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SOME GEOMETRIC INEQUALITIES OF RADON — MITRINOVIĆ

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Abstract. Some Mitrinović type inequalities for general convex polygons are presented. The main tool in the proofs is the Radon inequality.

1. Introduction

Inequality of *J. Radon* has the statement:

If $n \in N^*$, $m \in R_+$, $x_k \in R_+$, $y_k \in R_+^*$, $\forall k = \overline{1,n}$, $X_n = \sum\limits_{k=1}^{n} x_k$, $Y_n = \sum\limits_{k=1}^{n} y_k$, then:

$$\sum_{k=1}^{n} \frac{x_k^{m+1}}{y_k^m} \geqslant \frac{X_n^{m+1}}{Y_n^m} \tag{R}$$

Equality holds if and only if there exists $t \in R_+^*$ such that $x_k = t y_k$, $\forall k = \overline{1,n}$.

Inequality of *Dragoslav S. Mitrinović* has the statement:

In any triangle with perimeter $2p$, circumscribed of circle $C(I;r)$ occurs the inequality

$$p \geqslant 3r\sqrt{3} \tag{M}$$

Equality holds if and only if the triangle is equilateral.

2. Results

The purpose of this article is to establish some geometric inequalities (other then $[1]$) on *(M)-type*, in convex polygons, used the *(R)-inequality*.

For any convex polygon $A_1A_2...A_n, n \geqslant 3$ we denoted by S the area of the polygon, by $2p$ the perimeter of the polygon, by a_k the side of the length $[A_kA_{k+1}]$ $\left(k = \overline{1,n}\right)$, $A_{n+1} \equiv A_1$, and for any point M from inside the polygon we denoted $T_k = pr_{A_KA_{K+1}}M$, $d_k = MT_k$, $S_k = area\,[A_kMA_{k+1}]$, $\forall k = \overline{1,n}$, and $u_k = \mu\left(\angle A_kMA_{k+1}\right)$, $v_k = \mu\left(\angle A_kMT_k\right)$, $w_k = \mu\left(\angle T_kMA_{k+1}\right)$ the measures of angles $\angle A_kMA_{k+1}$, $\angle A_kMT_k$, respectively $\angle T_kMA_{k+1}$ in radians.

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LEMMA. *If $A, B, A \neq B$ are points in the same plane and $M \notin AB$, $T = pr_{AB}M$, $d = MT$, then:*

$$\frac{AB}{d} = \frac{AB}{MT} \geqslant 2\,\mathrm{tg}\,\frac{u}{2},$$

where $u = \mu(\angle AMB)$ is the measure of the angle $\angle AMB$ in radians.

Proof. We denoted $v = \mu(\angle AMT)$ and $w = \mu(\angle TMB)$.

We have the following possibilities:

i) $T \in (AB)$. In this case $\mathrm{tg}\,v = \frac{AT}{MT}$ and $\mathrm{tg}\,w = \frac{TB}{MT}$, so

$$\frac{AB}{d} = \frac{AB}{MT} = \frac{AT + TB}{MT} = \mathrm{tg}\,v + \mathrm{tg}\,w.$$

Since the function $f : \left[0, \frac{\pi}{2}\right] \to R_+$, $f(x) = \mathrm{tg}\,x$, is convex on $\left[0, \frac{\pi}{2}\right]$ it follows by *Jensen's inequality* that

$$\frac{AB}{d} = \mathrm{tg}\,v + \mathrm{tg}\,w \geqslant 2\,\mathrm{tg}\,\frac{v+w}{2} = 2\,\mathrm{tg}\,\frac{u}{2}.$$

ii) $T \equiv A$. In this case $\mathrm{tg}\,w = \frac{AB}{d}$, and $\mathrm{tg}\,v = \frac{AT}{d} = \frac{TT}{d} = 0$. Also we have: $\mathrm{tg}\,u = \mathrm{tg}\,w$, so $\frac{AB}{d} = \mathrm{tg}\,u = \mathrm{tg}\,2\left(\frac{u}{2}\right) = \frac{2\,\mathrm{tg}\,\frac{u}{2}}{1-\mathrm{tg}^2\frac{u}{2}} \geqslant 2\,\mathrm{tg}\,\frac{u}{2} \Leftrightarrow \mathrm{tg}^3\frac{u}{2} \geqslant 0$, which is true.

