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PERTURBATION BOUNDS FOR SINGULAR VALUES OF MATRICES WITH SINGLETONS

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Abstract. This paper considers bounds for the singular values of a matrix with a row- or column-singleton if such an element is perturbed. A multiplicative approach to such perturbations may lead to sharper bounds than the usual additive Weyl bounds. These bounds immediately give the interlace property for the singular values. These results are used for perturbation bounds if a tridiagonal matrix with a zero diagonal is perturbed in the last pair of off-diagonal elements. Such perturbation bounds are then applied to bound perturbations of the eigenvalues and the first component of the eigenvectors, which has a direct application in the computation of the Gaussian quadrature formulae for compression splines.

1. Introduction

Many problems of numerical linear algebra lead to either the eigensystem computation of tridiagonal matrices, or the singular value decomposition (SVD) of bidiagonal matrices. Usually, the perturbations of such matrices are viewed in the context of computation of the eigenvalues (or singular values) when *all* nonzero elements of a matrix are perturbed.

In this paper we consider perturbations of the singular values of matrices with singletons, if a singleton is the only perturbed element. We say that the element $b_{k,\ell}$ is a row-singleton of a matrix B if the k-th row is zero, except at the position (k,ℓ). The element $b_{k,\ell}$ is a column-singleton of B if $\overline{b}_{\ell,k}$ is a row-singleton in B^*.

The first and the last diagonal elements of the bidiagonal matrices are column- and row-singletons, respectively. Perturbations in singular values of a bidiagonal matrix can be interpreted as perturbations of eigenvalues of the tridiagonal matrix with zero diagonal elements. Such perturbation can be of great interest in the algorithm for computation of nodes of the Gaussian quadratures for the compression splines [1] on the interval $[-1,1]$.

The compression splines are spanned in each interval by the *compression powers*. Compression powers are a basis of functions that consists of polynomials and two trigonometric functions,

$$\{1,x,\ldots,x^{m-3},\sin(px),\cos(px)\},$$

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where $m \geqslant 2$, and $p > 0$ is a given compression parameter. The compression parameter p need not be the same in each interval. If $p < \pi$, for given $n \in \mathbb{N}$ there exists a unique Gaussian quadrature formula of order n,

$$Q[f] = \sum_{i=1}^{n} w_i f(x_i),$$

where $x_i \in \langle -1, 1 \rangle$ are nodes, and w_i are positive weights for $i = 1, \ldots, n$. Such formula is exact for all functions f that are linear combination of compression powers, i.e.,

$$\{1, x, \ldots, x^{2n-3}, \sin(px), \cos(px)\}.$$

Typically, the nodes x_i and the weights w_i can be calculated from the Jacobi matrix

$$J_n = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & 0 & \cdots & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & \ddots & \vdots \\ 0 & \sqrt{\beta_2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \alpha_{n-2} & \sqrt{\beta_{n-1}} \\ 0 & \cdots & 0 & \sqrt{\beta_{n-1}} & \alpha_{n-1} \end{bmatrix}. \tag{1.1}$$

The nodes x_i of the integration formula are eigenvalues of the Jacobi matrix, while the weights w_i are

$$w_i = \beta_0 v_{1,i}^2,$$

where $v_{1,i}$, $i = 1, \ldots, n$ are the first components of the normalized eigenvectors of J_n. The normalization factor β_0 is the integral of the weight function w over the domain of the integration.

From the symmetry of the nodes in the case of Gaussian quadratures for compression splines, it follows that

$$\alpha_i = 0, \quad i = 0, \ldots, n-1$$

in (1.1), while all β_i, except β_{n-1}, are known and equal

$$\beta_i = \frac{i^2}{4i^2 - 1}.$$

Eigenvalues of the tridiagonal matrix are iteratively computed by the Golub–Welsch algorithm [8], to determine the value of the unknown parameter β_{n-1}. If the parameter p is changed, the derived bounds can be used as endpoints of the interval where new β_{n-1} lies.

