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AN INEQUALITY ON JORDAN–VON NEUMANN CONSTANT AND JAMES CONSTANT ON $Z_{p,q}$ SPACE

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Abstract. Let $\lambda > 0$, $Z_{p,q}$ denote $\mathbb{R}^2$ endowed with the norm

$$|x|_{p,q} = (\|x\|_p^2 + \lambda \|x\|_q^2)^{\frac{1}{2}}.$$

Recently, James constant $J(Z_{p,q})$ and von Neumann-Jordan constant $C_{NJ}(Z_{p,q})$ have been investigated under the two cases of a space $2 \leqslant p \leqslant q \leqslant \infty$ and $1 \leqslant p \leqslant q \leqslant 2$. In this note, we show an inequality on these two constants under the case of $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$. As an application, we give a sufficient condition for the space $Z_{p,q}$ with uniform normal structure.

2. Introduction and preliminaries

Let X be a non-trivial Banach space, and B_X and S_X denote the unit ball and unit sphere of X, respectively. Many recent studies have focused on the von Neumann-Jordan (NJ) constant and James constant (cf. [1–18]). The constant

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S_X\}$$

is called the non-square or James constant of X. It is well known that([5-6])

(i) $\sqrt{2} \leqslant J(X) \leqslant 2$.

(ii) $J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in B_X\}$.

(iii) If $1 \leqslant p \leqslant \infty$ and $\dim L_p(\mu) \geqslant 2$, then $J(L_p(\mu)) = \max\{2^{1/p}, 2^{1-1/p}\}$.

The von Neumann-Jordan constant of a Banach space X was introduced by Clarkson [3] as the smallest constant C for which

$$\frac{1}{C} \leqslant \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leqslant C$$

holds for all $x, y \in X$ with $(x,y) \neq (0,0)$. An equivalent definition of the NJ constant is found in [8] as the following form:

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X\right\}.$$

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Now let us collect some properties of these constants in [1, 3, 8, 9, 16]:

(1) $C_{NJ}(X) = C_{NJ}(X^*)$.

(2) $1 \leqslant C_{NJ}(X) \leqslant 2$; X is a Hilbert space if and only if $C_{NJ}(X) = 1$.

(3) X is uniformly non-square if and only if $C_{NJ}(X) < 2$.

(4) For any non-trivial Banach space X,

$$\frac{J(X)^2}{2} \leqslant C_{NJ}(X) \leqslant J(X). \tag{1.1}$$

(5) If $1 \leqslant p \leqslant \infty$ and $\dim L_p(\mu) \geqslant 2$, then

$$C_{NJ}(L_p(\mu)) = \max\{2^{\frac{2}{p}-1}, 2^{1-\frac{2}{p}}\}.$$

(6) X has uniform normal structure if $C_{NJ}(X) < (3+\sqrt{5})/4$.

It can be recalled that a norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x,y)\| = \|(|x|,|y|)\|$ for arbitrary $(x,y) \in \mathbb{R}^2$, and to be normalized if $\|(1,0)\| = \|(0,1)\| = 1$.

Let $\lambda > 0$, and $Z_{p,q}$ denote $\mathbb{R}^2$ endowed with the norm

$$|x|_{p,q} = (\|x\|_p^2 + \lambda \|x\|_q^2)^{\frac{1}{2}},$$

then by the definition, it is clear that $|\cdot|_{p,q}$ is absolute and $\|\cdot\|_{p,q} =: \frac{|\cdot|_{p,q}}{\sqrt{1+\lambda}}$ is an absolute normalized norm.

Recently, we gave the exact values of James constant $J(Z_{p,q})$ and von Neumann-Jordan constant $C_{NJ}(Z_{p,q})$ under the cases of $2 \leqslant p \leqslant q \leqslant \infty$ and $1 \leqslant p \leqslant q \leqslant 2$ as follows.

(i) If $2 \leqslant p \leqslant q \leqslant \infty$, then

$$J(Z_{p,q}) = 2\sqrt{\frac{\lambda+1}{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}}, \qquad C_{NJ}(Z_{p,q}) = \frac{2(\lambda+1)}{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}.$$

(ii) If $1 \leqslant p \leqslant q \leqslant 2$, then

$$J(Z_{p,q}) = \sqrt{\frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{\lambda+1}}, \qquad C_{NJ}(Z_{p,q}) = \frac{2^{\frac{2}{p}} + \lambda 2^{\frac{2}{q}}}{2(\lambda+1)}.$$

For $2 \leqslant p \leqslant q \leqslant \infty$ and $1 \leqslant p \leqslant q \leqslant 2$, James constant $J(Z_{p,q})$ and von Neumann-Jordan constant $C_{NJ}(Z_{p,q})$ have been investigated. It is natural to ask whether the related question holds under the case of $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$. In this note, we consider these two constants under the case of $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$ and give an inequality, along with a consequence.

