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CHEBYSHEV TYPE INEQUALITIES FOR THE SAIGO FRACTIONAL INTEGRALS AND THEIR q–ANALOGUES

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Abstract. The aim of the present paper is to obtain certain new integral inequalities involving the Saigo fractional integral operator. It is also shown how the various inequalities considered in this paper admit themselves of q-extensions which are capable of yielding various results in the theory of q-integral inequalities.

1. Introduction

Our work in the present paper is based on a celebrated functional introduced by Chebyshev [4], which is defined by

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1.1}$$

where f and g are two integrable functions which are synchronous on $[a,b]$, i.e.

$$\{(f(x)-f(y))\,(g(x)-g(y))\}\geqslant 0, \tag{1.2}$$

for any $x,y\in[a,b]$.

The functional (1.1) has applications in numerical quadrature, transform theory, probability and in statistical problems. Motivated by these applications, researchers have used the functional (1.1) in the theory of fractional integral inequalities (see [3], [5] and [7]). Recently, Belarbi and Dahmani [3], Dahmani *et al.* [5], and Kalla and Rao [7] established certain integral inequalities by using known fractional integral operators. Also, Öğünmez and Özkan [9] derived certain integral inequalities involving the fractional q-integral operators.

The object of the present investigation is to obtain certain Chebyshev type integral inequalities involving the Saigo fractional integral operators ([12]). Further, we consider the q-extensions of the main results, and point out also their relevances with other related results.

Before stating the fractional integral inequalities, we mention below the definitions and notations of some well-known operators of fractional calculus.

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DEFINITION 1. A real-valued function $f(t)$ $(t > 0)$ is said to be in the space $\mathbb{C}_\mu$ $(\mu \in \mathbb{R})$, if there exists a real number $p > \mu$ such that $f(t) = t^p \phi(t)$; where $\phi(t) \in \mathbb{C}(0,\infty)$.

DEFINITION 2. Let $\alpha > 0$, $\beta, \eta \in \mathbb{R}$, then the Saigo fractional integral $I_{0,t}^{\alpha,\beta,\eta}$ of order α for a real-valued continuous function $f(t)$ is defined by ([12], see also [8, p. 19], [11]):

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right) f(\tau)d\tau, \tag{1.3}$$

where, the function ${}_2F_1(-)$ in the right-hand side of (1.3) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \tag{1.4}$$

and $(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1.$$

The integral operator $I_{0,t}^{\alpha,\beta,\eta}$ includes both the Riemann-Liouville and the Erdélyi-Kober fractional integral operators given by the following relationships:

$$R^\alpha\{f(t)\} = I_{0,t}^{\alpha,-\alpha,\eta}\{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau \quad (\alpha > 0) \tag{1.5}$$

and

$$I^{\alpha,\eta}\{f(t)\} = I_{0,t}^{\alpha,0,\eta}\{f(t)\} = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\,\tau^\eta f(\tau)d\tau \tag{1.6}$$

$$(\alpha > 0, \quad \eta \in \mathbb{R}).$$

For $f(t) = t^\mu$ in (1.3), we get the known formula [12]:

$$I_{0,t}^{\alpha,\beta,\eta}\{t^\mu\} = \frac{\Gamma(\mu+1)\Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta)\Gamma(\mu+1+\alpha+\eta)} t^{\mu-\beta}, \tag{1.7}$$

$$(\alpha > 0, \quad \min(\mu, \mu-\beta+\eta) > -1, \quad t > 0)$$

which shall be used in the sequel.

2. Fractional integral inequalities

In this section, we establish Chebyshev type integral inequalities for the synchronous functions involving the Saigo fractional integral operator (1.3).

THEOREM 1. *Let f and g be two synchronous functions on $[0,\infty)$, then*

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)\} \geqslant \frac{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\,t^{\beta}}{\Gamma(1-\beta+\eta)}\, I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}\; I_{0,t}^{\alpha,\beta,\eta}\{g(t)\}, \tag{2.1}$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\beta<1$, $\beta-1<\eta<0$.