Now supposte that A is tridiagonal Hermitian matrix of order n with zero diagonal elements and nonzero elements on the superdiagonal. Then A has rank either n, if n is even or $n-1$ if n is odd. A behavior of its eigenvalues can be seen as a behavior of the singular values of matrix B, where

$$PAP^T = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}. \tag{1.2}$$

If n is even, $n = 2s$, then B is of order s and

$$B = \begin{bmatrix} b_1 & b_2 & & & \\ & b_3 & b_4 & & \\ & & \ddots & \ddots & \\ & & & b_{2s-3} & b_{2s-2} \\ & & & & b_{2s-1} \end{bmatrix}, \qquad P = [e_{s+1}, e_1, e_{s+2}, e_2, \dots, e_{2s}, e_s], \tag{1.3}$$

where e_i are vectors of the canonical basis. If n is odd, $n = 2s+1$, then B is $s \times (s+1)$ matrix and

$$B = \begin{bmatrix} b_1 & b_2 & & & \\ & b_3 & b_4 & & \\ & & \ddots & \ddots & \\ & & & b_{2s-1} & b_{2s} \end{bmatrix}, \qquad P = [e_{s+1}, e_1, e_{s+2}, e_2, \dots, e_{s+1}]. \tag{1.4}$$

If the singular values of B are ordered decreasingly and denoted by σ_i,

$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_s \geqslant 0,$$

it is easy to show that the eigenvalues of A from (1.2) are $\pm\sigma_i$, $i = 1, \dots, s$ if n is even, and $\pm\sigma_i$, $i = 1, \dots, s$ and 0 if n is odd. Thus, a perturbation of the last off-diagonal element of A corresponds to a perturbation of the singletons b_{2s-1} in (1.3) if n is even, or of b_{2s} in (1.4) if n is odd.

The rest of the paper is organized as follows. In Section 2 we present bounds for the singular values of a matrix if its singleton is perturbed. The results obtained in Section 2 are then applied in Section 3 on a symmetric tridiagonal matrix with a zero diagonal, perturbed on the last off-diagonal element. Together with the bounds from the $\sin\vartheta$ theorems, bounds for eigenvalues of a perturbed tridiagonal matrix can used to obtain bounds for the first component of the normalized eigenvector of a perturbed matrix A'. In the final section we give some numerical examples which show how tight the obtained bounds can be.

2. Interlace properties for the singular values of perturbed bidiagonal matrices

Demmel and Kahan [4] showed that small relative perturbations in the entries of a bidiagonal matrix cause small relative perturbations in its singular values. This result is extended by Demmel and Gragg [3] to biacyclic matrices, i.e., matrices whose associated bipartite graph is acyclic.

Suppose that the nontrivial elements of a biacyclic matrix B are denoted by $b_{k,\ell}$, where (k,ℓ) denote the position of a nontrivial element, while the elements of a perturbed matrix B' are denoted by $b'_{k,\ell}$. The singular values of B, B' are denoted by σ_i, σ'_i, respectively.

THEOREM 2.1. (Demmel–Gragg) *Let B be biacyclic, and let $B' = B + \delta B$ be a componentwise relative perturbation of B, i.e., $b'_{k,\ell} = \alpha_{k,\ell} b_{k,\ell}$ for all k and ℓ, where*

$\alpha_{k\ell} \neq 0$. *Then for all singular values*

$$\frac{1}{1+\eta}\sigma_i \leqslant \sigma_i' \leqslant (1+\eta)\sigma_i,$$

where

$$\eta = \prod_{b_{k,\ell}\neq 0} \max\{|\alpha_{k,\ell}|, 1/|\alpha_{k,\ell}|\} - 1.$$

To prove the previous Theorem, Demmel and Gragg also proved the following lemma.

LEMMA 2.2. (Demmel–Gragg) *Let B be biacyclic, and let $B' = B$ except for $b'_{k,\ell}$, where $b'_{k,\ell} = \alpha b_{k,\ell}$, $\alpha \neq 0$. Then for all singular values*

$$\min\{|\alpha|, 1/|\alpha|\}\sigma_i \leqslant \sigma_i' \leqslant \max\{|\alpha|, 1/|\alpha|\}\sigma_i.$$

In order to prove the modifications of the previous Lemma, we need the following theorem, given in [9, Theorem 3.3.16], and just for completeness, we present it here.