3. Main results and proofs

Now, we prove the following inequalities on $J(Z_{p,q})$ and $C_{NJ}(Z_{p,q})$.

THEOREM 2.1. *Let* $\lambda > 0, Z_{p,q} = \mathbb{R}^2$ *endowed with the norm*

$$|x|_{p,q} = (\|x\|_p^2 + \lambda \|x\|_q^2)^{\frac{1}{2}}.$$

If $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$, *then*

$$\max\left\{\frac{2(\lambda+1)}{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}, \frac{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}{2(\lambda+1)}\right\} \leqslant C_{NJ}(Z_{p,q}) \leqslant \frac{2^{\frac{2}{p}}+2\lambda}{2^{\frac{2}{q}}\lambda+2}. \tag{2.1}$$

$$\max\left\{2\sqrt{\frac{\lambda+1}{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}}, \sqrt{\frac{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}{\lambda+1}}\right\} \leqslant J(Z_{p,q}) \leqslant \sqrt{\frac{2(2^{\frac{2}{p}}+2\lambda)}{2^{\frac{2}{q}}\lambda+2}}. \tag{2.2}$$

Proof. (i) First, we show that (2.1) is valid.

Let $x, y \in Z_{p,q}$, by the value of $C_{NJ}(Z_{p,2})$ and $C_{NJ}(Z_{2,q})$ as above, we have

$$\begin{aligned}
&|x+y|_{p,q}^2 + |x-y|_{p,q}^2 \\
=\ & \|x+y\|_p^2 + \lambda\|x+y\|_q^2 + \|x-y\|_p^2 + \lambda\|x-y\|_q^2 \\
=\ & |x+y|_{p,2}^2 + |x-y|_{p,2}^2 + |x+y|_{2,q}^2 + |x-y|_{2,q}^2 - (1+\lambda)(\|x+y\|_2^2 + \|x-y\|_2^2) \\
\leqslant\ & \frac{2^{\frac{2}{p}}+2\lambda}{2(\lambda+1)} 2(|x|_{p,2}^2 + |y|_{p,2}^2) + \frac{2(1+\lambda)}{2+\lambda 2^{\frac{2}{q}}} 2(|x|_{2,q}^2 + |y|_{2,q}^2) - 2(1+\lambda)(\|x\|_2^2 + \|y\|_2^2) \\
=\ & \frac{2^{\frac{2}{p}}+2\lambda}{\lambda+1}(\|x\|_p^2 + \|y\|_p^2) + \frac{4(1+\lambda)}{2+\lambda 2^{\frac{2}{q}}}\lambda(\|x\|_q^2 + \|y\|_q^2) \\
&+ \left[\frac{\lambda(2\lambda+2^{\frac{2}{p}})}{1+\lambda} + \frac{4(1+\lambda)}{2+\lambda 2^{\frac{2}{q}}} - 2(1+\lambda)\right](\|x\|_2^2 + \|y\|_2^2).
\end{aligned}$$

Let

$$\alpha = \frac{2(2^{\frac{2}{p}}+2\lambda)}{2+\lambda 2^{\frac{2}{q}}} - \frac{2\lambda+2^{\frac{2}{p}}}{1+\lambda}, \qquad \beta = \frac{2^{\frac{2}{q}}(2^{\frac{2}{p}}-2)\lambda}{2^{\frac{2}{q}}\lambda+2},$$

we have

$$\begin{aligned}
\alpha+\beta &= \frac{2(2^{\frac{2}{p}}+2\lambda)}{2+\lambda 2^{\frac{2}{q}}} - \frac{2\lambda+2^{\frac{2}{p}}}{1+\lambda} + \frac{2^{\frac{2}{q}}(2^{\frac{2}{p}}-2)\lambda}{2^{\frac{2}{q}}\lambda+2} \\
&= 2^{\frac{2}{p}} + \frac{4\lambda - 2^{1+\frac{2}{q}}\lambda}{2+\lambda 2^{\frac{2}{q}}} - \frac{2\lambda+2^{\frac{2}{p}}}{1+\lambda} \\
&= \frac{\lambda(2\lambda+2^{\frac{2}{p}})}{1+\lambda} + \frac{4(1+\lambda)}{2+\lambda 2^{\frac{2}{q}}} - 2(1+\lambda).
\end{aligned}$$