iii) $T \equiv B$.This case is analogous with ii).Hence the conclusion follows.

iv) $A \in (TB)$. We have $\mathrm{tg}\,v = \frac{TA}{MT} = \frac{TA}{d}$ and $\mathrm{tg}\,w = \mathrm{tg}\,(u+v) = \frac{TB}{MT} = \frac{TB}{d}$.
Hence,

$$\frac{AB}{d} = \frac{TB - TA}{d} = \mathrm{tg}\,(u+v) - \mathrm{tg}\,v \geqslant \mathrm{tg}\,u$$
$$\Leftrightarrow \frac{\mathrm{tg}\,u + \mathrm{tg}\,v}{1 - \mathrm{tg}\,u\,\mathrm{tg}\,v} - \mathrm{tg}\,u - \mathrm{tg}\,v \geqslant 0 \Leftrightarrow \mathrm{tg}\,u\,\mathrm{tg}\,v(\mathrm{tg}\,u + \mathrm{tg}\,v) \geqslant 0,$$

which is true.

Therefore, $\frac{AB}{d} \geqslant \mathrm{tg}\,u \geqslant 2\,\mathrm{tg}\,\frac{u}{2}$.

v) $B \in (AT)$. This case is analogous with iv). Hence the conclusion follows.

And we are done

THEOREM 1. *If $A_1A_2...A_n$ $(n \geqslant 3)$ is a convex polygon and M is a point inside this polygon, then:*

$$\sum_{k=1}^{n} \frac{a_k}{d_k} \geqslant 2n\,\mathrm{tg}\,\frac{\pi}{n} \tag{1}$$

Proof. By lemma we have:

$$\frac{a_k}{d_k} \geqslant 2\,\mathrm{tg}\,\frac{u_k}{2}, \quad \forall k = \overline{1,n}$$

and then

$$\sum_{k=1}^{n} \frac{a_k}{d_k} \geqslant 2 \sum_{k=1}^{n} \operatorname{tg} \frac{u_k}{2}.$$

Since the function $f : \left[0, \frac{\pi}{2}\right] \to R_+$, $f(x) = \operatorname{tg} \frac{x}{2}$, is convex on $\left[0, \frac{\pi}{2}\right]$ by Jensen's inequality follows that:

$$\sum_{k=1}^{n} \operatorname{tg} \frac{u_k}{2} \geqslant n \operatorname{tg} \left(\frac{1}{2n} \sum_{k=1}^{n} u_k \right) = n \operatorname{tg} \frac{2\pi}{2n} = n \operatorname{tg} \frac{\pi}{n}. \quad \square$$

OBSERVATION 1.1. If the polygon $A_1A_2...A_n$ is circumscribed in the circle $C(I;r)$ and $M \equiv I$, then: $d_k = r$, $\forall k = \overline{1,n}$, and (1) becomes:

$$\frac{1}{r} \sum_{k=1}^{n} a_k = \frac{2p}{r} \geqslant 2n \operatorname{tg} \frac{\pi}{n} \Leftrightarrow p \geqslant nr \operatorname{tg} \frac{\pi}{n} \tag{1'}$$

OBSERVATION 1.2. The inequality (1) is a generalization of the *(M)-inequality*, and if the polygon is the triangle ABC (1) becomes:

$$\frac{a}{d_a} + \frac{b}{d_b} + \frac{c}{d_c} \geqslant 6 \operatorname{tg} \frac{\pi}{3} = 6\sqrt{3} \tag{1''}$$

and if $M \equiv I$, then we obtain the *(M)-inequality*.

THEOREM 2. *If $A_1A_2...A_n$ $(n \geqslant 3)$ is a convex polygon, M is a point inside this polygon and $m \in R_+$, then:*

$$\sum_{k=1}^{n} \frac{a_k}{d_k^m} \geqslant 2 \cdot \frac{p^{m+1}}{S^m} \tag{2}$$

Proof. We have:

$$\sum_{k=1}^{n} \frac{a_k}{d_k^m} = \sum_{k=1}^{n} \frac{a_k^{m+1}}{(a_k d_k)^m} = \sum_{k=1}^{n} \frac{a_k^{m+1}}{2^m S_k^m} = \frac{1}{2^m} \sum_{k=1}^{n} \frac{a_k^{m+1}}{S_k^m},$$

and used the *(R)-inequality*, we obtain

$$\sum_{k=1}^{n} \frac{a_k}{d_k^m} \geqslant \frac{1}{2^m} \frac{\left(\sum\limits_{k=1}^{n} a_k\right)^{m+1}}{\left(\sum\limits_{k=1}^{n} S_k\right)^m} = \frac{1}{2^m} \cdot \frac{2^{m+1} p^{m+1}}{S^m} = 2 \cdot \frac{p^{m+1}}{S^m} \quad \square$$

