Journal of Mathematical Inequalities
Volume 7, Number 2 (2013), 255–269 doi:10.7153/jmi-07-24

ON GENERALIZATIONS OF SOME CLASSICAL INTEGRAL INEQUALITIES

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(*Communicated by A. Vukelić*)

Abstract. A unified treatment for generalizations of the midpoint, trapezoid, averaged midpoint-trapezoid and Simpson type inequalities is obtained. Various error bounds for these generalizations are established.

1. Introduction and preliminary results

Let $f:[a,b]\to\mathbf{R}$ be such that the $(n-1)$th derivative $f^{(n-1)}$ $(n\geqslant 1)$ is absolutely continuous on $[a,b]$. Then we have the identities (see e. g., [2], [3], [4], [8], [9], [16]):

$$\int_a^b f(x)\,dx=(b-a)f\left(\frac{a+b}{2}\right)+\sum_{k=1}^{[\frac{n-1}{2}]}\frac{(b-a)^{2k+1}}{(2k+1)!2^{2k}}f^{(2k)}\left(\frac{a+b}{2}\right)+(-1)^n\int_a^b M_n(x)\,df^{(n-1)}(x), \tag{1}$$

where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$ and

$$M_n(x):=\begin{cases}\dfrac{(x-a)^n}{n!}, & \text{if } x\in\left[a,\dfrac{a+b}{2}\right],\\[2mm] \dfrac{(x-b)^n}{n!}, & \text{if } x\in\left(\dfrac{a+b}{2},b\right];\end{cases} \tag{2}$$

$$\int_a^b f(x)\,dx=\frac{b-a}{2}[f(a)+f(b)]-\sum_{k=1}^{[\frac{n-1}{2}]}\frac{k(b-a)^{2k+1}}{(2k+1)!2^{2k-1}}f^{(2k)}\left(\frac{a+b}{2}\right)+(-1)^n\int_a^b T_n(x)\,df^{(n-1)}(x), \tag{3}$$

where

$$T_n(x):=\begin{cases}\dfrac{(x-a)^n}{n!}-\dfrac{(b-a)(x-a)^{n-1}}{2(n-1)!} & \text{if } x\in\left[a,\dfrac{a+b}{2}\right],\\[2mm] \dfrac{(x-b)^n}{n!}+\dfrac{(b-a)(x-b)^{n-1}}{2(n-1)!} & \text{if } x\in\left(\dfrac{a+b}{2},b\right],\end{cases} \tag{4}$$

Mathematics subject classification (2010): 26D15.
Keywords and phrases: Midpoint type inequality, trapezoid type inequality, Simpson type inequality, averaged midpoint-trapezoid type inequality, absolutely continuous.

and

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big] + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\Big(\frac{a+b}{2}\Big) + (-1)^n \int_a^b S_n(x)\,df^{(n-1)}(x), \tag{5}$$

where

$$S_n(x) := \begin{cases} \dfrac{(x-a)^n}{n!} - \dfrac{(b-a)(x-a)^{n-1}}{6(n-1)!} & \text{if } x \in \Big[a, \dfrac{a+b}{2}\Big], \\ \dfrac{(x-b)^n}{n!} + \dfrac{(b-a)(x-b)^{n-1}}{6(n-1)!} & \text{if } x \in \Big(\dfrac{a+b}{2}, b\Big]. \end{cases} \tag{6}$$

In general, it is not difficult to get

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big] + \sum_{k=1}^{[\frac{n-1}{2}]} \frac{[1-(2k+1)\theta](b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\Big(\frac{a+b}{2}\Big) + (-1)^n \int_a^b K_n(x,\theta)\,df^{(n-1)}(x), \tag{7}$$

where $\theta \in [0,1]$ and

$$K_n(x,\theta) := \begin{cases} \dfrac{(x-a)^n}{n!} - \dfrac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!} & \text{if } x \in \Big[a, \dfrac{a+b}{2}\Big], \\ \dfrac{(x-b)^n}{n!} + \dfrac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!} & \text{if } x \in \Big(\dfrac{a+b}{2}, b\Big]. \end{cases} \tag{8}$$

The purpose of this paper is to provide a unified treatment for generalizations of some classical integral inequalities by using the integral identity (7).