Applying

$$\|x\|_2 \leqslant \|x\|_p,\ \|y\|_2 \leqslant \|y\|_p \quad \text{and} \quad \|x\|_2 \leqslant 2^{\frac{1}{2}-\frac{1}{q}}\|x\|_q,\ \|y\|_2 \leqslant 2^{\frac{1}{2}-\frac{1}{q}}|y\|_q,$$

we can obtain

$$\begin{aligned}
&|x+y|_{p,q}^2 + |x-y|_{p,q}^2 \\
\leqslant &\left[\frac{2^{\frac{2}{p}}+2\lambda}{\lambda+1}+\alpha\right](\|x\|_p^2+\|y\|_p^2)+\left[\frac{4(1+\lambda)\lambda}{2+\lambda 2^{\frac{2}{q}}}+\beta 2^{1-\frac{2}{q}}\right](\|x\|_q^2+\|y\|_q^2) \\
\leqslant &\, 2\frac{2^{\frac{2}{p}}+2\lambda}{2^{\frac{2}{q}}\lambda+2}(|x|_{p,q}^2+|y|_{p,q}^2).
\end{aligned}$$

So

$$C_{NJ}(Z_{p,q}) \leqslant \frac{2^{\frac{2}{p}}+2\lambda}{2^{\frac{2}{q}}\lambda+2},$$

and by (1.1), we also have

$$J(Z_{p,q}) \leqslant \sqrt{2C_{NJ}(Z_{p,q})} \leqslant \sqrt{\frac{2(2^{\frac{2}{p}}+2\lambda)}{2^{\frac{2}{q}}\lambda+2}}.$$

Now let $x_0=(a,a)$, and $y_0=(a,-a)$, where $a=\frac{1}{\sqrt{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}}$. Then $\|x_0\|_\lambda = \|y_0\|_\lambda = 1, x_0+y_0=(2a,0)$, and $x_0-y_0=(0,2a)$. Hence,

$$C_{NJ}(Z_{p,q}) \geqslant \frac{2(\lambda+1)}{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}.$$

and

$$J(Z_{p,q}) \geqslant 2\sqrt{\frac{\lambda+1}{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}}.$$

If let $x_1=(b,0)$, and $y_1=(0,-b)$, where $b=\frac{1}{\sqrt{\lambda+1}}$.
Then $\|x_1\|_\lambda=\|y_1\|_\lambda=1, x_1+y_1=(b,-b)$, and $x_1-y_1=(b,b)$.
We also have

$$C_{NJ}(Z_{p,q}) \geqslant \frac{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}{2(\lambda+1)},$$

and

$$J(Z_{p,q}) \geqslant \sqrt{\frac{2^{\frac{2}{p}}+\lambda 2^{\frac{2}{q}}}{\lambda+1}}$$

This completes the proof of Theorem 2.1. □

REMARK. We remark that the left inequality in (2.2) can also be obtained by the following equality: (see [17])

$$J(Z_{p,q}) = \max_{0\leqslant t\leqslant \frac{1}{2}} \frac{2-2t}{\psi_{p,q}(t)}\psi_{p,q}\Big(\frac{1}{2-2t}\Big),$$

where $\psi_{p,q}(t) = \|(1-t,t)\|_{p,q}$ for all $t\in[0,1]$. And the left inequality in (2.1) can also be proved by the results as follows: (see [18])

$$\max\{M_1^2, M_2^2\} \leqslant C_{NJ}(Z_{p,q}) \leqslant M_1^2 M_2^2,$$

where $M_1 = \max\limits_{0\leqslant t\leqslant 1} \frac{\psi_{p,q}(t)}{\psi_2(t)}, M_2 = \max\limits_{0\leqslant t\leqslant 1} \frac{\psi_2(t)}{\psi_{p,q}(t)}$ and $\psi_2(t) = \|(1-t,t)\|_2$ for all $t\in[0,1]$. As an application of Theorem 2.1, we give a sufficient condition for the space $Z_{p,q}$ with uniform normal structure. We have the following results.

THEOREM 2.2.

(1) *If* $\frac{\ln 2}{\ln 2-\ln(\sqrt{5}-1)} < p \leqslant 2 < \frac{\ln 2}{\ln(\sqrt{5}-1)} < q$, *and* $0<\lambda<\frac{2-(3-\sqrt{5})2^{\frac{2}{p}}}{6-2\sqrt{5}-2^{\frac{2}{q}}}$, *then* $Z_{p,q}$ *has uniform normal structure.*

(2) *If* $1 \leqslant p < \frac{\ln 2}{\ln 2-\ln(\sqrt{5}-1)} < 2 \leqslant q < \frac{\ln 2}{\ln(\sqrt{5}-1)}$, *and* $\lambda>\frac{2-(3-\sqrt{5})2^{\frac{2}{p}}}{6-2\sqrt{5}-2^{\frac{2}{q}}}$, *then* $Z_{p,q}$ *has uniform normal structure.*

