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SHARP INEQUALITIES RELATED TO ONE–PARAMETER MEAN AND GINI MEAN

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Abstract. In the present paper, we answer the question: For $\alpha+\beta\in(0,1)$, what are the greatest values p,s_1 and the least values q,s_2 such that the inequalities

$$J_p(a,b)\leqslant A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\leqslant J_q(a,b)$$

and

$$G_{s_1,1}(a,b)\leqslant A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\leqslant G_{s_2,1}(a,b)$$

hold for all $a,b>0$ with $a\neq b$? where $J_p(a,b)$, $A(a,b)$, $G(a,b)$, $H(a,b)$ and $G_{s,1}(a,b)$ are the one-parameter mean, arithmetic mean, geometric mean, harmonic mean and Gini mean for two positive numbers a and b, respectively.

1. Introduction

For $p,s\in\mathrm{R}$, the one-parameter mean $J_p(a,b)$, Gini mean $G_{s,1}(a,b)$, arithmetic mean $A(a,b)$, geometric mean $G(a,b)$ and harmonic mean $H(a,b)$ of two positive numbers a and b are defined by

$$J_p(a,b)=\begin{cases}\frac{p(a^{p+1}-b^{p+1})}{(p+1)(a^p-b^p)}, & a\neq b, p\neq 0,-1,\\ \frac{a-b}{\log a-\log b}, & a\neq b, p=0,\\ \frac{ab(\log a-\log b)}{a-b}, & a\neq b, p=-1,\\ a, & a=b,\end{cases} \tag{1.1}$$

$$G_{s,1}(a,b)=\left(\frac{a^s+b^s}{a+b}\right)^{\frac{1}{s-1}},\ s\neq 1, \tag{1.2}$$

$A(a,b)=\frac{a+b}{2}$, $G(a,b)=\sqrt{ab}$ and $H(a,b)=\frac{2ab}{a+b}$, respectively.

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Recently, the one-parameter mean $J_p(a,b)$ and Gini mean $G_{s,1}(a,b)$ have been the subject of intensive research. In particular, many remarkable inequalities and properties for them can be found in the literature [1–8].

It is well-known that the one-parameter mean $J_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathrm{R}$ for fixed $a,b>0$ with $a \neq b$. Many means are special cases of the one-parameter mean, for example

$$J_1(a,b)=\frac{a+b}{2}=A(a,b), \text{ the arithmetic mean,}$$

$$J_{\frac{1}{2}}(a,b)=\frac{a+\sqrt{ab}+b}{3}=H_e(a,b), \text{ the Heronian mean,}$$

$$J_{-\frac{1}{2}}(a,b)=\sqrt{ab}=G(a,b), \text{ the geometric mean,}$$

and

$$J_{-2}(a,b)=\frac{2ab}{a+b}=H(a,b), \text{ the harmonic mean.}$$

For $r \in \mathrm{R}$, let us introduce the power mean $M_r(a,b)$ of order r of two positive number a and b

$$M_r(a,b)=\begin{cases} (\frac{a^r+b^r}{2})^{\frac{1}{r}}, & r \neq 0, \\ \sqrt{ab}, & r=0. \end{cases}$$

The main properties of the power mean are given in [9]. In particular, $M_r(a,b)$ is continuous and strictly increasing with respect to $r \in R$ for fixed $a,b>0$ with $a \neq b$.

In [10], Alzer and Janous established the following sharp double inequality

$$M_{\frac{\log 2}{\log 3}}(a,b)<\frac{2}{3}J_1(a,b)+\frac{1}{3}J_{-\frac{1}{2}}(a,b)<M_{\frac{2}{3}}(a,b)$$

for all $a,b>0$ with $a \neq b$.

In [11], Mao proved

$$M_{\frac{1}{3}}(a,b) \leqslant \frac{1}{3}J_1(a,b)+\frac{2}{3}J_{-\frac{1}{2}}(a,b) \leqslant M_{\frac{1}{2}}(a,b)$$

for all $a,b>0$, and $M_{\frac{1}{3}}(a,b)$ is the best possible lower power mean bound for the sum $\frac{1}{3}J_1(a,b)+\frac{2}{3}J_{-\frac{1}{2}}(a,b)$.