For convenience, we shall first collect some technical results related to the function $K_n(x,\theta)$ which will be used in the proofs of our theorems.

By elementary calculus, it is not difficult to get the following results:

$$\int_a^b K_n(x,\theta)\,dx = \frac{[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+1}} = \begin{cases} 0, & n \text{ odd}, \\ \dfrac{[1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^n}, & n \text{ even}. \end{cases} \tag{9}$$

$$\int_a^b |K_n(x,\theta)|\,dx = \begin{cases} \dfrac{[1-(n+1)\theta+2n^n\theta^{n+1}](b-a)^{n+1}}{(n+1)!2^n}, & n < \dfrac{1}{\theta}, \\ \dfrac{[(n+1)\theta-1](b-a)^{n+1}}{(n+1)!2^n}, & n \geqslant \dfrac{1}{\theta}. \end{cases} \tag{10}$$

$$\int_a^b K_n^2(x,\theta)\,dx = \frac{[(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2](b-a)^{2n+1}}{(4n^2-1)(n!)^2 2^{2n}}. \tag{11}$$

$$\sup_{x\in[a,b]} |K_n(x,\theta)| = \begin{cases} \dfrac{\max\{|1-n\theta|,(n-1)^{n-1}\theta^n\}(b-a)^n}{(n!)2^n}, & n < \dfrac{1}{\theta}+1, \\ \dfrac{(n\theta-1)(b-a)^n}{(n!)2^n}, & n \geqslant \dfrac{1}{\theta}+1. \end{cases} \tag{12}$$

In what follows, we will use the notations

$$D_n := \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}$$

and

$$G_n := \sum_{k=1}^{[\frac{n-1}{2}]} \frac{[1-(2k+1)\theta](b-a)^{2k+1}}{(2k+1)!2^{2k}} f^{(2k)}\left(\frac{a+b}{2}\right).$$

2. Bounds in terms of L_2 norm

THEOREM 1. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ $(n\geqslant 1)$ is absolutely continuous on $[a,b]$. If $f^{(n)}\in L_2[a,b]$, then for any $\theta\in[0,1]$ we have*

$$\begin{aligned} \Big|\int_a^b f(x)\,dx - \frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\Big| \\ \leqslant \frac{(b-a)^{n+\frac{1}{2}}}{n!2^n}\sqrt{\frac{(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2}{4n^2-1}}\,\|f^{(n)}\|_2, \end{aligned} \tag{13}$$

where $\|f^{(n)}\|_2 := [\int_a^b |f^{(n)}(x)|^2\,dx]^{\frac{1}{2}}$ is the usual Lebesgue norm on $L_2[a,b]$.

Proof. By using the identity (7) and Cauchy-Schwarz inequality, we get

$$\begin{aligned} \Big|\int_a^b f(x)\,dx - \frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\Big| \\ = \Big|\int_a^b K_n(x,\theta)f^{(n)}(x)\,dx\Big| \leqslant \Big[\int_a^b K_n^2(x,\theta)\,dx\Big]^{\frac{1}{2}}\|f^{(n)}\|_2. \end{aligned} \tag{14}$$

Consequently, the inequality (13) follows from (14) and (11). □

REMARK 1. It is not difficult to find that the inequality (13) is sharp in the sense that we can choose f to attain the equality in (13). In fact, we may find a function

$f:[a,b]\to\mathbf{R}$ such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ as

$$f^{(n-1)}(x)=\begin{cases}\dfrac{(x-a)^{n+1}}{(n+1)!}-\dfrac{\theta(b-a)(x-a)^n}{2n!} & \text{if } x\in\left[a,\dfrac{a+b}{2}\right],\\[2mm] \dfrac{(x-b)^{n+1}}{(n+1)!}+\dfrac{\theta(b-a)(x-b)^n}{2n!} & \text{if } x\in\left(\dfrac{a+b}{2},b\right]\end{cases}$$