THEOREM 2.3. *Let $F, G \in \mathbb{C}^{m\times n}$ be given matrices and let $q = \min\{m,n\}$. The following inequalities hold for decreasingly ordered singular values of F, G, $F+G$ and FG^**

(a) $\sigma_{i+j-1}(F+G) \leqslant \sigma_i(F) + \sigma_j(G)$,

(b) $\sigma_{i+j-1}(FG^*) \leqslant \sigma_i(F)\sigma_j(G)$

for $1 \leqslant i, j \leqslant q$ and $i+j \leqslant q+1$. In particular, for $j = 1$

(c) $|\sigma_i(F+G) - \sigma_i(F)| \leqslant \sigma_1(G)$, *for* $i = 1, \ldots, q$,

(d) $\sigma_i(FG^*) \leqslant \sigma_i(F)\sigma_1(G)$, *for* $i = 1, \ldots, q$.

Now we describe how the perturbation of a singleton in a matrix is reflected in its singular values.

THEOREM 2.4. *Suppose that row- (column-) singleton $b_{k,\ell}$ of a matrix $B \in \mathbb{C}^{m\times n}$ is perturbed such that $B' = B$ except on position (k,ℓ), where $b'_{k,\ell} = \alpha b_{k,\ell}$, and α is a nonzero constant. Then, singular values of B and B' satisfy*

$$\min\{1, |\alpha|\}\sigma_i \leqslant \sigma_i' \leqslant \max\{1, |\alpha|\}\sigma_i.$$

Proof. Suppose that $b_{k,\ell}$ is a row-singleton. Then B' can be written as $B' = DB$, where

$$D = \operatorname{diag}(I_{k-1}, \alpha, I_{m-k}). \tag{2.1}$$

A direct application of Theorem 2.3(d) gives $\sigma_i' \leqslant \sigma_1(D)\sigma_i$. If $|\alpha| \leqslant 1$ then $\sigma_1(D) = 1$, otherwise $\sigma_1(D) = |\alpha|$,

$$\sigma_i' \leqslant \max\{1, |\alpha|\}\sigma_i. \tag{2.2}$$

Matrices B and B' can swap roles, i.e., $B = D^{-1}B'$. If $|\alpha| > 1$ then $\sigma_1(D^{-1}) = 1$, otherwise $\sigma_1(D) = 1/|\alpha|$, and Theorem 2.3(d) gives

$$\sigma_i \leqslant \sigma_1(D^{-1})\sigma_i' = \max\{1, 1/|\alpha|\}\sigma_i'. \tag{2.3}$$

Note that

$$\frac{1}{\max\{1, 1/|\alpha|\}} = \min\{1, |\alpha|\},$$

and (2.3) can be written as

$$\min\{1, |\alpha|\}\sigma_i \leqslant \sigma_i'. \tag{2.4}$$

Now (2.2) and (2.4) complete the proof for a row-singleton.

Suppose that $b_{k,\ell}$ is a column-singleton. Then B' can be written as $B' = B\Delta$, where

$$\Delta = \operatorname{diag}(I_{\ell-1}, \alpha, I_{n-\ell}).$$

Since the singular values of matrix B and B^* are the same, we have

$$\sigma_i([B']^*) = \sigma_i' \leqslant \sigma_1(\Delta^*)\sigma_i(B^*) = \max\{1, |\alpha|\}\sigma_i.$$

From Theorem 2.3(d), applied on $B^* = \Delta^{-*}[B']^*$, it follows

$$\min\{1, |\alpha|\}\sigma_i \leqslant \sigma_i',$$

and that completes the proof. □

If the perturbation in the last theorem is such that $|b'_{k,\ell}|$ is bigger than $|b_{k,\ell}|$, all the singular values of B' become bigger, and vice-versa.

THEOREM 2.5. *Suppose that row (column)-singleton $b_{k,\ell}$ of a matrix $B \in \mathbb{C}^{m\times n}$ is perturbed such that $B' = B$ except on position (k,ℓ), where $b'_{k,\ell} = \alpha b_{k,\ell}$ and α is a nonzero constant.*

1. *If $\alpha < 1$, then singular values of B and B' satisfy*

$$\sigma_{i+1} \leqslant \sigma_i' \leqslant \sigma_i, \tag{2.5}$$

for $i = 1, \ldots, n$, where we adopt the notational convention $\sigma_{n+1} = 0$.

2. *If $\alpha \geqslant 1$, then singular values of B and B' satisfy*

$$\sigma_i \leqslant \sigma_i' \leqslant \sigma_{i-1}, \tag{2.6}$$

for $i = 1, \dots, n$, where we adopt the notational convention $\sigma_{n+1} = \infty$.

Proof. Note that the right-hand side inequality in (2.5) and the left-hand side inequality in (2.6) are consequences of Theorem 2.4.