In [12], Wang, Qiu and Chu proved

$$J_{3\alpha-2}(a,b)<\alpha A(a,b)+(1-\alpha)H(a,b)<J_{\frac{\alpha}{2-\alpha}}(a,b) \tag{1.3}$$

for all $a,b>0$ with $a \neq b$, and $J_{3\alpha-2}(a,b)$ and $J_{\frac{\alpha}{2-\alpha}}(a,b)$ are the best possible lower and upper one-parameter mean bounds for the sum $\alpha A(a,b)+(1-\alpha)H(a,b)$.

For some results related to Gini mean, we refer the reader to [4] and the references therein.

The purpose of this paper is to answer the question: what are the greatest value p, s_1 and the least value q, s_2 such that the double inequalities

$$J_p(a,b) \leqslant A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \leqslant J_q(a,b)$$

and

$$G_{s_1,1}(a,b) \leqslant A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \leqslant G_{s_2,1}(a,b)$$

hold for all $a,b>0$ with $a \neq b$ and $\alpha,\beta>0$ with $\alpha+\beta<1$?

2. Some Lemmas

In order to establish our main results, we need the following lemmas.

LEMMA 2.1. *For $t>1$, one has*

$$g(t) = -(t^2+4t+1)\log t + 3(t^2-1) < 0. \tag{2.1}$$

Proof. Simple computations lead to

$$\lim_{t\to 1^+} g(t) = 0, \tag{2.2}$$

$$g'(t) = -2(t+2)\log t + 5t - \frac{1}{t} - 4 \tag{2.3}$$

$$\lim_{t\to 1^+} g'(t) = 0, \tag{2.4}$$

$$g''(t) = -2\log t - \frac{4}{t} + \frac{1}{t^2} + 3, \tag{2.5}$$

$$\lim_{t\to 1^+} g''(t) = 0, \tag{2.6}$$

$$g'''(t) = \frac{-2(t-1)^2}{t^3} < 0. \tag{2.7}$$

(2.1) follows from (2.2)–(2.7). □

LEMMA 2.2. *For $t>1$ and $0<\lambda<2$, let*

$$g_{1,\lambda}(t) = (2-\lambda)t^{3\lambda-2} + \lambda t^{3(\lambda-1)} + 2(1-\lambda)t^{\frac{3\lambda}{2}} + 2(\lambda-1)t^{\frac{3\lambda-4}{2}} - \lambda t - 2 + \lambda.$$

Then

$$g_{1,\lambda}(t) \begin{cases} >0, \ \lambda \in \left(\frac{2}{3},1\right) \cup \left(\frac{4}{3},2\right), \\ <0, \ \lambda \in \left(0,\frac{2}{3}\right) \cup \left(1,\frac{4}{3}\right). \end{cases} \tag{2.8}$$

Proof. Simple computations lead to

$$g_{1,\lambda}(1) = 0, \tag{2.9}$$

$$\begin{aligned} g'_{1,\lambda}(t) = &(2-\lambda)(3\lambda-2)t^{3(\lambda-1)} + 3\lambda(\lambda-1)t^{3\lambda-4} \\ &+3\lambda(1-\lambda)t^{\frac{3\lambda-2}{2}} + (\lambda-1)(3\lambda-4)t^{\frac{3(\lambda-2)}{2}} - \lambda, \end{aligned} \tag{2.10}$$

$$g'_{1,\lambda}(1) = 0, \tag{2.11}$$

$$g''_{1,\lambda}(t) = \frac{3}{2}t^{\frac{3\lambda-8}{2}} g_{2,\lambda}(t), \tag{2.12}$$