for n is odd and

$$f^{(n-1)}(x)=\begin{cases}\dfrac{(x-a)^{n+1}}{(n+1)!}-\dfrac{\theta(b-a)(x-a)^n}{2n!}-\dfrac{[1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+1}} \\ \qquad\qquad\qquad\qquad\qquad \text{if } x\in\left[a,\dfrac{a+b}{2}\right],\\[2mm] \dfrac{(x-b)^{n+1}}{(n+1)!}+\dfrac{\theta(b-a)(x-b)^n}{2n!}+\dfrac{[1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+1}} \\ \qquad\qquad\qquad\qquad\qquad \text{if } x\in\left(\dfrac{a+b}{2},b\right]\end{cases}$$

for n is even. Both imply that

$$f^{(n)}(x)=\begin{cases}\dfrac{(x-a)^{n}}{n!}-\dfrac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!} & \text{if } x\in\left[a,\dfrac{a+b}{2}\right],\\[2mm] \dfrac{(x-b)^{n}}{n!}+\dfrac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!} & \text{if } x\in\left(\dfrac{a+b}{2},b\right]\end{cases}$$

which is just the function defined in (8) and satisfies the condition of Theorem 1.

REMARK 2. For $\theta=0,\,1,\,\frac{1}{3},\,\frac{1}{2}$ in (13) we get inequalities related to the midpoint, trapezoid, Simpson and averaged midpoint-trapezoid quadrature formulae, respectively.

3. For functions whose $(n-1)$th derivatives are Lipschitzian type

Recall that a function $f:[a,b]\to\mathbf{R}$ is said to be L-Lipschitzian on $[a,b]$ if

$$|f(x)-f(y)|\leqslant L|x-y|$$

for all $x,y\in[a,b]$, where $L>0$ is given, and, it is said to be (l,L)-Lipschitzian on $[a,b]$ (see e.g. [11]), if

$$l(x_2-x_1)\leqslant f(x_2)-f(x_1)\leqslant L(x_2-x_1)$$

for all $a\leqslant x_1\leqslant x_2\leqslant b$, where $l,L\in\mathbf{R}$ with $l<L$ (the condition has also been considered in [6] and [7] independently).

Clearly, an L-Lipschitzian function is a $(-L,L)$-Lipschitzian function.

It is well known (see e.g. [5]) that if $h,g:[a,b]\to\mathbf{R}$ are such that h is Riemann-integral on $[a,b]$ and g is L-Lipschitzian on $[a,b]$, then $\int_a^b h(t)\,dg(t)$ exists and

$$\left|\int_a^b h(x)\,dg(x)\right|\leqslant L\int_a^b |h(x)|\,dx. \tag{15}$$

THEOREM 2. *Let* $f:[a,b]\to\mathbf{R}$ *be such that* $f^{(n-1)}$ $(n\geqslant 1)$ *is* (l,L)*-Lipschitzian on* $[a,b]$*. Then for any* $\theta\in[0,1]$ *we have*

$$\begin{aligned}\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\\ -\frac{(l+L)[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+2}}\Big|\\ \leqslant\frac{L-l}{2}\times\begin{cases}\dfrac{[1-(n+1)\theta+2n^n\theta^{n+1}](b-a)^{n+1}}{(n+1)!2^n}, & n<\dfrac{1}{\theta},\\ \dfrac{[(n+1)\theta-1](b-a)^{n+1}}{(n+1)!2^n}, & n\geqslant\dfrac{1}{\theta}.\end{cases}\end{aligned}\tag{16}$$

Proof. By (7) and (9) we get

$$\begin{aligned}(-1)^n\int_a^b K_n(x,\theta)\,d\Big[f^{(n-1)}(x)-\frac{l+L}{2}x\Big]\\ =\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]\\ -G_n-\frac{(l+L)[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+2}},\end{aligned}$$

then notice that $f^{(n-1)}(x)-\frac{l+L}{2}x$ is $\frac{L-l}{2}$-Lipschitzian on $[a,b]$ and by using (15), we have

$$\begin{aligned}\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\\ -\frac{(l+L)[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+2}}\Big|\\ \leqslant\frac{L-l}{2}\int_a^b|K_n(x,\theta)|\,dx,\end{aligned}\tag{17}$$