Proof. By hypothesis, the functions f and g are synchronous functions on $[0,\infty)$, therefore, for all τ, $\rho\geqslant 0$, we have

$$\{(f(\tau)-f(\rho))\,(g(\tau)-g(\rho))\}\geqslant 0, \tag{2.2}$$

which implies that

$$f(\tau)g(\tau)+f(\rho)g(\rho)\geqslant f(\tau)g(\rho)+f(\rho)g(\tau). \tag{2.3}$$

Consider

$$\begin{aligned} F(t,\tau) &= \frac{t^{-\alpha-\beta}\,(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right) \qquad (\tau\in(0,t);\ t>0) \qquad (2.4)\\ &= \frac{1}{\Gamma(\alpha)}\,\frac{(t-\tau)^{\alpha-1}}{t^{\alpha+\beta}}+\frac{(\alpha+\beta)(-\eta)}{\Gamma(\alpha+1)}\,\frac{(t-\tau)^{\alpha}}{t^{\alpha+\beta+1}}\\ &\quad+\frac{(\alpha+\beta)(\alpha+\beta+1)(-\eta)(-\eta+1)}{\Gamma(\alpha+2)}\,\frac{(t-\tau)^{\alpha+1}}{t^{\alpha+\beta+2}}+\cdots. \end{aligned}$$

We observe that the function $F(t,\tau)$ remains positive, for all $\tau\in(0,t)$ $(t>0)$ since each term of the above series is positive in view of the conditions stated with Theorem 1.

Multiplying both sides of (2.3) by $F(t,\tau)$ (defined above by (2.4)) and integrating with respect to τ from 0 to t, and using (1.3), we get

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)\}+f(\rho)g(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{1\}\geqslant g(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}+f(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{g(t)\}. \tag{2.5}$$

Next, multiplying both sides of (2.5) by $F(t,\rho)$ $(\rho\in(0,t),\ t>0)$, where $F(t,\rho)$ is given by (2.4), and integrating with respect to ρ from 0 to t, and using formula (1.7), we arrive at the desired result (2.1). □

THEOREM 2. *Let f and g be two synchronous functions on $[0,\infty)$, then*

$$\begin{aligned} &\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\,t^{\beta}}\,I_{0,t}^{\gamma,\delta,\zeta}\{f(t)g(t)\}+\frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)\,t^{\delta}}\,I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)\}\\ &\qquad\geqslant I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}\;I_{0,t}^{\gamma,\delta,\zeta}\{g(t)\}+I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\}\;I_{0,t}^{\alpha,\beta,\eta}\{g(t)\}, \qquad (2.6)\end{aligned}$$

for all $t>0,\ \alpha>\max\{0,-\beta\},\ \gamma>\max\{0,-\delta\},\ \beta,\ \delta<1,\ \beta-1<\eta<0,\ \delta-1<\zeta<0$.

Proof. Multiplying both sides of (2.5) by

$$\frac{t^{-\gamma-\delta}(t-\rho)^{\gamma-1}}{\Gamma(\gamma)}\,{}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)\qquad(\rho\in(0,t);\ t>0),$$

which (in view of the arguments mentioned above in the proof of Theorem 1) remains positive under the conditions stated with Theorem 2. Integrating the resulting inequality so obtained with respect to ρ from 0 to t, we get

$$\begin{aligned}&I_{0,t}^{\gamma,\delta,\zeta}\{1\}I_{0,t}^{\alpha,\beta,\eta}\{f(t)g(t)\}+I_{0,t}^{\alpha,\beta,\eta}\{1\}I_{0,t}^{\gamma,\delta,\zeta}\{f(t)g(t)\}\\&\geqslant I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}\ I_{0,t}^{\gamma,\delta,\zeta}\{g(t)\}+I_{0,t}^{\gamma,\delta,\zeta}\{f(t)\}\ I_{0,t}^{\alpha,\beta,\eta}\{g(t)\},\end{aligned}$$

which on using (1.7) readily yields the desired result (2.6). □

REMARK 1. It may be noted that the inequalities (2.1) and (2.6) are reversed if the functions are asynchronous on $[0,\infty)$, i.e.

$$\{(f(x)-f(y))(g(x)-g(y))\}\leqslant 0, \tag{2.7}$$

for any $x,y\in[0,\infty)$.