$$\begin{aligned} g_{2,\lambda}(t) = (\lambda-1)\Big[&2(2-\lambda)(3\lambda-2)t^{\frac{3\lambda}{2}} + 2\lambda(3\lambda-4)t^{\frac{3\lambda-2}{2}} \\ &-\lambda(3\lambda-2)t^2 + (3\lambda-4)(\lambda-2)\Big], \end{aligned} \tag{2.13}$$

$$g_{2,\lambda}(1) = 0, \tag{2.14}$$

$$g'_{2,\lambda}(t) = t g_{3,\lambda}(t), \tag{2.15}$$

$$g_{3,\lambda}(t) = \lambda(3\lambda-2)(\lambda-1)\left[3(2-\lambda)t^{\frac{3\lambda-4}{2}} + (3\lambda-4)t^{\frac{3(\lambda-2)}{2}} - 2\right], \tag{2.16}$$

$$g_{3,\lambda}(1) = 0, \tag{2.17}$$

$$g'_{3,\lambda}(t) = \frac{3}{2}t^{\frac{3\lambda-8}{2}} g_{4,\lambda}(t), \tag{2.18}$$

$$g_{4,\lambda}(t) = \lambda(3\lambda-2)(\lambda-1)(3\lambda-4)(2-\lambda)(t-1)\begin{cases} >0,\ \lambda \in \left(\frac{2}{3},1\right) \cup \left(\frac{4}{3},2\right), \\ <0,\ \lambda \in \left(0,\frac{2}{3}\right) \cup \left(1,\frac{4}{3}\right). \end{cases} \tag{2.19}$$

Lemma 2.2 follows from inequalities (2.9)–(2.19). □

LEMMA 2.3. *For $t>1$ and $\lambda \in (0,2)$, one has*

$$g_{5,\lambda}(t) = (2-\lambda)t^{\lambda} + \lambda t^{\lambda-1} - \lambda t - 2 + \lambda \begin{cases} <0,\ \lambda \in (0,1), \\ >0,\ \lambda \in (1,2). \end{cases} \tag{2.20}$$

Proof. Simple computations yield

$$g_{5,\lambda}(1) = 0, \tag{2.21}$$

$$g'_{5,\lambda}(t) = \lambda\left[(2-\lambda)t^{\lambda-1} + (\lambda-1)t^{\lambda-2} - 1\right], \tag{2.22}$$

$$g'_{5,\lambda}(1) = 0, \tag{2.23}$$

$$g''_{5,\lambda}(t) = \lambda(\lambda-1)(2-\lambda)t^{\lambda-3}(t-1)\begin{cases} <0,\ \lambda \in (0,1), \\ >0,\ \lambda \in (1,2). \end{cases} \tag{2.24}$$

(2.20) follows from (2.21)–(2.24). □

LEMMA 2.4. *For $t>1$ and $\lambda\in(0,2)$, one has*

$$g_{6,\lambda}(t)=(3-\lambda)t^{\lambda-1}+(\lambda-1)t^{\lambda-2}-(\lambda-1)t+(\lambda-3)\begin{cases}>0,\ \lambda\in(0,1),\\ <0,\ \lambda\in(1,2).\end{cases} \tag{2.25}$$

Proof. Simple computations yield

$$g_{6,\lambda}(1)=0, \tag{2.26}$$

$$g'_{6,\lambda}(t)=(\lambda-1)\left[(3-\lambda)t^{\lambda-2}+(\lambda-2)t^{\lambda-3}-1\right], \tag{2.27}$$

$$g'_{6,\lambda}(1)=0, \tag{2.28}$$

$$g''_{6,\lambda}(t)=(\lambda-1)(\lambda-2)(\lambda-3)t^{\lambda-4}(1-t)\begin{cases}>0,\ \lambda\in(0,1),\\ <0,\ \lambda\in(1,2).\end{cases} \tag{2.29}$$

(2.25) follows from (2.26)–(2.29). □

3. Main results

THEOREM 3.1. *Let $0<\alpha,\beta<1$ satisfy $\alpha+\beta<1$. Then for any $a,b>0$ with $a\neq b$, we have*