On the other hand, B can be written as $B = B' + C$, where

$$c_{i,j} = \begin{cases} b_{k,\ell} - b'_{k,\ell} & \text{if } i = k \text{ and } j = \ell, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Note that C is of rank at most one, and

$$\sigma_1(C) = |b_{k,\ell} - b'_{k,\ell}|, \quad \sigma_2(C) = \cdots = \sigma_n(C) = 0.$$

By substituting $j = 2$, $F = B'$, and $G = C$ in Theorem 2.3(a), we obtain

$$\sigma_{i+1} \leqslant \sigma_i' + \sigma_2(C) = \sigma_i'.$$

To prove the left-hand side of relation (2.6), B' can be written as $B' = B - C$, where C is defined by (2.7). By substituting $j = 2$, $F = B$, and $G = -C$ in Theorem 2.3(a), we obtain

$$\sigma_{i+1}' \leqslant \sigma_i' + \sigma_2(-C) = \sigma_i'.$$

By translation of the index $i \to i-1$, we obtain the right-hand side of (2.6). □

From the proof of the previous Theorem, notice that the left-hand side inequality in (2.5) is obtained by additive perturbation, while the right-hand side of (2.5) is obtained by multiplicative perturbation. In (2.6) the left-hand side inequality is obtained by multiplicative, while the right-hand side is obtained by additive perturbation.

Finally, note that (2.5) and (2.6) represent the interlace property for the singular values of matrices B and B'. In between any neighboring pair of the singular values of B lies exactly one singular value of B', and vice-versa.

Note that Theorem 2.3(c) is standard Weyl perturbation bound, i.e.,

$$\sigma_i - |b_k' - b_k| \leqslant \sigma_i' \leqslant \sigma_i + |b_k' - b_k|, \quad i = 1, \dots, n. \tag{2.8}$$

In Section 4 we show that, in some cases, the bounds from Theorem 2.4 and Theorem 2.5 can be tighter than the bounds from the relation (2.8).

Obviously, all bounds presented in this section hold for matrices B from (1.3) and (1.4). Moreover, these bounds hold for the eigenvalues of a matrix A from (1.2).

3. Perturbation bounds for the first components of eigenvectors

It is quite interesting question how the change of the eigenvalues of a Hermitian matrix $A \in \mathbb{R}^{n\times n}$, caused by the Hermitian perturbation of $a_{n-1,n}$ and $a_{n,n-1}$, reflects on the change of the eigenvectors of $A' = D^T AD$.

This problem is solved by the variants of the absolute or relative $\sin\vartheta$ theorems. We use these theorems to determine the bounds for the first components of the eigenvectors of a symmetric, tridiagonal A (with a zero diagonal) from (1.2). If A is irreducible, i.e., if the off-diagonal elements of A are nontrivial, the eigenvalues of A are distinct, and each eigenvector spans the one-dimensional subspace. Therefore it is sufficient to analyze $\sin\vartheta_i$ rather than to consider angles between subspaces. For example, matrices generated by three-term recurrence relation for orthogonal polynomials, such as the Jacobi matrix J_n from (1.1), are irreducible.

The following theorem is proved in [7, Theorem 2.2]. It bounds the angle ϑ_i between subspaces spanned by an exact eigenvector and the corresponding perturbed eigenvector in terms of the relative gap.

THEOREM 3.1. *Let $A' = A + \delta A = D^T AD$, where D is a nonsingular matrix, and let*

$$\beta = \|D - I\|_2, \qquad \gamma = \|D^T D - I\|_2, \qquad \delta = \|D^T D\|_2 \|D^{-T} D^{-1} - I\|_2.$$

Then

$$\sin\vartheta_i \leqslant \frac{\delta}{\rho_i - \gamma} + \beta,$$

where the relative gap ρ_i is defined by

$$\rho_i = \min\left\{2, \min_{j\neq i} \frac{|\lambda_j - \lambda_i|}{|\lambda_i|}\right\},$$

provided that $\rho_i > \gamma$.

For a tridiagonal matrix A with zero diagonal, and perturbed matrix A' such that $A' = A$ except on positions $(n-1,n)$ and $(n,n-1)$, we have the following corollary, very similar to [7, Corollary 4.4].