REMARK 2. For $\alpha=\gamma,\ \beta=\delta,\ \eta=\zeta$, Theorem 2 immediately reduces to Theorem 1.

THEOREM 3. *Let* $(f_i)_{i=1,\cdots,n}$ *be* n *positive increasing functions on* $[0,\infty)$, *then*

$$I_{0,t}^{\alpha,\beta,\eta}\left\{\prod_{i=1}^{n}f_i(t)\right\}\geqslant\left[\frac{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\,t^{\beta}}{\Gamma(1-\beta+\eta)}\right]^{n-1}\prod_{i=1}^{n}I_{0,t}^{\alpha,\beta,\eta}\{f_i(t)\}, \tag{2.8}$$

for all $t>0,\ \alpha>\max\{0,-\beta\},\ \beta<1,\ \beta-1<\eta<0$.

Proof. We prove this theorem by induction. Clearly, for $n=1$ in (2.8), we have

$$I_{0,t}^{\alpha,\beta,\eta}\{f_1(t)\}\geqslant I_{0,t}^{\alpha,\beta,\eta}\{f_1(t)\}\qquad(t>0,\ \ \alpha>0).$$

Next, for $n=2$, in (2.8), we get

$$I_{0,t}^{\alpha,\beta,\eta}\{f_1(t)f_2(t)\}\geqslant\frac{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\,t^{\beta}}{\Gamma(1-\beta+\eta)}\,I_{0,t}^{\alpha,\beta,\eta}\{f_1(t)\}\ I_{0,t}^{\alpha,\beta,\eta}\{f_2(t)\}$$

$$(t>0,\ \ \alpha>0),$$

which holds in view of (2.1) of Theorem 1.

By the induction principle, we suppose that the inequality

$$I_{0,t}^{\alpha,\beta,\eta}\left\{\prod_{i=1}^{n-1}f_i(t)\right\}\geqslant\left[\frac{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\,t^{\beta}}{\Gamma(1-\beta+\eta)}\right]^{n-2}\prod_{i=1}^{n-1}I_{0,t}^{\alpha,\beta,\eta}\{f_i(t)\}, \tag{2.9}$$

holds true for some positive integer $n \geqslant 2$.

Now $(f_i)_{i=1,\cdots,n}$ are increasing functions implies that the function $\prod_{i=1}^{n-1} f_i(t)$ is also an increasing function. Therefore, we can apply inequality (2.1) of Theorem 1 to the functions $\prod_{i=1}^{n-1} f_i(t) = g$ and $f_n = f$ to get

$$I_{0,t}^{\alpha,\beta,\eta}\left\{\prod_{i=1}^{n} f_i(t)\right\} \geqslant \frac{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\, t^{\beta}}{\Gamma(1-\beta+\eta)}\, I_{0,t}^{\alpha,\beta,\eta}\left\{\prod_{i=1}^{n-1} f_i(t)\right\} I_{0,t}^{\alpha,\beta,\eta}\left\{f_n(t)\right\},$$

provided that $t > 0,\ \alpha > \max\{0, -\beta\},\ \beta < 1,\ \beta - 1 < \eta < 0$.

Making use of (2.9) now, this last inequality above leads to the result (2.8), which proves Theorem 3. $\square$

By setting $\beta = 0$ (and $\delta = 0$ additionally for Theorem 2), and using the relation (1.6), Theorems 1 to 3 yield the following integral inequalities involving the Erdélyi-Kober type fractional integral operator defined by (1.6):

COROLLARY 1. *Let f and g be two synchronous functions on $[0,\infty)$, then*

$$I^{\alpha,\eta}\left\{f(t)g(t)\right\} \geqslant \frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\, I^{\alpha,\eta}\left\{f(t)\right\} I^{\alpha,\eta}\left\{g(t)\right\}, \tag{2.10}$$

for all $t > 0,\ \alpha > 0,\ -1 < \eta < 0$.

