2(t) = \ln \frac{\left(\sinh^2 t - t^2\right)\cosh t}{t^2\cosh t - t\sinh t},$$
$$f_3(t) = \ln \frac{t\cosh t}{\sinh t}$$

and notice that

$$f_2\left(0^+\right) = \lim_{t\to 0^+} f_2(t) = 0,$$
$$f_3\left(0^+\right) = \lim_{t\to 0^+} f_3(t) = 0.$$

Then $f_1(t)$ can be written as

$$f_1(t) = \frac{f_2(t) - f_2(0^+)}{f_3(t) - f_3(0^+)}.$$

By Lemma 1, in order to prove that f_1 is strictly increasing on $(0,\infty)$, it suffices to show that the function

$$t \mapsto \frac{f_2'(t)}{f_3'(t)} := f_4(t)$$

is strictly increasing.

Differentiation yields

$$\begin{aligned} f_4(t) &= \frac{\frac{d}{dt}\left(\ln\left(\left(\sinh^2 t - t^2\right)\cosh t\right) - \ln\left(t\left(t\cosh t - \sinh t\right)\right)\right)}{\frac{d}{dt}\left(\ln\left(t\cosh t\right) - \ln\sinh t\right)} \\ &= \frac{2t(\sinh t\cosh t)\left(\sinh 4t + 2t^2\sinh 4t - 4t\cosh 4t - 2\sinh 2t + 8t^2\sinh 2t + 4t\cosh 2t - 8t^3\right)}{(\sinh 2t - 2t)\left(t^2\cosh 4t - t\sinh 4t - 4t^4\cosh 2t + 4t^3\sinh 2t + 2t\sinh 2t - t^2 - 4t^4\right)} \\ &:= \frac{A(t)}{B(t)}. \end{aligned}$$

Using "product into sum" formula for hyperbolic functions leads to

$$\begin{aligned} A(t) = {} & \left(t - 4t^3\right) + \left(2t^2\sinh 2t - 8t^4\sinh 2t - \tfrac{1}{2}t\cosh 2t - t^3\cosh 2t\right) \\ & + \left(2t^2\sinh 4t - t\cosh 4t + 4t^3\cosh 4t\right) \\ & + \left(-2t^2\sinh 6t + \tfrac{1}{2}t\cosh 6t + t^3\cosh 6t\right), \\ B(t) = {} & \left(-t + 8t^5\right) + \left(-\tfrac{11}{2}t^2\sinh 2t - 12t^4\sinh 2t + \tfrac{1}{2}t\cosh 2t + 8t^5\cosh 2t\right) \\ & + \left(2t^2\sinh 4t - 2t^4\sinh 4t + t\cosh 4t\right) + \left(\tfrac{1}{2}t^2\sinh 6t - \tfrac{1}{2}t\cosh 6t\right). \end{aligned}$$

From which we easily obtain the Taylor series of $A(t)$ and $B(t)$:

$$A(t) = \sum_{n=5}^{\infty} \tilde{a}_n t^{2n+1} \quad \text{and} \quad B(t) = \sum_{n=5}^{\infty} \tilde{b}_n t^{2n+1},$$

where

$$\begin{aligned}\tilde{a}_n &= \tfrac{2^{2n-1}}{(2n)!}\big(\big(22n^2-16n^3-3n-1\big)+2^{2n-1}\big(4n^2+2n-4\big)\\ &\quad +3^{2n-2}\big(2n^2-13n+9\big)\big),\\ \tilde{b}_n &= \tfrac{2^{2n-1}}{(2n)!}\big(16n^4-72n^3+80n^2-35n+1\big)+2^{2n-3}\big(-4n^3+6n^2+14n+16\big)\\ &\quad +3^{2n-1}\left(n-3\right).\end{aligned}$$