COROLLARY 3.2. *Let A be a symmetric tridiagonal matrix with a zero diagonal, and let $A' = A + \delta A$ equal A, except for the off-diagonal elements $a_{n-1,n} = a_{n,n-1}$, that are perturbed to $\alpha a_{n-1,n} = \alpha a_{n,n-1}$ for some $\alpha \neq 0$. Then*

$$\sin\vartheta_i \leqslant \frac{\delta}{\rho_i - \gamma} + \beta,$$

where

$$\beta = |\alpha - 1|, \quad \gamma = |\alpha^2 - 1|, \quad \delta = \max\{\alpha^2, 1\}\frac{|\alpha^2 - 1|}{\alpha^2},$$

provided that $\rho_i > \gamma$.

Proof. The proof immediately follows from Theorem 3.1, by defining D as in (2.1),

$$D = \operatorname{diag}(I_n, \alpha).$$

Then

$$\begin{aligned}
\beta &= \|D - I\|_2 = |\alpha - 1|, \\
\gamma &= \|D^T D - I\|_2 = |\alpha^2 - 1|, \\
\delta &= \|D^T D\|_2 \|D^{-T} D^{-1} - I\|_2 = \max\{\alpha^2, 1\} \left| \frac{1}{\alpha^2} - 1 \right| = \max\{\alpha^2, 1\} \frac{|\alpha^2 - 1|}{\alpha^2},
\end{aligned}$$

and that completes the proof. □

REMARK 3.3. For D in the proof of Corollary 3.2, we can take different D's. For example, if $\alpha > 0$, Eisenstat and Ipsen in [7, Corollary 4.4] set

$$D = \operatorname{diag}(\ldots, \sqrt{1/\alpha}, \sqrt{\alpha}, \sqrt{1/\alpha}, \sqrt{\alpha}, \sqrt{\alpha})$$

and obtain

$$\sin \vartheta_i \leqslant \frac{\eta(1+\eta)}{\rho_i - \eta} + \frac{\eta}{2}, \tag{3.1}$$

where $\eta = \max\{\alpha, 1/\alpha\} - 1$, provided that $\rho_i > \eta$.

Note that the bound from Corollary 3.2 could be better than the bound in (3.1). For example, if $\lambda = 1/2$ then

$$\beta = \frac{1}{2}, \quad \gamma = \frac{3}{4}, \quad \delta = 3, \quad \eta = 1,$$

The bound from Corollary 3.2 gives

$$\sin \vartheta_i \leqslant \frac{3}{\rho_i - 3/4} + \frac{1}{2}, \tag{3.2}$$

while the bound from (3.1) gives

$$\sin \vartheta_i \leqslant \frac{2}{\rho_i - 1} + \frac{1}{2}. \tag{3.3}$$

Note that the bound in (3.2) is valid for $\rho_i > 3/4$, while the bound from (3.3) is valid for $\rho_i > 1$. Moreover, even if both inequalities are well-defined, (3.2) gives better bound than (3.3) if $\rho_i < 3/2$.

There are numerous other possibilities how to bound $\sin \vartheta_i$. For example, Dhillon and Parlett in [6, Theorem 1] cite the following theorem of Temple from the 1930s.

THEOREM 3.4. *Let $A = A^T$ be a real matrix that has a simple eigenvalue λ_i with normalized eigenvector v_i. For any unit vector w and a scalar μ, closer to λ_i than to any other eigenvalue,*

$$|\sin \sphericalangle(v_i, w)| \leqslant \frac{\|Aw - \mu w\|_2}{\operatorname{gap}(\mu)},$$

where $\mathrm{gap}(\mu) = \min\{|\lambda_j - \mu| : \lambda_j \neq \lambda_i, \lambda_j \in \mathit{spectrum}(A)\}$. *In addition, the error in the eigenvalue is bounded by the residual norm, i.e.,*

$$|\mu - \lambda_i| \leqslant \|Aw - \mu w\|_2.$$

In Theorem 3.4, one should use the Kato–Temple (see for example [2]) inequality to estimate the spectrum. In this case,

$$|\mu - \lambda_i| \leqslant \min\left\{\frac{\|Aw - \mu w\|_2^2}{\mathrm{gap}(\mu)}, \|Aw - \mu w\|_2\right\}.$$