COROLLARY 2. *Let f and g be two synchronous on $[0,\infty)$, then*

$$\begin{aligned} &\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha+\eta)}\, I^{\gamma,\zeta}\left\{f(t)g(t)\right\} + \frac{\Gamma(1+\zeta)}{\Gamma(1+\gamma+\zeta)}\, I^{\alpha,\eta}\left\{f(t)g(t)\right\} \\ &\qquad \geqslant I^{\alpha,\eta}\left\{f(t)\right\} I^{\gamma,\zeta}\left\{g(t)\right\} + I^{\gamma,\zeta}\left\{f(t)\right\} I^{\alpha,\eta}\left\{g(t)\right\}, \end{aligned} \tag{2.11}$$

for all $t > 0,\ \alpha,\ \gamma > 0,\ -1 < max(\eta,\zeta) < 0$.

COROLLARY 3. *Let $(f_i)_{i=1,\cdots,n}$ be n positive increasing functions on $[0,\infty)$, then*

$$I^{\alpha,\eta}\left\{\prod_{i=1}^{n} f_i(t)\right\} \geqslant \left[\frac{\Gamma(1+\alpha+\eta)}{\Gamma(1+\eta)}\right]^{n-1} \prod_{i=1}^{n} I^{\alpha,\eta}\left\{f_i(t)\right\}, \tag{2.12}$$

for all $t > 0,\ \alpha > 0,\ -1 < \eta < 0$.

Next, if we replace β by $-\alpha$ (and δ by $-\gamma$ additionally for Theorem 2), and make use of the relation (1.5), then Theorems 1 to 3 corresponds to the known results due to Belarbi and Dahmani [3, pp. 2–4, Theorems 3.1 to 3.3].

3. Fractional q-integral inequalities

In this section, we establish some fractional q-integral inequalities which may be regarded as q-extensions of the results derived in the previous section. For the convenience of the reader, we deem it proper to give here basic definitions and related details of the q-calculus.

The q-shifted factorial is defined for $\alpha, q \in \mathbb{C}$ as a product of n factors by

$$(\alpha;q)_n = \begin{cases} 1 & ; \quad n=0 \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}) & ; \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha};q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)} \quad (n>0), \tag{3.2}$$

where the q-gamma function is defined by ([6, p. 16, eqn. (1.10.1)])

$$\Gamma_q(t) = \frac{(q;q)_\infty (1-q)^{1-t}}{(q^t;q)_\infty} \quad (0<q<1). \tag{3.3}$$

We note that

$$\Gamma_q(1+t) = \frac{(1-q^t)\,\Gamma_q(t)}{1-q}, \tag{3.4}$$

and if $|q|<1$, the definition (3.1) remains meaningful for $n=\infty$, as a convergent infinite product given by

$$(\alpha;q)_\infty = \prod_{j=0}^{\infty}(1-\alpha q^j). \tag{3.5}$$

Also, the q-binomial expansion is given by

$$(x-y)_\nu = x^\nu(-y/x;q)_\nu = x^\nu \prod_{n=0}^{\infty}\left[\frac{1-(y/x)q^n}{1-(y/x)q^{\nu+n}}\right]. \tag{3.6}$$

Let $t_0 \in \mathbb{R}$, then we define a specific time scale

$$\mathbb{T}_{t_0} = \{t; t=t_0 q^n, n \text{ a nonnegative integer}\} \cup \{0\}, \quad 0<q<1, \tag{3.7}$$

and for convenience sake, we denote $\mathbb{T}_{t_0}$ by $\mathbb{T}$ throughout this paper.

The Jackson's q-derivative and q-integral of a function f defined on $\mathbb{T}$ are, respectively, given by (see [6, pp. 19, 22])

$$D_{q,t}f(t) = \frac{f(t)-f(tq)}{t(1-q)} \qquad (t \neq 0, \;\; q \neq 1) \tag{3.8}$$

and

$$\int_0^t f(\tau)\,d_q\tau = t\,(1-q)\sum_{k=0}^{\infty} q^k f(tq^k). \tag{3.9}$$

DEFINITION 3. The Riemann-Liouville fractional q-integral operator of a function $f(t)$ of order α (due to Agarwal [1]) is given by

$$I_q^{\alpha}\{f(t)\} = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^t (q\tau/t;q)_{\alpha-1} f(\tau)\,d_q\tau \qquad (\alpha>0, \;\; 0<q<1), \tag{3.10}$$

where

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \quad (\alpha \in \mathbb{R}). \tag{3.11}$$