OBSERVATION 2.1. If the polygon $A_1A_2...A_n$ $(n \geqslant 3)$ is circumscribed in the circle $C(I;r)$, then $S = pr$, and (2) becomes:

$$\sum_{k=1}^{n} \frac{a_k}{d_k^m} \geqslant 2 \cdot \frac{p^{m+1}}{p^m r^m} = \frac{2p}{r^m} \tag{2'}$$

If $M \equiv I$ then by $(1')$, we deduce that:

$$\sum_{k=1}^{n} \frac{a_k}{d_k^m} \geqslant \frac{2n}{r^{m-1}} \operatorname{tg} \frac{\pi}{n} \tag{2''}$$

THEOREM 3. *If $A_1A_2...A_n$ $(n \geqslant 3)$ is a convex polygon, M is a point inside this polygon, and $m \in R_+$, then:*

$$\sum_{k=1}^{n} \frac{a_k}{d_k^{m+1}} \geqslant \frac{2^m n^{m+1}}{p} \operatorname{tg}^{m+1} \frac{\pi}{n} \tag{3}$$

Proof. We have: $\sum\limits_{k=1}^{n} \frac{a_k}{d_k^{m+1}} = \sum\limits_{k=1}^{n} \frac{1}{a_k^m}\left(\frac{a_k}{d_k}\right)^{m+1}$, where we apply the *(R)-inequality* and we obtain $\sum\limits_{k=1}^{n} \frac{a_k}{d_k^{m+1}} \geqslant \frac{\left(\sum\limits_{k=1}^{n} \frac{a_k}{d_k}\right)^{m+1}}{\sum\limits_{k=1}^{n} a_k} = \frac{1}{2p}\left(\sum\limits_{k=1}^{n} \frac{a_k}{d_k}\right)^{m+1}$, then by (1) we deduce the conclusion. □

OBSERVATION 3.1. The relation (3) is also written as:

$$p \sum_{k=1}^{n} \frac{a_k}{d_k^{m+1}} \geqslant 2^m n^{m+1} \operatorname{tg}^{m+1} \frac{\pi}{n} \tag{3'}$$

If the polygon $A_1A_2...A_n$ $(n \geqslant 3)$ is circumscribed in the circle $C(I;r)$, then $(3')$ becomes:

$$2p^2 \cdot \frac{1}{r^{m+1}} \geqslant 2^m n^{m+1} \operatorname{tg}^{m+1} \frac{\pi}{n} \Leftrightarrow p^2 \geqslant 2^{m-1}\left(nr \operatorname{tg} \frac{\pi}{n}\right)^{m+1} \tag{3''}$$

THEOREM 4. *If $A_1A_2...A_n$ $(n \geqslant 3)$ is a convex polygon, M is a point inside this polygon and $m \in R_+$, then for any $x, y \in R_+^*$ such that $2px > y \max\limits_{1 \leqslant k \leqslant n} a_k$, occurs the inequality:*

$$\sum_{k=1}^{n} \frac{a_k}{d_k^m (2px - ya_k)^{m+1}} \geqslant \frac{n^{m+1}}{2^m S^m (nx - y)^{m+1}} \tag{4}$$

Proof. We have:

$$\sum_{k=1}^{n} \frac{a_k}{d_k^m (2px - ya_k)^{m+1}} = \sum_{k=1}^{n} \frac{a_k^{m+1}}{a_k^m d_k^m (2px - a_k y)^{m+1}} = \frac{1}{2^m} \sum_{k=1}^{n} \frac{1}{S_k^m}\left(\frac{a_k}{2px - ya_k}\right)^{m+1},$$

where we apply the *(R)-inequality* and we obtain that

$$\sum_{k=1}^{n} \frac{a_k}{d_k^m (2px - ya_k)^{m+1}} \geqslant \frac{1}{2^m} \cdot \frac{\left(\sum\limits_{k=1}^{n} \frac{a_k}{2px - ya_k}\right)^{m+1}}{\left(\sum\limits_{k=1}^{n} S_k\right)^m} = \frac{1}{2^m S^m}\left(\sum_{k=1}^{n} \frac{a_k}{2px - ya_k}\right)^{m+1}.$$