and hence the inequality (16) follows from (17) and (10). □

REMARK 3. It is not difficult to find that the inequality (16) is sharp in the sense that we can choose f to attain the equality in (16). In fact, if n is odd, we may construct f, such that

$$f^{(n-1)}(x)=\begin{cases}l(t-a), & a\leqslant x<a+\dfrac{n\theta}{2}(b-a),\\ L(t-a)-\dfrac{n\theta(L-l)}{2}(b-a), & a+\dfrac{n\theta}{2}(b-a)\leqslant x<\dfrac{a+b}{2},\\ l\Big(t-\dfrac{a+b}{2}\Big)+\dfrac{(1-n\theta)L+n\theta l}{2}(b-a), & \dfrac{a+b}{2}\leqslant x<b-\dfrac{n\theta}{2}(b-a),\\ L(t-b)+\dfrac{L+l}{2}(b-a), & b-\dfrac{n\theta}{2}(b-a)\leqslant x\leqslant b\end{cases}$$

for $n < \frac{1}{\theta}$ and

$$f^{(n-1)}(x) = \begin{cases} l(t-a), & a \leqslant x < \frac{a+b}{2}, \\ L(t-a) - \frac{L-l}{2}(b-a), & \frac{a+b}{2} \leqslant x \leqslant b \end{cases}$$

for $n \geqslant \frac{1}{\theta}$. If n is even, we may construct f, such that

$$f^{(n-1)}(x) = \begin{cases} l(t-a), & a \leqslant x < a+\frac{n\theta}{2}(b-a), \\ L(t-a) - \frac{n\theta(L-l)}{2}(b-a), & a+\frac{n\theta}{2}(b-a) \leqslant x < b-\frac{n\theta}{2}(b-a), \\ l(t-b)+[(1-n\theta)L+n\theta l](b-a), & b-\frac{n\theta}{2}(b-a) \leqslant x \leqslant b \end{cases}$$

for $n < \frac{1}{\theta}$ and

$$f^{(n-1)}(x) = \begin{cases} l(t-a), & a \leqslant x < \frac{a+b}{2}, \\ -L(t-a) + \frac{L+l}{2}(b-a), & \frac{a+b}{2} \leqslant x \leqslant b \end{cases}$$

for $n \geqslant \frac{1}{\theta}$.

Clearly, the above all $f^{(n-1)}$ are absolutely continuous on $[a,b]$ with

$$f^{(n)}(x) = \begin{cases} l, & a < x < a + \frac{n\theta}{2}(b-a), \\ L, & a + \frac{n\theta}{2}(b-a) < x < \frac{a+b}{2}, \\ l, & \frac{a+b}{2} < x < b - \frac{n\theta}{2}(b-a), \\ L, & b - \frac{n\theta}{2}(b-a) < x < b \end{cases}$$

for n odd and $n < \frac{1}{\theta}$,

$$f^{(n)}(x) = \begin{cases} l, & a < x < \frac{a+b}{2}, \\ L, & \frac{a+b}{2} < x < b \end{cases}$$

for n odd and $n \geqslant \frac{1}{\theta}$ as well as

$$f^{(n)}(x) = \begin{cases} l, & a < x < a + \frac{n\theta}{2}(b-a), \\ L, & a + \frac{n\theta}{2}(b-a) < x < b - \frac{n\theta}{2}(b-a), \\ l, & b - \frac{n\theta}{2}(b-a) < x < b \end{cases}$$

for n even and $n < \frac{1}{\theta}$,

$$f^{(n)}(x) = \begin{cases} l, & a < x < \dfrac{a+b}{2}, \\ -L, & \dfrac{a+b}{2} < x \leqslant b \end{cases}$$

for n even and $n \geqslant \frac{1}{\theta}$, which satisfy the condition of Theorem 2.