DEFINITION 4. For $\alpha > 0$, $\eta \in \mathbb{R}$ and $0 < q < 1$, the basic analogue of the Kober fractional integral operator (cf. [2]) is given by

$$I_q^{\alpha,\eta}\{f(t)\} = \frac{t^{-\eta-1}}{\Gamma_q(\alpha)} \int_0^t (q\tau/t;q)_{\alpha-1} \tau^\eta f(\tau)\, d_q\tau. \tag{3.12}$$

DEFINITION 5. For $\alpha > 0$, $\beta \in \mathbb{R}$, a basic analogue of the Saigo's fractional integral operator (introduced by Purohit and Yadav [10]) is given for $|\tau/t| < 1$ by

$$I_q^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)} \times \int_0^t (q\tau/t;q)_{\alpha-1}\, \mathscr{T}_{q,\frac{q^{\alpha+1}\tau}{t}} \left({}_2\Phi_1\left[q^{\alpha+\beta},\, q^{-\eta}; q^\alpha; q, q\right]\right) f(\tau)\, d_q\tau, \tag{3.13}$$

where η is any non-negative integer, and the function ${}_2\Phi_1(-)$ (see [6]) and the q-translation operator occurring in the right-hand side of (3.13) are, respectively, defined by

$${}_2\Phi_1[a,b;c;q,t] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} t^n \qquad (|q| < 1,\ \ |t| < 1) \tag{3.14}$$

and

$$\mathscr{T}_{q,\tau}(f(t)) = \sum_{n=-\infty}^{+\infty} A_n t^n (\tau/t;q)_n, \tag{3.15}$$

where $(A_n)_{n\in\mathbb{Z}}$ $(\mathbb{Z} = 0, \pm 1, \pm 2, \cdots)$ is any bounded sequence of real or complex numbers.

Following [10], when $f(t) = t^\mu$, we obtain

$$I_q^{\alpha,\beta,\eta}\{t^\mu\} = \frac{\Gamma_q(\mu+1)\Gamma_q(\mu+1-\beta+\eta)}{\Gamma_q(\mu+1-\beta)\Gamma_q(\mu+1+\alpha+\eta)} t^{\mu-\beta}, \tag{3.16}$$

$$(0 < q < 1, \quad \min(\mu, \mu-\beta+\eta) > -1, \quad t > 0).$$

We now state and prove the q-integral inequalities which can be treated as the q-analogues of the inequalities (2.1), (2.6) and (2.8).

THEOREM 4. *Let f and g be two synchronous functions on $\mathbb{T}$, then*

$$I_q^{\alpha,\beta,\eta}\{f(t)g(t)\} \geqslant \frac{\Gamma_q(1-\beta)\Gamma_q(1+\alpha+\eta)\, t^\beta}{\Gamma_q(1-\beta+\eta)} I_q^{\alpha,\beta,\eta}\{f(t)\}\, I_q^{\alpha,\beta,\eta}\{g(t)\}, \tag{3.17}$$

for all $t > 0$, $0 < q < 1$, $\alpha > \max\{0, -\beta\}$, $\beta < 1$, $\eta - \beta > -1$.

Proof. Since the functions f and g are synchronous functions on $\mathbb{T}$ for all τ, $\rho \geqslant 0$, therefore the inequality (1.2) is satisfied. Now, on multiplying both sides of (1.2) (or, equivalently (2.3)) by

$$\frac{t^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\,(q\tau/t;q)_{\alpha-1}\,\mathscr{T}_{q,\frac{q^{\alpha+1}\tau}{t}}\left({}_2\Phi_1\left[q^{\alpha+\beta},\,q^{-\eta};q^{\alpha};q,q\right]\right),\tag{3.18}$$

$$(\tau\in(0,t),\;\; t>0),$$

and taking q-integration with respect to τ from 0 to t, then on using Definition 5, we get

$$I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}+f(\rho)g(\rho)\,I_q^{\alpha,\beta,\eta}\{1\}\geqslant g(\rho)\,I_q^{\alpha,\beta,\eta}\{f(t)\}+f(\rho)\,I_q^{\alpha,\beta,\eta}\{g(t)\}.\tag{3.19}$$

It may be observed that the function (3.18) remains positive for all values of $\tau\in(0,t)$ $(t>0)$ and under the conditions imposed with Theorem 4.