It is clear that

$$\tilde{a}_n = \frac{2^{2n-1}}{(2n)!}a_n, \quad \tilde{b}_n = \frac{2^{2n-1}}{(2n)!}b_n, \quad \tilde{a}_n/\tilde{b}_n = a_n/b_n,$$

where a_n, b_n are defined by (2.2), (2.3), respectively. By Lemma 4, it is seen that $\tilde{b}_n > 0$ and $\tilde{a}_n/\tilde{b}_n$ is strictly increasing for $n \geqslant 5$, from which and Lemma 2 it follows that $A(t)/B(t)$ is strictly increasing on $(0,\infty)$. This reveals that f_4, that is, f_2'/f_3' is also strictly increasing on $(0,\infty)$. Thus, by Lemma 1 the function f_1 is strictly increasing on $(0,\infty)$.

Limit calculation gives

$$\lim_{t\to 0} f_1(t) = \frac{8}{5} \quad \text{and} \quad \lim_{t\to\infty} f_1(t) = \infty,$$

which completes the proof. □

LEMMA 6. *Let the function $t \mapsto F_p(t)$ be defined on $(0,\infty)$ by*

$$F_p(t) = \frac{t\cosh t}{\sinh t} - 1 - \tfrac{1}{p}\ln\left(\tfrac{1}{2}(\cosh t)^p + \tfrac{1}{2}\left(\frac{\sinh t}{t}\right)^p\right). \tag{2.5}$$

Then

$$\lim_{t\to 0^+} \tfrac{F_p(t)}{t^4} = -\tfrac{5p-8}{360}, \tag{2.6}$$

$$\lim_{t\to\infty} F_p(t) = \begin{cases} \tfrac{1}{p}\ln 2 - (1-\ln 2) & \text{if } p > 0 \\ \infty & \text{if } p \leqslant 0 \end{cases}, \tag{2.7}$$

where $F_0(t) := \lim_{p\to 0} F_p(t)$.

Proof. Using power series expansion we have

$$F_p(t) = -\tfrac{5p-8}{360}t^4 + O\left(t^6\right),$$

which yields (2.6).

To obtain (2.7), we write $F_p(t)$ as

$$F_p(t) = \frac{2t}{e^{2t}-1} - \ln\frac{1-e^{-2t}}{2} - 1 - \frac{1}{p}\ln\left(\frac{1}{2}\left(\frac{1+e^{-2t}}{1-e^{-2t}}\right)^p + \frac{1}{2}\left(\frac{1}{t}\right)^p\right),$$

from which (2.7) easily follows. □

LEMMA 7. *Let the function $t \mapsto F_p(t)$ be defined on $(0,\infty)$ by (2.5). Then F_p is strictly increasing with respect to t on $(0,\infty)$ if and only if $p \leqslant 8/5$.*

Proof. Differentiation and simplifying yield

$$F_p'(t) = \frac{\left(\sinh^2 t - t^2\right)(\cosh t)\left(\frac{1}{t}\sinh t\right)^p - \left(t^2\cosh t - t\sinh t\right)\cosh^p t}{t\left(\cosh t \sinh^2 t\right)\left(\left(\frac{1}{t}\sinh t\right)^p + \cosh^p t\right)}$$
$$= \frac{(t\cosh t - \sinh t)\left(\frac{1}{t}\sinh t\right)^p}{\left(\cosh t\sinh^2 t\right)\left(\left(\frac{1}{t}\sinh t\right)^p + \cosh^p t\right)} \times \left(\frac{\left(\sinh^2 t - t^2\right)\cosh t}{t(t\cosh t - \sinh t)} - \left(\frac{t\cosh t}{\sinh t}\right)^p\right).$$

Clearly, $\sinh^2 t - t^2$, $t\cosh t - \sinh t > 0$ for $t > 0$, so $F_p'(t)$ can be written as

$$F_p'(t) = \frac{(t\cosh t - \sinh t)\left(\frac{1}{t}\sinh t\right)^p}{\left(\cosh t\sinh^2 t\right)\left(\left(\frac{1}{t}\sinh t\right)^p + \cosh^p t\right)} \times \left(\ln\frac{t\cosh t}{\sinh t}\right) \times L\left(\frac{\left(\sinh^2 t - t^2\right)\cosh t}{t(t\cosh t - \sinh t)}, \left(\frac{t\cosh t}{\sinh t}\right)^p\right) \times (f_1(t) - p), \tag{2.8}$$

where $f_1(t)$ is defined by (2.4) and $L(x,y)$ is the logarithmic mean of positive numbers x and y.