COROLLARY 1. *Let $f:[a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ $(n \geqslant 1)$ is L-Lipschitzian on $[a,b]$. Then for any $\theta \in [0,1]$ we have sharp inequalities*

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{2}\Big[\theta f(a) + 2(1-\theta) f\Big(\frac{a+b}{2}\Big) + \theta f(b)\Big] - G_n \right| \\ &\leqslant L \times \begin{cases} \dfrac{[1-(n+1)\theta + 2n^n\theta^{n+1}](b-a)^{n+1}}{(n+1)!2^n}, & n < \dfrac{1}{\theta}, \\ \dfrac{[(n+1)\theta - 1](b-a)^{n+1}}{(n+1)!2^n}, & n \geqslant \dfrac{1}{\theta}. \end{cases} \end{aligned} \tag{18}$$

Proof. It is immediate by taking $l = -L$ in Theorem 2. □

COROLLARY 2. *Let $f:[a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ $(n \geqslant 1)$ is absolutely continuous on $[a,b]$. If $f^{(n)} \in L_\infty[a,b]$, then for any $\theta \in [0,1]$ we have sharp inequalities*

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{2}\Big[\theta f(a) + 2(1-\theta) f\Big(\frac{a+b}{2}\Big) + \theta f(b)\Big] - G_n \right| \\ &\leqslant \|f^{(n)}\|_\infty \times \begin{cases} \dfrac{[1-(n+1)\theta + 2n^n\theta^{n+1}](b-a)^{n+1}}{(n+1)!2^n}, & n < \dfrac{1}{\theta}, \\ \dfrac{[(n+1)\theta - 1](b-a)^{n+1}}{(n+1)!2^n}, & n \geqslant \dfrac{1}{\theta}, \end{cases} \end{aligned} \tag{19}$$

where $\|f^{(n)}\|_\infty := ess\sup_{x\in[a,b]} |f^{(n)}(x)|$ is the usual Lebesgue norm on $L_\infty[a,b]$.

Proof. It is immediate by taking $L = \|f^{(n)}\|_\infty$ in Theorem 2. □

REMARK 4. For $\theta = 0,\ 1,\ \frac{1}{3},\ \frac{1}{2}$ in (16), (18) and (19) we get inequalities related to the midpoint, trapezoid, Simpson and averaged midpoint-trapezoid quadrature formulae, respectively.

4. For functions whose $(n-1)$th derivatives are of bounded variation

It is well known (see e.g. [17]) that if $h,g:[a,b] \to \mathbf{R}$ are such that h is continuous on $[a,b]$ and g is of bounded variation on $[a,b]$, then $\int_a^b h(t)\,dg(t)$ exists and

$$\left| \int_a^b h(t)\,dg(t) \right| \leqslant \sup_{t\in[a,b]} |h(t)| \bigvee_a^b (g). \tag{20}$$

Clearly, it is not difficult to find that the above argument is also valid for h is piecewise continuous on $[a,b]$.

THEOREM 3. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}(n\geqslant 1)$ is a continuous function of bounded variation on $[a,b]$. Then for any $\theta\in[0,1]$ we have*

$$\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\Big| \leqslant \bigvee_a^b(f^{(n-1)})\times\begin{cases}\dfrac{\max\{|1-n\theta|,(n-1)^{n-1}\theta^n\}(b-a)^n}{(n!)2^n}, & n<\dfrac{1}{\theta}+1,\\ \dfrac{(n\theta-1)(b-a)^n}{(n!)2^n}, & n\geqslant\dfrac{1}{\theta}+1.\end{cases} \tag{21}$$

Proof. By using the identity (7) and the inequality (20) we get

$$\begin{aligned}&\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\Big|\\&=\Big|\int_a^b K_n(x,\theta)\,df^{(n-1)}(x)\Big|\leqslant\sup_{x\in[a,b]}|K_n(x,\theta)|\bigvee_a^b(f^{(n-1)}).\end{aligned} \tag{22}$$

Consequently, the inequality (21) follows from (22) and (12). □

COROLLARY 3. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ $(n\geqslant 1)$ is absolutely continuous on $[a,b]$. If $f^{(n)}\in L_1[a,b]$, then for any $\theta\in[0,1]$ we have*

$$\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\Big| \leqslant \|f^{(n)}\|_1\times\begin{cases}\dfrac{\max\{|1-n\theta|,(n-1)^{n-1}\theta^n\}(b-a)^n}{(n!)2^n}, & n<\dfrac{1}{\theta}+1,\\ \dfrac{(n\theta-1)(b-a)^n}{(n!)2^n}, & n\geqslant\dfrac{1}{\theta}+1\end{cases} \tag{23}$$

where $\|f^{(n)}\|_1:=\int_a^b|f^{(n)}(x)|\,dx$ is the usual Lebesgue norm on $L_1[a,b]$.