Next, multiplying both sides of (3.19) by

$$\frac{t^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}\,(q\rho/t;q)_{\alpha-1}\,\mathscr{T}_{q,\frac{q^{\alpha+1}\rho}{t}}\left({}_2\Phi_1\left[q^{\alpha+\beta},\,q^{-\eta};q^{\alpha};q,q\right]\right)\tag{3.20}$$

$$(\rho\in(0,t),\;\; t>0),$$

and noting that the function (3.20) is also positive for all $\rho\in(0,t)$ $(t>0)$ and under the conditions imposed with Theorem 4, we perform q-integration in the resulting inequality with respect to ρ from 0 to t, using the formula (3.16), the desired result (3.17) is thus easily arrived at. □

THEOREM 5. *Let f and g be two synchronous functions on $\mathbb{T}$, then*

$$\begin{aligned}\frac{\Gamma_q(1-\beta+\eta)}{\Gamma_q(1-\beta)\Gamma_q(1+\alpha+\eta)\,t^{\beta}}\,&I_q^{\gamma,\delta,\zeta}\{f(t)g(t)\}\\&+\frac{\Gamma_q(1-\delta+\zeta)}{\Gamma_q(1-\delta)\Gamma_q(1+\gamma+\zeta)\,t^{\delta}}\,I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}\\&\geqslant I_q^{\alpha,\beta,\eta}\{f(t)\}\;I_q^{\gamma,\delta,\zeta}\{g(t)\}+I_q^{\gamma,\delta,\zeta}\{f(t)\}\;I_q^{\alpha,\beta,\eta}\{g(t)\},\end{aligned}\tag{3.21}$$

for all $t>0,\; 0<q<1,\; \alpha>\max\{0,-\beta\},\; \gamma>\max\{0,-\delta\},\; \beta,\; \delta<1,\; \eta-\beta,\; \zeta-\delta>-1.$

Proof. To prove the above theorem, we start with the inequality (3.19). On multiplying both sides of the inequality (3.19) by

$$\frac{t^{-\delta-1}q^{-\zeta(\gamma+\delta)}}{\Gamma_q(\gamma)}\,(q\rho/t;q)_{\gamma-1}\,\mathscr{T}_{q,\frac{q^{\gamma+1}\rho}{t}}\left({}_2\Phi_1\left[q^{\gamma+\delta},\,q^{-\zeta};q^{\gamma};q,q\right]\right),\tag{3.22}$$

$$(\rho\in(0,t),\;\; t>0),$$

and taking basic integration with respect to ρ from 0 to t, we get

$$\begin{aligned}I_q^{\gamma,\delta,\zeta}(1)\,I_q^{\alpha,\beta,\eta}\{f(t)g(t)\}+I_q^{\alpha,\beta,\eta}(1)\,I_q^{\gamma,\delta,\zeta}\{f(t)g(t)\}\\ \geqslant I_q^{\alpha,\beta,\eta}f(t)\;I_q^{\gamma,\delta,\zeta}g(t)+I_q^{\gamma,\delta,\zeta}f(t)\;I_q^{\alpha,\beta,\eta}g(t),\end{aligned}$$

which yields the desired result by taking into account (3.16). □

REMARK 3. The inequalities (3.17) and (3.21) are reversed if the functions are asynchronous on $\mathbb{T}$.

REMARK 4. Evidently, when $\alpha=\gamma,\ \beta=\delta,\ \eta=\zeta$, then Theorem 5 leads to Theorem 4.

THEOREM 6. *Let* $(f_i)_{i=1,\cdots,n}$ *be* n *positive increasing functions on* $\mathbb{T}$*, then*

$$I_q^{\alpha,\beta,\eta}\left\{\prod_{i=1}^{n} f_i(t)\right\} \geqslant \left[\frac{\Gamma_q(1-\beta)\Gamma_q(1+\alpha+\eta)\,t^{\beta}}{\Gamma_q(1-\beta+\eta)}\right]^{n-1} \prod_{i=1}^{n} I_q^{\alpha,\beta,\eta}\left\{f_i(t)\right\}, \tag{3.23}$$

for all $t>0,\ 0<q<1,\ \alpha>\max\{0,-\beta\},\ \beta<1,\ \eta-\beta>-1.$

Proof. By applying the induction method and Theorem 4, one can easily establish the above theorem. Therefore, we omit the further details of the proof of this theorem. □