(i) $J_{\frac{6\alpha+3\beta-4}{2}}(a,b)=A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)=J_{\frac{2\alpha+\beta-2}{2}}(a,b)$, *for* $2\alpha+\beta=1$,

(ii) $J_{\frac{6\alpha+3\beta-4}{2}}(a,b)<A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)<J_{\frac{2\alpha+\beta-2}{2}}(a,b)$, *for* $2\alpha+\beta\in(0,1)$,

(iii) $J_{\frac{2\alpha+\beta-2}{2}}(a,b)<A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)<J_{\frac{6\alpha+3\beta-4}{2}}(a,b)$, *for* $2\alpha+\beta\in(1,2)$.

The numbers $\frac{6\alpha+3\beta-4}{2}$ *and* $\frac{2\alpha+\beta-2}{2}$ *in* (ii) *and* (iii) *are optimal.*

It is obvious that $A^{\alpha}G^{\beta}H^{1-\alpha-\beta}=A^{\delta}H^{1-\delta}$, where $\delta=\alpha+\frac{\beta}{2}$, $J_{\frac{6\alpha+3\beta-4}{2}}=J_{3\delta-2}$ and $J_{\frac{2\alpha-\beta-2}{2}}=J_{\delta-1}$. Therefore, the statement of Theorem 3.1 can be written as

THEOREM 3.1'. *Let $0<\delta<1$. Then for any $a,b>0$ with $a\neq b$, we have*

(i′) $J_{3\delta-2}(a,b)=A^{\delta}(a,b)H^{1-\delta}(a,b)=J_{\delta-1}(a,b)$, *for* $\delta=\frac{1}{2}$,

(ii′) $J_{3\delta-2}(a,b)<A^{\delta}(a,b)H^{1-\delta}(a,b)<J_{\delta-1}(a,b)$, *for* $\delta\in\left(0,\frac{1}{2}\right)$,

(iii′) $J_{\delta-1}(a,b)<A^{\delta}(a,b)H^{1-\delta}(a,b)<J_{3\delta-2}(a,b)$, *for* $\delta\in\left(\frac{1}{2},1\right)$.

The numbers $3\delta-2$ and $\delta-1$ in (ii') *and* (iii') *are optimal.*

It is obvious that

$$A^{\delta}(a,b)H^{1-\delta}(a,b)<\delta A(a,b)+(1-\delta)H(a,b)$$

for all $a,b>0$ with $a\neq b$ and all $0<\delta<1$l. Combining Theorem 3.1' with (1.3) one obtain

COROLLARY 3.1. *Let $0<\delta<1$. Then for any $a,b>0$ with $a\neq b$, we have*

$$\begin{aligned} J_{3\delta-2}(a,b) &< A^{\delta}(a,b)H^{1-\delta}(a,b) < \delta A(a,b)+(1-\delta)H(a,b) \\ &< J_{\frac{\delta}{2-\delta}}(a,b), \ \ for\ 0<\delta<\frac{1}{2}, \\ J_{\delta-1}(a,b) &< A^{\delta}(a,b)H^{1-\delta}(a,b) < \delta A(a,b)+(1-\delta)H(a,b) \\ &< J_{\frac{\delta}{2-\delta}}(a,b), \ \ for\ \frac{1}{2}<\delta<1. \end{aligned}$$

Proof of Theorem 3.1. Without loss of generality, we assume $a>b$. Let $t=\frac{a}{b}>1$ and $\lambda=2\alpha+\beta$.