Necessity. If F_p is strictly increasing on $(0,\infty)$, then $F_p'(t) > 0$, hence

$$\lim_{t\to 0^+} \frac{F_p'(t)}{t^3} \geqslant 0. \tag{2.9}$$

On the other hand, applying L'Hospital rule to the right hand of (2.6) we get

$$\lim_{t\to 0^+} \frac{F_p(t)}{t^4} = \lim_{t\to 0^+} \frac{F_p'(t)}{4t^3} = -\frac{5p-8}{360}. \tag{2.10}$$

(2.9) and (2.10) lead to $p \leqslant 8/5$.

Sufficiency. Suppose that $p \leqslant 8/5$. We now prove that $F_p'(t) > 0$. Since $t\cosh t - \sinh t > 0$ and $\ln\frac{t\cosh t}{\sinh t} > 0$ for $t > 0$, it needs to show that $(f_1(t) - p) > 0$. By Lemma 5, we have

$$f_1(t) = \frac{\ln\left(\left(\sinh^2 t - t^2\right)\cosh t\right) - \ln\left(t(t\cosh t - \sinh t)\right)}{\ln(t\cosh t) - \ln\sinh t} > \frac{8}{5},$$

that is, $f_1(t) - p > 0$ for $t > 0$.

Our required result follows. □

3. Proofs of Main Results

Proof of Theorem 1. By symmetry, we assume that $x > y > 0$. We have

$$I\left(e^t, e^{-t}\right) = \exp\left(t\coth t - 1\right), \quad A\left(e^t, e^{-t}\right) = \cosh t, \quad L\left(e^t, e^{-t}\right) = \frac{\sinh t}{t},$$

where $t = \frac{1}{2}\ln(x/y) > 0$. Due to Lemma 3, in order to prove that the inequality (1.11) holds if and only if $p \leqslant 8/5$, it is enough to show that

$$t\coth t - 1 > \tfrac{1}{p}\ln\left(\tfrac{1}{2}(\cosh t)^p + \tfrac{1}{2}\left(\tfrac{\sinh t}{t}\right)^p\right), \tag{3.1}$$

that is, $F_p(t) > 0$ holds for all $t > 0$ if and only if $p \leqslant 8/5$, where $F_p(t)$ is defined by (2.5).

Necessity. If $F_p(t) > 0$ holds for all $t > 0$, then by Lemma 6 we have

$$\begin{cases} \lim_{t\to 0^+} \frac{F_p(t)}{t^4} = -\frac{5p-8}{360} \geqslant 0, \\ \lim_{t\to\infty} F_p(t) = \frac{1}{p}\ln 2 - (1-\ln 2) \geqslant 0 \text{ if } p > 0 \end{cases}$$

or

$$\begin{cases} \lim_{t\to 0^+} \frac{F_p(t)}{t^4} = -\frac{5p-8}{360} \geqslant 0, \\ \lim_{t\to\infty} F_p(t) = \infty \geqslant 0 \text{ if } p \leqslant 0. \end{cases}$$

Solving the inequalities for p yields $p \leqslant 8/5$.

Sufficiency. If $p \leqslant 8/5$, then from Lemma 7 it is seen that F_p is increasing with respect to t, hence we get

$$F_p(t) > \lim_{t\to 0^+} F_p(t) = 0,$$

which completes the proof of sufficiency.