Instead of any normalized vector w and any scalar μ, closer to λ_i than to any other eigenvalue, we can take the eigenpair (λ_i', v_i') of matrix $A' = DAD$, for D diagonal and sufficiently close to the identity matrix. In this case we obtain

$$|\sin\sphericalangle(v_i, v_i')| \leqslant \frac{\|Av_i' - \lambda_i' v_i'\|_2}{\mathrm{gap}(\lambda_i)} = \frac{\|Av_i' - DADv_i'\|_2}{\mathrm{gap}(\lambda_i)} \leqslant \frac{\|A - DAD\|_2}{\mathrm{gap}(\lambda_i)}.$$

Theorem 3.4 is in [5, 6] also used to prove that the algorithm `MRRR` applied to the symmetric indefinite factorization of a tridiagonal matrix A, $A = LDL^T$, can compute numerically orthogonal approximation of the eigenvectors at the cost of $\mathcal{O}(n^2)$.

Now, we can bound the first component of the eigenvectors of the symmetric tridiagonal matrix A' with zero diagonal elements. Interpreted in terms of the integration formula, this bounds are in fact perturbation bounds for weights in the integration formula.

PROPOSITION 3.5. *Suppose that* v_i *is the unit eigenvector of* A, *and* v_i' *is the unit eigenvector of* A'. *If the angle* ϑ_i, *between subspaces spanned by* v_i *and* v_i', *is bounded, i.e., if* $0 \leqslant \sin\vartheta_i \leqslant \zeta$, *where*

$$\zeta \leqslant \frac{v_{1,i}^2}{1 + v_{1,i}^2}, \tag{3.4}$$

then the nonnegative first component $v_{1,i}'$ *of the eigenvector* v_i' *is bounded by*

$$\sqrt{1-\zeta^2}\, v_{1,i} - \zeta \leqslant v_{1,i}' \leqslant v_{1,i} + \zeta. \tag{3.5}$$

Proof. Suppose that v_i is the unit eigenvector of A, and v_i' is the unit eigenvector of A'. Then, in the $v_i v_i'$ plane, v_i' can be written as

$$v_i' = \cos\vartheta_i v_i + \sin\vartheta_i z,$$

where z is a unit vector orthogonal to v_i in the $v_i v_i'$ plane. We can always take nonnegative first component of the normalized eigenvector v_i (if v_i is a unit eigenvector, $-v_i$ is also a unit eigenvector). Since ϑ_i is the angle between subspaces, then $0 \leqslant \vartheta_i \leqslant \pi/2$. Thus, if the right-hand sides in Corollary 3.2 and Remark 3.3 are denoted by ζ, i.e., if $0 \leqslant \sin\vartheta_i \leqslant \zeta$, then $1 \geqslant \cos\vartheta_i \geqslant \sqrt{1-\zeta^2}$. Furthermore, the first component z_1 of the unit vector z satisfy $-1 \leqslant z_1 \leqslant 1$. The absolute value of the first component $v'_{1,i}$ of v'_i can be bounded from above

$$|v'_{1,i}| \leqslant \cos\vartheta_i v_{1,i} + \sin\vartheta_i |z_1| \leqslant v_{1,i} + \zeta. \tag{3.6}$$

On the other hand, we have

$$v'_{1,i} = \cos\vartheta_i v_{1,i} + \sin\vartheta_i z_1 \geqslant \sqrt{1-\zeta^2} v_{1,i} - \sin\vartheta_i \geqslant \sqrt{1-\zeta^2} v_{1,i} - \zeta. \tag{3.7}$$

If ζ is small enough, i.e., if $\sqrt{1-\zeta^2} v_{1,i} - \zeta \geqslant 0$, or in other words, if (3.4) holds, from (3.7) we have

$$|v'_{1,i}| \geqslant \sqrt{1-\zeta^2} v_{1,i} - \zeta.$$

To conclude, if ζ satisfies (3.4), from (3.6) and (3.7) it follows (3.5). $\square$

4. Numerical examples

In this section we present a few numerical examples that show how good the new bounds are. To ensure the accuracy of the computed eigenvalues, they are computed symbolically by Wolfram Mathematica and then rounded to 16 digits. Our bounds are especially good for the smallest singular value of a matrix.