For $\lambda=2\alpha+\beta=1$, (i) follows from

$$\begin{aligned} J_{\frac{6\alpha+3\beta-4}{2}}(a,b) &= J_{\frac{2\alpha+\beta-2}{2}}(a,b) = J_{-\frac{1}{2}}(a,b) \\ &= (ab)^{\frac{1}{2}} = \left(\frac{a+b}{2}\right)^{\alpha}(ab)^{\frac{\beta}{2}}\left(\frac{2ab}{a+b}\right)^{1-\alpha-\beta} \\ &= A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b). \end{aligned}$$

We now prove that for $\lambda=2\alpha+\beta=\frac{2}{3}$,

$$J_{\frac{6\alpha+3\beta-4}{2}}(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b). \tag{3.1}$$

In fact, let

$$\begin{aligned} f_1(t) &= \log J_{\frac{3\lambda-4}{2}}(a,b) - \log A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \\ &= \log\left[\frac{t\log t}{t-1}\right] - \frac{1}{3}\log\frac{2t^2}{t+1}. \end{aligned} \tag{3.2}$$

Simple computations lead to

$$\lim_{t\to 1^+} f_1(t) = 0, \tag{3.3}$$

$$f_1'(t) = \frac{g(t)}{3t(t+1)(t-1)\log t} < 0, \ \ \text{for } t>1, \tag{3.4}$$

here we have used Lemma 2.1, then (3.1) follows from (3.2)–(3.4).

In the following, we prove that for $\lambda=2\alpha+\beta=\frac{4}{3}$,

$$A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) < J_{\frac{6\alpha+3\beta-4}{2}}(a,b). \tag{3.5}$$

In fact, let

$$\begin{aligned} f_2(t) &= \log J_{\frac{3\lambda-4}{2}}(a,b) - \log A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) \\ &= \log\left[\frac{t-1}{\log t}\right] - \frac{1}{3}\log\frac{t(t+1)}{2}. \end{aligned} \tag{3.6}$$

Simple computations lead to

$$\lim_{t\to 1^+} f_2(t) = 0, \tag{3.7}$$

$$f_2'(t) = \frac{-g(t)}{3t(t^2-1)\log t} > 0, \tag{3.8}$$

here we have used Lemma 2.1. (3.5) follows from (3.6)–(3.8).

For $\lambda = 2\alpha+\beta \neq \frac{2}{3}$ or $\frac{4}{3}$, we prove

$$J_{\frac{6\alpha+3\beta-4}{2}}(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b), \quad \text{for } 2\alpha+\beta \in (0,1), \tag{3.9}$$

and

$$A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) < J_{\frac{6\alpha+3\beta-4}{2}}(a,b), \quad \text{for } 2\alpha+\beta \in (1,2). \tag{3.10}$$

Let

$$\begin{aligned} f_3(t) &= \log\left[J_{\frac{3\lambda-4}{2}}(a,b)\right] - \log\left[A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\right] \\ &= \log\frac{(3\lambda-4)(t^{\frac{3\lambda-2}{2}}-1)}{(3\lambda-2)(t^{\frac{3\lambda-4}{2}}-1)} - (\lambda-1)\log\frac{t+1}{2} - \frac{2-\lambda}{2}\log t. \end{aligned} \tag{3.11}$$

Then

$$\lim_{t\to 1^+} f_3(t) = 0, \tag{3.12}$$

$$f_3'(t) = \frac{g_{1,\lambda}(t)}{2t(t+1)(t^{\frac{3\lambda-4}{2}}-1)(t^{\frac{3\lambda-2}{2}}-1)}. \tag{3.13}$$

Since

$$\left(t^{\frac{3\lambda-4}{2}}-1\right)\left(t^{\frac{3\lambda-2}{2}}-1\right)\begin{cases} >0, \ \lambda \in (0,\frac{2}{3})\cup(\frac{4}{3},2), \\ <0, \ \lambda \in (\frac{2}{3},\frac{4}{3}). \end{cases} \tag{3.14}$$

then (3.9) and (3.10) follow from (3.11)–(3.14) and Lemma 2.2.