Proof. It is immediate from Theorem 3, since if $f^{(n-1)}(n\geqslant 1)$ is absolutely continuous on $[a,b]$ then we certainly have

$$\bigvee_a^b(f^{(n-1)})=\|f^{(n)}\|_1. \quad \square$$

REMARK 5. For $\theta=0,1,\frac{1}{3},\frac{1}{2}$ in (21) and (23) we get inequalities related to the midpoint, trapezoid, Simpson and averaged midpoint-trapezoid quadrature formulae, respectively.

5. Non symmetric bounds

THEOREM 4. *Let* $f:[a,b]\to\mathbf{R}$ *be such that* $f^{(n-1)}$ $(n\geqslant 1)$ *is absolutely continuous with* $\gamma_n\leqslant f^{(n)}(x)\leqslant\Gamma_n$ *a.e. on* $[a,b]$, *where* $\gamma_n,\Gamma_n\in\mathbf{R}$ *are constants. Then for any* $\theta\in[0,1]$ *we have*

$$\begin{aligned}
\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\\
-\frac{(\gamma_n+\Gamma_n)[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+2}}\Big|\\
\leqslant\frac{\Gamma_n-\gamma_n}{2}\times\begin{cases}\dfrac{[1-(n+1)\theta+2n^n\theta^{n+1}](b-a)^{n+1}}{(n+1)!2^n}, & n<\dfrac{1}{\theta},\\[2mm] \dfrac{[(n+1)\theta-1](b-a)^{n+1}}{(n+1)!2^n}, & n\geqslant\dfrac{1}{\theta}.\end{cases}
\end{aligned}\tag{24}$$

Proof. By (7) and (9) we get

$$\begin{aligned}
(-1)^n\int_a^b K_n(x,\theta)\Big[f^{(n)}(x)-\frac{\gamma_n+\Gamma_n}{2}\Big]\,dx\\
=\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]\\
-G_n-\frac{(\gamma_n+\Gamma_n)[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+2}},
\end{aligned}$$

then notice that $|f^{(n)}(x)-\frac{\gamma_n+\Gamma_n}{2}|\leqslant\frac{\Gamma_n-\gamma_n}{2}$ a.e. on $[a,b]$, we have

$$\begin{aligned}
\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\\
-\frac{(\gamma_n+\Gamma_n)[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+2}}\Big|\\
\leqslant\frac{\Gamma_n-\gamma_n}{2}\int_a^b|K_n(x,\theta)|\,dx,
\end{aligned}\tag{25}$$

and hence the inequality (24) follows from (25) and (10). □

REMARK 6. It is not difficult to find that the inequality (24) is sharp in the sense that we can choose f to attain the equality in (24). The arguments are similar to that in Remark 3 and so are omitted here.

THEOREM 5. *Let* $f:[a,b]\to\mathbf{R}$ *be such that* $f^{(n-1)}$ $(n\geqslant 1)$ *is absolutely continuous with* $\gamma_n\leqslant f^{(n)}(x)$ *a.e. on* $[a,b]$, *where* $\gamma_n\in\mathbf{R}$ *is a constant. Then for any*

$\theta \in [0,1]$ *we have*

$$\begin{aligned}&\Big|\int_a^b f(x)\,dx - \frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\\&\qquad - \frac{\gamma_n[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+1}}\Big|\\&\leqslant (D_n-\gamma_n)\times\begin{cases}\dfrac{\max\{|1-n\theta|,(n-1)^{n-1}\theta^n\}(b-a)^{n+1}}{(n!)2^n}, & n<\dfrac{1}{\theta}+1,\\ \dfrac{(n\theta-1)(b-a)^{n+1}}{(n!)2^n}, & n\geqslant\dfrac{1}{\theta}+1.\end{cases}\end{aligned} \tag{26}$$