We now, briefly consider some consequences of the theorems derived in this section. If we set $\beta=0$ (and additionally $\delta=0$ for Theorem 5), and make use of the known result [10, p. 38, eqn. (3.7)], namely

$$I_q^{\alpha,0,\eta}\left\{f(t)\right\} = I_q^{\alpha,\eta}\left\{f(t)\right\}, \tag{3.24}$$

(with suitable changes for the parameters in Theorem 5) then Theorems 4 to 6 yield the following q-integral inequalities involving Erdélyi-Kober type fractional integral operators:

COROLLARY 4. *Let* f *and* g *be two synchronous functions on* $\mathbb{T}$*, then*

$$I_q^{\alpha,\eta}\left\{f(t)g(t)\right\} \geqslant \frac{\Gamma_q(1+\alpha+\eta)}{\Gamma_q(1+\eta)}\, I_q^{\alpha,\eta}\left\{f(t)\right\}\, I_q^{\alpha,\eta}\left\{g(t)\right\}, \tag{3.25}$$

for all $t>0,\ 0<q<1,\ \alpha>0$ *and* η *is any non-negative integer.*

COROLLARY 5. *Let* f *and* g *be two synchronous functions on* $\mathbb{T}$*, then*

$$\begin{aligned}\frac{\Gamma_q(1+\eta)}{\Gamma_q(1+\alpha+\eta)}\, I_q^{\gamma,\zeta}\left\{f(t)g(t)\right\} + \frac{\Gamma_q(1+\zeta)}{\Gamma_q(1+\gamma+\zeta)}\, I_q^{\alpha,\eta}\left\{f(t)g(t)\right\} \\ \geqslant I_q^{\alpha,\eta}\left\{f(t)\right\}\, I_q^{\gamma,\zeta}\left\{g(t)\right\} + I_q^{\gamma,\zeta}\left\{f(t)\right\}\, I_q^{\alpha,\eta}\left\{g(t)\right\},\end{aligned} \tag{3.26}$$

where $t>0,\ 0<q<1,\ \alpha,\ \gamma>0,\ \eta,\ \zeta$ *are any non-negative integers.*

COROLLARY 6. *Let* $(f_i)_{i=1,\cdots,n}$ *be* n *positive increasing functions on* $\mathbb{T}$*, then*

$$I_q^{\alpha,\eta}\left\{\prod_{i=1}^{n} f_i(t)\right\} \geqslant \left[\frac{\Gamma_q(1+\alpha+\eta)}{\Gamma_q(1+\eta)}\right]^{n-1} \prod_{i=1}^{n} I_q^{\alpha,\eta} f_i(t), \tag{3.27}$$

where $t > 0,\ 0 < q < 1,\ \alpha > 0$ *and* η *is any non-negative integer.*

We observe that, if we replace β by $-\alpha$ and δ by $-\gamma$, and make use of the relation [10, p. 38, eqn. (3.8)], and note the following relations:

$$I_q^{\alpha,-\alpha,\eta}\{f(t)\} = I_q^{\alpha}\{f(t)\} \tag{3.28}$$

and

$$I_q^{\gamma,-\gamma,\zeta}\{f(t)\} = I_q^{\gamma}\{f(t)\}, \tag{3.29}$$

then, Theorems 4 to 6 reduce to the known q-integral inequalities due to Öğünmez and Özkan [9, pp. 4–6, Theorems 3.1 to 3.3], involving the Riemann-Liouville type of fractional q-integral operator.

Finally, it is interesting to observe that, if we let $q \to 1^-$, and use the limit formulas:

$$\operatorname*{Lim}_{q\to 1^-} \frac{(q^{\alpha};q)_n}{(1-q)^n} = (\alpha)_n \tag{3.29}$$

and

$$\operatorname*{Lim}_{q\to 1^-} \Gamma_q(\alpha) = \Gamma(\alpha), \tag{3.30}$$

the results of Section 3 then correspond to the results obtained in Section 2.

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