We now in a position to prove

$$A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) < J_{\frac{2\alpha+\beta-2}{2}}(a,b), \quad \text{for } 2\alpha+\beta \in (0,1), \tag{3.15}$$

and

$$J_{\frac{2\alpha+\beta-2}{2}}(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b), \quad \text{for } 2\alpha+\beta \in (1,2). \tag{3.16}$$

In fact, let

$$\begin{aligned} f_4(t) &= \log[J_{\frac{\lambda-2}{2}}(a,b)] - \log[A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)] \\ &= \log\frac{(\lambda-2)(t^{\frac{\lambda}{2}}-1)}{\lambda(t^{\frac{\lambda-2}{2}}-1)} - (\lambda-1)\log\frac{t+1}{2} - \frac{2-\lambda}{2}\log t. \end{aligned} \tag{3.17}$$

Simple computations yield

$$\lim_{t\to 1^+} f_4(t) = 0, \tag{3.18}$$

$$f_4'(t) = \frac{g_{5,\lambda}(t)}{2t(t+1)(t^{\frac{\lambda}{2}}-1)(t^{\frac{\lambda-2}{2}}-1)}, \tag{3.19}$$

(3.15) and (3.16) follow from (3.17)–(3.19), Lemma 2.3 and the fact

$$\left(t^{\frac{\lambda}{2}}-1\right)\left(t^{\frac{\lambda-2}{2}}-1\right) < 0.$$

The inequalities (ii) and (iii) follow from (3.1), (3.5), (3.9), (3.10), (3.15) and (3.16).

At last, we prove that the parameters $\frac{2-\lambda}{2}$ and $\frac{3\lambda-4}{2}$ cannot be improved in either case. For $|\varepsilon| \ll 1$,

$$\begin{aligned} &\lim_{t\to+\infty} \frac{J_{\frac{\lambda-2}{2}+\varepsilon}(t,1)}{A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)} \\ &= \lim_{t\to+\infty} \frac{(\lambda+2\varepsilon-2)(t^{\frac{\lambda+2\varepsilon}{2}}-1)}{(\lambda+2\varepsilon)(t^{\frac{\lambda+2\varepsilon-2}{2}}-1)\left(\frac{t+1}{2}\right)^{\lambda-1} t^{1-\frac{\lambda}{2}}} = \begin{cases} +\infty, & \text{for } \varepsilon > 0, \\ 0, & \text{for } \varepsilon < 0. \end{cases} \end{aligned} \tag{3.20}$$

(3.20) implies that for $\varepsilon > 0$, there exists a sufficiently large $T_1 = T_1(\varepsilon,\alpha,\beta) > 1$, such that $J_{\frac{\lambda-2}{2}+\varepsilon}(t,1) > A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1)$ for $t \in (T_1,+\infty)$, and for $\varepsilon < 0$, there exists a sufficiently large $T_2 = T_2(\varepsilon,\alpha,\beta) > 1$, such that $A^{\alpha}(t,1)G^{\beta}(t,1)H^{1-\alpha-\beta}(t,1) > J_{\frac{\lambda-2}{2}+\varepsilon}(t,1)$ for $t \in (T_2,+\infty)$. In the end, for $|\varepsilon| \ll 1$,

$$\begin{aligned} &J_{\frac{3\lambda-4}{2}+\varepsilon}(1+x,1) - A^{\alpha}(1+x,1)G^{\beta}(1+x,1)H^{1-\alpha-\beta}(1+x,1) \\ &= \frac{1}{(3\lambda+2\varepsilon-2)\left[(1+x)^{\frac{3\lambda+2\varepsilon-4}{2}}-1\right]}\Bigg\{(3\lambda+2\varepsilon-4)\left[(1+x)^{\frac{3\lambda+2\varepsilon-2}{2}}-1\right] \\ &\quad -(3\lambda+2\varepsilon-2)\left[(1+x)^{\frac{3\lambda+2\varepsilon-4}{2}}-1\right]\left(\frac{2+x}{2}\right)^{\lambda-1}(1+x)^{\frac{2-\lambda}{2}}\Bigg\} \\ &= \frac{1}{(3\lambda+2\varepsilon-2)\left[(1+x)^{\frac{3\lambda+2\varepsilon-4}{2}}-1\right]}\left[\frac{1}{24}(3\lambda+2\varepsilon-2)(3\lambda+2\varepsilon-4)\varepsilon x^3 + o(x^3)\right]. \end{aligned} \tag{3.21}$$