Proof. By (7) and (9) we get

$$\begin{aligned}&(-1)^n\int_a^b K_n(x,\theta)[f^{(n)}(x)-\gamma_n]\,dx\\&\quad=\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]\\&\qquad -G_n-\frac{\gamma_n[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+1}},\end{aligned}$$

then we have

$$\begin{aligned}&\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\\&\qquad-\frac{\gamma_n[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+1}}\Big|\\&\quad\leqslant\sup_{x\in[a,b]}|K_n(x,\theta)|\int_a^b|f^{(n)}(x)-\gamma_n|\,dx,\end{aligned} \tag{27}$$

and notice that $f^{(n)}(x)-\gamma_n\geqslant 0$ a.e. on $[a,b]$, the inequality (26) follows from (27) and (12). □

THEOREM 6. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ $(n\geqslant 1)$ is absolutely continuous with $f^{(n)}(x)\leqslant\Gamma_n$ a.e. on $[a,b]$, where $\Gamma_n\in\mathbf{R}$ is a constant. Then for any $\theta\in[0,1]$ we have*

$$\begin{aligned}&\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\Big|\\&\leqslant(\Gamma_n-D_n)\times\begin{cases}\dfrac{\max\{|1-n\theta|,(n-1)^{n-1}\theta^n\}(b-a)^{n+1}}{(n!)2^n}, & n<\dfrac{1}{\theta}+1,\\ \dfrac{(n\theta-1)(b-a)^{n+1}}{(n!)2^n}, & n\geqslant\dfrac{1}{\theta}+1.\end{cases}\end{aligned} \tag{28}$$

Proof. By (7) and (9) we get

$$
\begin{aligned}
(-1)^n \int_a^b & K_n(x,\theta)[f^{(n)}(x)-\Gamma_n]\,dx \\
&= \int_a^b f(x)\,dx - \frac{b-a}{2}\Big[\theta f(a) + 2(1-\theta) f\Big(\frac{a+b}{2}\Big) + \theta f(b)\Big] \\
&\quad - G_n - \frac{\Gamma_n[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+1}},
\end{aligned}
$$

then we have

$$
\begin{aligned}
\Big| \int_a^b & f(x)\,dx - \frac{b-a}{2}\Big[\theta f(a) + 2(1-\theta) f\Big(\frac{a+b}{2}\Big) + \theta f(b)\Big] - G_n \\
&- \frac{\Gamma_n[1+(-1)^n][1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+1}} \Big| \\
&\leqslant \sup_{x\in[a,b]} |K_n(x,\theta)| \int_a^b |f^{(n)}(x)-\Gamma_n|\,dx,
\end{aligned}
\tag{29}
$$

and notice that $\Gamma_n - f^{(n)}(x) \geqslant 0$ a.e. on $[a,b]$, the inequality (28) follows from (29) and (12). □

REMARK 7. For $\theta = 0,\ 1,\ \frac{1}{3},\ \frac{1}{2}$ in (24), (26) and (28) we get inequalities related to the midpoint, trapezoid, Simpson and averaged midpoint-trapezoid quadrature formulae, respectively.

6. Another sharp bound

Now we turn to consider another sharp bound which was first appeared in [1] in 2001, then in [18, 10, 12, 13, 14, 15] in recent years. Put

$$
\sigma(f) := \|f\|_2^2 - \frac{1}{b-a}\Big(\int_a^b f(t)\,dt\Big)^2. \tag{30}
$$

We have

THEOREM 7. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L_2[a,b]$ where n is an odd integer. Then for any $\theta\in[0,1]$ we have*

$$
\begin{aligned}
\Big| \int_a^b & f(x)\,dx - \frac{b-a}{2}\Big[\theta f(a) + 2(1-\theta) f\Big(\frac{a+b}{2}\Big) + \theta f(b)\Big] - G_n \Big| \\
&\leqslant \frac{(b-a)^{n+\frac{1}{2}}}{2^n n!} \sqrt{\frac{(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2}{4n^2-1}} \sqrt{\sigma(f^{(n)})}.
\end{aligned}
\tag{31}
$$

Inequality (31) is the best possible in the sense that the constant

$$
\frac{1}{2^n n!}\sqrt{\frac{(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2}{4n^2-1}}
$$

cannot be replaced by a smaller one.