By the monotonicity of F_{p_0} where $p_0 = 8/5$ and note that (2.7), we find that

$$0 = \lim_{t\to 0^+} F_{p_0}(t) < F_{p_0}(t) < \lim_{t\to\infty} F_{p_0}(t) = \frac{1}{p_0}\ln 2 - (1-\ln 2),$$

that is, inequalities (1.12) hold, and $c_0 = \exp\left(\frac{1}{p_0}\ln 2 - (1-\ln 2)\right) = 2^{13/8}e^{-1}$ is clearly the best possible constant. $\square$

Proof of Theorem 2. In order to prove that the reverse inequality in (1.11) holds if and only $p \geqslant \tilde{p}_0 = (\ln 2)/(1-\ln 2)$, it suffices to show that $F_p(t) < 0$ holds for all $t > 0$ if and only if $p \geqslant \tilde{p}_0$, where F_p is defined by (2.5).

Necessity. If $F_p(t) < 0$ holds for all $t > 0$, then by Lemma 6 we have

$$\begin{cases} \lim_{t\to 0^+} \frac{F_p(t)}{t^4} = -\frac{5p-8}{360} \leqslant 0, \\ \lim_{t\to\infty} F_p(t) = \frac{1}{p}\ln 2 - (1-\ln 2) \leqslant 0 \text{ if } p > 0, \end{cases}$$

which yields $p \geqslant (\ln 2)/(1-\ln 2) = \tilde{p}_0$.

Sufficiency. Suppose that $p \geqslant \tilde{p}_0$. Since the function $p \mapsto F_p(t)$ is decreasing, so it needs to check that $F_p(t) < 0$ holds for $p = \tilde{p}_0$.

From the proof of Lemma 6 it is easy to see that

$$\operatorname{sgn} F_p'(t) = \operatorname{sgn}(f_1(t) - p), \tag{3.2}$$

where $f_1(t)$ and $F_p(t)$ are defined by (2.4) and (2.5), respectively. By Lemma 4 $f_1(t)$ increases from $8/5$ to ∞ as t increases from 0 to ∞, and then, the function $t \mapsto (f_1(t) - \tilde{p}_0)$ is also increasing. Since

$$\lim_{t\to 0^+}(f_1(t) - \tilde{p}_0) = \frac{8}{5} - \frac{\ln 2}{1-\ln 2} < 0,$$
$$\lim_{t\to\infty}(f_1(t) - \tilde{p}_0) = \infty,$$

hence there is a unique number $t_0 \in (0,\infty)$ such that

$$f_1(t) - \tilde{p}_0 = 0,$$

and $f_1(t) - \tilde{p}_0 < 0$ for $t \in (0,t_0)$ and $f_1(t) - \tilde{p}_0 > 0$ for $t \in (t_0,\infty)$. From (3.2) which, in turn, implies that $F_{\tilde{p}_0}'(t) < 0$ for $t \in (0,t_0)$ and $F_{\tilde{p}_0}'(t) > 0$ for $t \in (t_0,\infty)$, consequently,

$$F_{\tilde{p}_0}(t_0) \leqslant F_{\tilde{p}_0}(t) < \lim_{t\to 0^+} F_{\tilde{p}_0}(t) = 0 \text{ for } t \in (0,t_0),$$
$$F_{\tilde{p}_0}(t_0) \leqslant F_{\tilde{p}_0}(t) < \lim_{t\to\infty} F_{\tilde{p}_0}(t) = \frac{1}{\tilde{p}_0}\ln 2 - (1-\ln 2) = 0 \text{ for } t \in (t_0,\infty),$$

that is, $F_{\tilde{p}_0}(t_0) \leqslant F_{\tilde{p}_0}(t) < 0$ for $t \in (0,\infty)$.

Solving the equation $f_1(t) - \tilde{p}_0 = 0$ by using mathematical computer software we find that $t = t_0 \approx 2.5444821555$, and

$$F_{\tilde{p}_0}(t_0) \approx -2.3940 \times 10^{-2}, \ \tilde{c}_0 = \exp\left(F_{\tilde{p}_0}(t_0)\right) \approx 0.97634,$$

which prove the sufficiency and the inequalities (1.13). □

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