we know from (3.21) that for $\varepsilon > 0$, there exists a sufficiently small $\delta_1 = \delta_1(\varepsilon) > 0$, such that $J_{\frac{3\lambda-4}{2}+\varepsilon}(1+x,1) > A^{\alpha}(1+x,1)G^{\beta}(1+x,1)H^{1-\alpha-\beta}(1+x,1)$ for $x \in (0,\delta_1)$, and for $\varepsilon < 0$, there exists a sufficiently small $\delta_2 = \delta_2(\varepsilon) > 0$, such that $J_{\frac{3\lambda-4}{2}+\varepsilon}(1+x,1) < A^{\alpha}(1+x,1)G^{\beta}(1+x,1)H^{1-\alpha-\beta}(1+x,1)$ for $x \in (0,\delta_2)$. □

THEOREM 3.2. *Let $0<\alpha,\beta<1$ satisfy $\alpha+\beta<1$. Then for any $a,b>0$ with $a\neq b$, we have*

(I) $G_{\frac{2\alpha+\beta-2}{2\alpha+\beta},1}(a,b)=A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)=G_{2\alpha+\beta-2,1}(a,b),$ *for* $2\alpha+\beta=1$,

(II) $G_{\frac{2\alpha+\beta-2}{2\alpha+\beta},1}(a,b)<A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)<G_{2\alpha+\beta-2,1}(a,b),$ *for* $2\alpha+\beta\in(0,1)$,

(III) $G_{2\alpha+\beta-2,1}(a,b)<A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)<G_{\frac{2\alpha+\beta-2}{2\alpha+\beta},1}(a,b),$ *for* $2\alpha+\beta\in(1,2)$.

The numbers $\frac{2\alpha+\beta-2}{2\alpha+\beta}$ *and* $2\alpha+\beta-2$ *in* (II) *and* (III) *are optimal.*

For $\delta=\alpha+\frac{\beta}{2}$, it is obvious that $G_{\frac{2\alpha+\beta-2}{2\alpha+\beta},1}(a,b)=G_{\frac{\delta-1}{\delta},1}(a,b)$ and $G_{2\alpha+\beta-2,1}(a,b)=G_{\frac{\delta-1}{\delta},1}(a,b)$. Therefore, the statement of Theorem 3.2 can be written as

THEOREM 3.2'. *Let $0<\delta<1$. Then for any $a,b>0$ with $a\neq b$, we have*

(I′) $G_{\frac{\delta-1}{\delta},1}(a,b)=A^{\delta}(a,b)H^{1-\delta}(a,b)=G_{2\delta-2,1}(a,b)$, *for* $\delta=\frac{1}{2}$,

(II′) $G_{\frac{\delta-1}{\delta},1}(a,b)<A^{\delta}(a,b)H^{1-\delta}(a,b)<G_{2\delta-2,1}(a,b)$, *for* $\delta\in\left(0,\frac{1}{2}\right)$,

(III′) $G_{2\delta-2,1}(a,b)<A^{\delta}(a,b)H^{1-\delta}(a,b)<G_{\frac{\delta-1}{\delta},1}(a,b)$, *for* $\delta\in\left(\frac{1}{2},1\right)$.

The numbers $\frac{\delta-1}{\delta}$ *and* $2\delta-2$ *in* (II') *and* (III') *are optimal.*

Proof of Theorem 3.2. We assume $a>b$. Let $t=\frac{a}{b}>1$ and $\lambda=2\alpha+\beta$. For $\lambda=2\alpha+\beta=1$, (I) follows from

$$\begin{aligned}G_{\frac{2\alpha+\beta-2}{2\alpha+\beta},1}(a,b)&=G_{2\alpha+\beta-2,1}(a,b)=G_{-1,1}(a,b)\\&=(ab)^{\frac{1}{2}}=\left(\frac{a+b}{2}\right)^{\alpha}(ab)^{\frac{\beta}{2}}\left(\frac{2ab}{a+b}\right)^{1-\alpha-\beta}\\&=A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b).\end{aligned}$$