EXAMPLE 4.1. Suppose that

$$B = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 10^{-2} \end{bmatrix},$$

$\Delta B = \mathrm{diag}(0,0,0,10^{-3})$, and $B' = B + \Delta B$. The smallest singular value of B is

$$\sigma_4 \approx 0.004999781256152754,$$

while the smallest singular value of B' is

$$\sigma'_4 \approx 0.005499708853659211.$$

Note that $\alpha = 10/11$ and from Theorem 2.4 we obtain the following bounds

$$0.004999781256152754 = \sigma_4 \leqslant \sigma'_4 \leqslant \alpha\sigma_4 = 0.005499759381768030,$$

that is better than the bounds from (2.8),

$$0.003999781256152754 = \sigma_4 - 10^{-3} \leqslant \sigma'_4 \leqslant \sigma_4 + 10^{-3} = 0.005999781256152754.$$

Teorem 2.5 can be useful if B has two very close singular values.

EXAMPLE 4.2. Suppose that

$$B = \begin{bmatrix} 1 & 10^{-3} & & \\ & 1 & 10^{-3} & \\ & & 1 & 10^{-3} \\ & & & 10^{-2} \end{bmatrix},$$

$\Delta B = \mathrm{diag}(0,0,0,-10^{-3})$, and $B' = B + \Delta B$. The two largest singular values of B are

$$\sigma_1 \approx 1.000707356616978, \quad \sigma_2 \approx 1.000000500024877,$$

while the largest singular value of B' is

$$\sigma_1' \approx 1.000707356614608.$$

Since $\alpha = 9/10$, from Theorem 2.5 we obtain

$$1.000000500024877 = \sigma_2 \leqslant \sigma_1' \leqslant \sigma_1 = 1.000707356616978$$

that is better than the Weyl bound (2.8)

$$0.9997073566169783 = \sigma_1 - 10^{-3} \leqslant \sigma_1' \leqslant \sigma_1 + 10^{-3} = 1.001707356616978.$$

Our final example is connected to the Gaussian quadrature formula for compression splines.

EXAMPLE 4.3. Suppose that the matrix A of order 6 is a matrix in the process of the computation of the Gaussian integration formula for compression splines,

$$A = \begin{bmatrix} 0 & \sqrt{\beta_1} & & & & \\ \sqrt{\beta_1} & 0 & \sqrt{\beta_2} & & & \\ & \sqrt{\beta_2} & & \sqrt{\beta_3} & & \\ & & \sqrt{\beta_3} & 0 & \sqrt{\beta_4} & \\ & & & \sqrt{\beta_4} & 0 & \sqrt{\beta_5} \\ & & & & \sqrt{\beta_5} & 0 \end{bmatrix},$$

where

$$\beta_i = \frac{i^2}{4i^2 - 1}, \quad i = 1, \ldots, 4,$$

with $\sqrt{\beta_5} = 1/2$ in some step of the computation. If we denote the positive eigenvalues of A by $\lambda_1 > \lambda_2 > \lambda_3$, and the corresponding negative eigenvalues by $\lambda_{-i} = -\lambda_i$, we have

$$\lambda_1 = 0.9320990567449202,$$
$$\lambda_2 = 0.6600705085369154,$$
$$\lambda_3 = 0.2379272646969216.$$

If β_5 is perturbed such that $\sqrt{\beta_5'} = \sqrt{\beta_5} + 10^{-3}$ then $\alpha = 501/500$, and positive eigenvalues of A' are

$$\lambda_1' = 0.9322455780588768,$$
$$\lambda_2' = 0.6605223882455808,$$
$$\lambda_3' = 0.2382025774989347.$$

Theorem 2.4 gives tight upper bounds for positive (and lower bounds for negative) eigenvalues,

$$\lambda_1 \leqslant \lambda_1' \leqslant \alpha\lambda_1 = 0.9339632548584101,$$
$$\lambda_2 \leqslant \lambda_2' \leqslant \alpha\lambda_2 = 0.6613906495539892,$$
$$\lambda_3 \leqslant \lambda_3' \leqslant \alpha\lambda_3 = 0.2384031192263154.$$

Now we can also look for the bounds for the first component of the eigenvectors. If we take, for example, λ_2, then the bound from Remark 3.3 is

$$\sin\vartheta_2 \leqslant 0.412120439695105.$$

If we know the first component $v_{1,2} = 0.4249189651634649$ of the eigenvector that corresponds to λ_2, from (3.5) if follows

$$0.4190252340367909 \leqslant |v_{1,2}'| \leqslant 0.4308053346443175,$$

while the exact value is $v_{1,2}' = \pm 0.4248361559047488$.

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