Also we have,

$$U_n = \sum_{k=1}^{n} \frac{a_k}{2px - ya_k} \Leftrightarrow yU_n = \sum_{k=1}^{n} \frac{ya_k}{2px - ya_k}$$
$$\Leftrightarrow yU_n + n = \sum_{k=1}^{n} \left(\frac{ya_k}{2px - ya_k} + 1 \right) = 2px \sum_{k=1}^{n} \frac{1}{2px - ya_k}.$$

By *Bergström's* inequality we deduce that:

$$\sum_{k=1}^{n} \frac{1}{2px - ya_k} \geqslant \frac{n^2}{\sum\limits_{k=1}^{n} (2px - ya_k)} = \frac{n^2}{2p(nx - y)}.$$

Therefore,

$$yU_n + n = 2px \sum_{k=1}^{n} \frac{1}{2px - ya_k} \geqslant 2px \cdot \frac{n^2}{2p(nx - y)} = \frac{n^2 x}{nx - y}$$
$$\Leftrightarrow yU_n \geqslant \frac{n^2 x}{nx - y} - n = \frac{ny}{nx - y} \Leftrightarrow U_n \geqslant \frac{n}{nx - y}.$$

Hence,

$$\sum_{k=1}^{n} \frac{a_k}{d_k^m (2px - ya_k)^{m+1}} \geqslant \frac{n^{m+1}}{2^m S^m (nx - y)^{m+1}} \quad \square$$

OBSERVATION 4.1. If the polygon from theorem 4 is circumscribed in the circle $C(I;r)$, then for any $m \in R_+$ and for any point M which is inside the polygon holds the inequality:

$$\sum_{k=1}^{n} \frac{a_k}{d_k^m (2px - ya_k)^{m+1}} \geqslant \frac{n^{m+1}}{2^m S^m (nx - y)^{m+1}} = \frac{n^{m+1}}{2^m p^m r^m (nx - y)^{m+1}} \tag{4'}$$

If $M \equiv I$, then

$$\sum_{k=1}^{n} \frac{a_k}{r^m (2px - ya_k)^{m+1}} \geqslant \frac{n^{m+1}}{2^m p^m r^m (nx - y)^{m+1}}$$
$$\Leftrightarrow p^m \sum_{k=1}^{n} \frac{a_k}{(2px - ya_k)^{m+1}} \geqslant \frac{n^{m+1}}{2^m (nx - y)^{m+1}} \tag{4''}$$

OBSERVATION 4.2. If the polygon from theorem 4 is the triangle ABC, then (4) becomes:

$$\frac{a}{d_a^m (2px - ay)^{m+1}} + \frac{b}{d_b^m (2px - by)^{m+1}} + \frac{c}{d_c^m (2px - cy)^{m+1}} \geqslant \frac{3^{m+1}}{2^m S^m (3x - y)^{m+1}} \tag{4'''}$$

Also $(4'')$ becomes:

$$p^m\left(\frac{a}{(2px-ya)^{m+1}}+\frac{b}{(2px-yb)^{m+1}}+\frac{c}{(2px-yc)^{m+1}}\right)\geqslant\frac{3^{m+1}}{2^m(3x-y)^{m+1}}.$$

If we take $m=0$, $x=y=1$, then by the last relation we obtain:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geqslant\frac{3}{2},$$

and this is *Nesbitt's inequality*.

THEOREM 5. *If a convex polyhedron has n faces ($n\geqslant 4$) which are convex polygons with their areas $S_k\left(k=\overline{1,n}\right)$, M is a point inside the polyhedron, d_k is the distance from M to the face with area $S_k\left(k=\overline{1,n}\right)$, V is the volume of the polyhedron, S is total area of the polyhedron and $m\in R_+$, then:*

$$\sum_{k=1}^{n}\frac{S_k}{d_k^m}\geqslant\frac{S^{m+1}}{3^mV^m} \tag{5}$$

Proof. We have:

$$\sum_{k=1}^{n}\frac{S_k}{d_k^m}=\sum_{k=1}^{n}\frac{S_k^{m+1}}{(d_kS_k)^m}=\sum_{k=1}^{n}\frac{S_k^{m+1}}{3^mV_k^m},$$

where V_k is the volume of the pyramid which has the vertex M and which has the base the polygon of the face with the area $S_k\left(k=\overline{1,n}\right)$.