We now prove (II) and (III). In fact, let

$$\begin{aligned}f_5(t)&=\log\left[G_{\lambda-2,1}(a,b)\right]-\log\left[A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\right]\\&=\frac{1}{\lambda-3}\log\frac{t^{\lambda-2}+1}{t+1}-\frac{2-\lambda}{2}\log t-(\lambda-1)\log\frac{t+1}{2}.\end{aligned}\tag{3.22}$$

Simple computations lead to

$$\lim_{t\to1^+}f_5(t)=0,\tag{3.23}$$

$$f_5'(t) = \frac{(\lambda-2)g_6(\lambda,t)}{2(\lambda-3)t(t+1)(t^{\lambda-2}+1)} \begin{cases} >0, \ \lambda \in (0,1), \\ <0, \ \lambda \in (1,2). \end{cases} \tag{3.24}$$

where we have used Lemma 2.4, and

$$\begin{aligned} f_6(t) &= \log \frac{G_{\frac{\lambda-2}{\lambda}}(a,b)}{A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)} \\ &= -\frac{\lambda}{2}\log\frac{t^{\frac{\lambda-2}{\lambda}}+1}{t+1} - \frac{2-\lambda}{2}\log t - (\lambda-1)\log\frac{t+1}{2}, \end{aligned} \tag{3.25}$$

$$\lim_{t\to 1^+} f_6(t) = 0, \tag{3.26}$$

$$f_6'(t) = \frac{(2-\lambda)(t^{\frac{2\lambda-2}{\lambda}}-1)}{2t(t+1)(t^{\frac{\lambda-2}{\lambda}}+1)} \begin{cases} <0, \ \lambda \in [0,1), \\ >0, \ \lambda \in (1,2]. \end{cases} \tag{3.27}$$

(II) and (III) follow from (3.22)–(3.27).

At last, we prove that the parameters $\frac{\lambda-2}{\lambda}$ and $\lambda-2$ cannot be improved in either case. These will follow from

$$\begin{aligned} &\lim_{t\to+\infty} \frac{G_{\frac{\lambda-2}{\lambda}+\varepsilon,1}(1,t)}{A^{\alpha}(1,t)G^{\beta}(1,t)H^{1-\alpha-\beta}(1,t)} \\ &= \lim_{t\to+\infty}\left\{\left(\frac{1+t}{2t}\right)^{1-\lambda}\left(1+t^{\frac{\lambda+\varepsilon\lambda-2}{\lambda}}\right)^{\frac{\lambda}{\varepsilon\lambda-2}}\left(\frac{1+t}{t}\right)^{\frac{\lambda}{2-\varepsilon\lambda}} t^{\frac{\varepsilon\lambda^2}{2(2-\varepsilon\lambda)}}\right\} \\ &= \begin{cases} +\infty, & \lambda \in [0,1), \\ 0, & \lambda \in (1,2]. \end{cases} \end{aligned}$$

and

$$\begin{aligned} &[G_{\lambda-2-\varepsilon}(1+x,1)]^{3-\lambda+\varepsilon} - \left[A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b)\right]^{3-\lambda+\varepsilon} \\ &= \frac{1}{(1+x)^{\lambda-2-\varepsilon}+1}\{(2+x)-[(1+x)^{\lambda-2-\varepsilon}+1]\}(1+x)^{\frac{(2-\lambda)(3-\lambda+\varepsilon)}{2}} \\ &\quad \times \left(1+\tfrac{x}{2}\right)^{(\lambda-1)(3-\lambda+\varepsilon)} \\ &= \frac{1}{(1+x)^{\lambda-2-\varepsilon}+1}\left[-\frac{(3-\lambda+\varepsilon)\varepsilon}{4}x^2+o(x^2)\right] \end{aligned}$$

valid for $|\varepsilon| \ll 1$. $\square$

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