Proof. (1) By use of $\frac{\ln 2}{\ln 2-\ln(\sqrt{5}-1)} < p$, we have $2^{\frac{1}{p}} < \frac{2}{\sqrt{5}-1}$, and $(3-\sqrt{5})2^{\frac{2}{p}} < 2$. On the other hand, applying $\frac{\ln 2}{\ln(\sqrt{5}-1)} < q$, we also have $6-2\sqrt{5} > 2^{\frac{2}{q}}$. Hence, by $\frac{6+2\sqrt{5}-4\cdot 2^{\frac{2}{p}}}{8-(3+\sqrt{5})2^{\frac{2}{q}}} = \frac{2-(3-\sqrt{5})2^{\frac{2}{p}}}{6-2\sqrt{5}-2^{\frac{2}{q}}}$, we have $0<\lambda<\frac{2-(3-\sqrt{5})2^{\frac{2}{p}}}{6-2\sqrt{5}-2^{\frac{2}{q}}}$ is equivalent to

$$\frac{2^{\frac{2}{p}}+2\lambda}{2^{\frac{2}{q}}\lambda+2} < \frac{3+\sqrt{5}}{4}.$$

By Theorem 2.1, we obtain that $C_{NJ}(Z_{p,q}) < \frac{3+\sqrt{5}}{4}$. Therefore, $Z_{p,q}$ has uniform normal structure.

(2) The proof of this part is similar to (1), so we omit it here. □

REFERENCES

[1] J. ALOSNSO, P. MATRÍN AND P. L. PAPINI, *Wheeling around von Neumann-Jordan constant in Banach spaces*, Studia Math. **188** (2008), 135–152.

[2] M. BARONTI. E. CASINI AND P. L. PAPINI, *Trangles inscribed in a semicricle in Minkowski planes and in normed spaces*, J. Math. Anal. Appl. **252** (2000), 124–146.

[3] J. A. CLARKSON, *The von Neumann-Jordan constant for the Lebesgue space*, Ann. of Math. **38** (1937), 114–115.

[4] S. DHOMPONGSA, P. PIRAISANGJUN AND S. SAEJUNG, *Generalized Jordan-von Neumann constants and uniform normal structure*, Bull. Austral. Math. Soc. **67** (2003), 225–240.

[5] J. GAO AND K. S. LAU, *On the geometry of spheres in normed linear spaces*, J. Austral. Math. Soc. Ser.A **48** (1990), 101–112.

[6] J. GAO AND K. S. LAU, *On two classes of Banach spaces with uniform normal structure*, Studia Math. **99** (1991), 41–56.

[7] P. JORDAN AND J. VON NEUMANN, *On inner products in linear metric spaces* , Ann. of Math. **36** (1935), 719–723.

[8] M. KATO, L. MALIGRANDA AND Y. TAKAHASHI, *On James and Jordan-von Neumann constants and the normal structure coefficient of Banach spaces*, Studia Math. **144** (2001), 275–292.

[9] M. KATO AND Y. TAKAHASHI, *Von Neumann-Jordan constant for Lebesgue-Bochner spaces* , J. Inequal. Appl. **2** (1998), 89–97.

[10] M. KATO AND L. MALIGRANDA, *On James and Jordan-von Neumann constants of Lorentz sequence spaces*, J. Math. Anal. Appl. **258** (2001), 457–465.

[11] L. MALIGRANDA, L. NIKOLOVA, L. E. PERSSON AND T. ZACHARIADES, *On n-th James and Khintchine constants of Banach spaces*, Math.Inequalities and Appl. **11** (2008), 1–22.

[12] S. SAEJUNG, *On James and von Neumann-Jordan constants and sufficient conditions for the fixed point property*, J. Math. Anal. Appl. **323** (2006), 1018–1024.

[13] F. WANG AND B. PANG, *Some inequalities concerning the James constant in Banach spaces*, J. Math. Anal. Appl. **353**, 1 (2009), 305–310.

[14] C. YANG AND F. WANG, *On a new geometric constant related to the von Neumann-Jordan constant*, J. Math. Anal. Appl. **324** (2006), 555–565.

[15] C. YANG, *A note of Jordan-von Neumann constant and James constant*, J. Math. Anal. Appl. **357**, 1 (2009), 98–102.

[16] C. YANG AND H. LI, *An inequality between Jordan-von Neumann constant and James constant*, Appl. Math. Lett. **23**, 3 (2010), 277–281.

[17] K. MITANI AND K.-S. SATIO, *The James constant of absolute norms on* $\mathbb{R}^2$, J. Nonlinear Convex Analysis **4** (2003), 399–410.

[18] K.-S. SATIO, M. KATO AND Y. TAKAHASHI, *Von Neumann-Jordan constant of absolute normalized norms on* $\mathbb{C}^2$, J. Math. Anal. Appl. **244**, 2 (2000), 515–532.

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