By *(R)-inequality*, we deduce that:

$$\sum_{k=1}^{n}\frac{S_k}{d_k^m}\geqslant\frac{\left(\sum\limits_{k=1}^{n}S_k\right)^{m+1}}{3^m\left(\sum\limits_{k=1}^{n}V_k\right)^m}=\frac{S^{m+1}}{3^mV^m}\qquad\square$$

OBSERVATION 5.1. If the convex polyhedron from theorem 5 is circumscribed in the sphere $S(I;r)$, then (5) becomes:

$$\sum_{k=1}^{n}\frac{S_k}{d_k^m}\geqslant\frac{S^{m+1}}{(Sr)^m}=\frac{S}{r^m} \tag{5'}$$

If $M\equiv I$, then $d_k=r$, $\forall k=\overline{1,n}$, and the inequality $(5')$ becomes a equality.

THEOREM 6. *If we have a convex polyhedron like in theorem 5 and $m\in R_+$, $x,y\in R_+^*$, such that $xS\geqslant y\max\limits_{1\leqslant k\leqslant n}S_k$, then:*

$$\sum_{k=1}^{n}\frac{S_k}{d_k^m(xS-yS_k)^{m+1}}\geqslant\frac{n^{m+1}}{3^mV^m(nx-y)^{m+1}} \tag{6}$$

Proof. We have:

$$\sum_{k=1}^{n} \frac{S_k}{d_k^m (xS - yS_k)^{m+1}} = \sum_{k=1}^{n} \frac{S_k^{m+1}}{(d_k S_k)^m (xS - yS_k)^{m+1}} = \sum_{k=1}^{n} \frac{1}{3^m V_k^m} \left(\frac{S_k}{xS - yS_k} \right)^{m+1},$$

where we apply the *(R)-inequality* and we obtain that:

$$\sum_{k=1}^{n} \frac{S_k}{d_k^m (xS - yS_k)^{m+1}} \geqslant \frac{\left(\sum_{k=1}^{n} \frac{S_k}{xS - yS_k} \right)^{m+1}}{3^m \left(\sum_{k=1}^{n} V_k \right)^m} = \frac{1}{3^m V^m} \left(\sum_{k=1}^{n} \frac{S_k}{xS - yS_k} \right)^{m+1}.$$

Also we have,

$$W_n = \sum_{k=1}^{n} \frac{S_k}{xS - yS_k} \Leftrightarrow yW_n = \sum_{k=1}^{n} \frac{yS_k}{xS - yS_k}$$
$$\Leftrightarrow yW_n + n = \sum_{k=1}^{n} \left(\frac{yS_k}{xS - yS_k} + 1 \right) = xS \sum_{k=1}^{n} \frac{1}{xS - yS_k}.$$

By Bergström's inequality we deduce that:

$$\sum_{k=1}^{n} \frac{1}{xS - yS_k} \geqslant \frac{n^2}{\sum_{k=1}^{n} (xS - yS_k)} = \frac{n^2}{(nx - y) S}.$$

So,

$$yW_n + n \geqslant x \cdot \frac{n^2}{nx - y} \Leftrightarrow yW_n \geqslant \frac{xn^2}{nx - y} - n = \frac{ny}{nx - y} \Leftrightarrow W_n \geqslant \frac{n}{nx - y}.$$

Therefore,

$$\sum_{k=1}^{n} \frac{S_k}{d_k^m (xS - yS_k)^{m+1}} \geqslant \frac{n^{m+1}}{3^m V^m (nx - y)^{m+1}} \qquad \square$$

OBSERVATION 6.1. If convex polyhedron from theorem 6 is circumscribed in the sphere $S(I;r)$, then the relation (6) becomes:

$$\sum_{k=1}^{n} \frac{S_k}{d_k^m (xS - yS_k)^{m+1}} \geqslant \frac{n^{m+1}}{S^m r^m (nx - y)^{m+1}} \tag{6'}$$

If in addition $M \equiv I$, then:

$$\sum_{k=1}^{n} \frac{S_k}{(xS - yS_k)^{m+1}} \geqslant \frac{n^{m+1}}{S^m (nx - y)^{m+1}} \tag{6''}$$

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