Proof. From (7), (9), (11) and (30),we can easily get

$$\begin{aligned}
&\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\Big|\\
&=\Big|\int_a^b K_n(x,\theta)f^{(n)}(x)\,dx\Big|\\
&=\Big|\int_a^b K_n(x,\theta)\Big[f^{(n)}(x)-\frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\Big]\,dx\Big|\\
&\leqslant\Big(\int_a^b K_n^2(x,\theta)\,dx\Big)^{\frac12}\Big(\int_a^b\Big[f^{(n)}(x)-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\Big]^2dx\Big)^{\frac12}\\
&=\Big(\frac{[(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2](b-a)^{2n+1}}{(4n^2-1)(n!)^2 2^{2n}}\Big)^{\frac12}\\
&\quad\times\Big(\|f^{(n)}\|_2^2-\frac{[f^{(n-1)}(b)-f^{(n-1)}(a)]^2}{b-a}\Big)^{\frac12}\\
&=\frac{(b-a)^{n+\frac12}}{2^n n!}\sqrt{\frac{(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2}{4n^2-1}}\sqrt{\sigma(f^{(n)})}.
\end{aligned}$$

We now suppose that (31) holds with a constant $C>0$ as

$$\begin{aligned}
&\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n\Big|\\
&\qquad\leqslant C(b-a)^{n+\frac12}\sqrt{\sigma(f^{(n)})}.
\end{aligned}\tag{32}$$

We may find a function $f:[a,b]\to\mathbf{R}$ such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ as

$$f^{(n-1)}(x)=\begin{cases}\dfrac{(x-a)^{n+1}}{(n+1)!}-\dfrac{\theta(b-a)(x-a)^n}{2n!} & \text{if } x\in\Big[a,\dfrac{a+b}{2}\Big],\\[2mm] \dfrac{(x-b)^{n+1}}{(n+1)!}+\dfrac{\theta(b-a)(x-b)^n}{2n!} & \text{if } x\in\Big(\dfrac{a+b}{2},b\Big].\end{cases}$$

It follows that

$$f^{(n)}(x)=\begin{cases}\dfrac{(x-a)^{n}}{n!}-\dfrac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!} & \text{if } x\in\Big[a,\dfrac{a+b}{2}\Big],\\[2mm] \dfrac{(x-b)^{n}}{n!}+\dfrac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!} & \text{if } x\in\Big(\dfrac{a+b}{2},b\Big]\end{cases}$$

which is just the function defined in (8). Thus by (7),(8) and (11), it is not difficult to find that the left-hand side of the inequality (32) becomes

$$L.H.S.(32)=\frac{[(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2](b-a)^{2n+1}}{(4n^2-1)(n!)^2 2^{2n}},\tag{33}$$

and the right-hand side of the inequality (32) is

$$R.H.S.(32) = \frac{C(b-a)^{2n+1}}{2^n n!}\sqrt{\frac{(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2}{4n^2-1}}. \tag{34}$$

From (32), (33) and (34), we find that $C \geqslant \frac{1}{2^n n!}\sqrt{\frac{(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2}{4n^2-1}}$, proving that the constant $\frac{1}{2^n n!}\sqrt{\frac{(2n-1)-(4n^2-1)\theta+(2n+1)n^2\theta^2}{4n^2-1}}$ is the best possible in (31).

THEOREM 8. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L_2[a,b]$ where n is an even integer. Then for any $\theta\in[0,1]$ we have*

$$\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n-\frac{[1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^n}D_n\Big|$$
$$\leqslant \frac{(b-a)^{n+\frac12}}{2^n(n+1)!}\sqrt{\frac{2n^3-n^2-(4n^4-5n^2+1)\theta+(2n^5+n^4-4n^3-2n^2+2n+1)\theta^2}{4n^2-1}}\sqrt{\sigma(f^{(n)})}. \tag{35}$$

Inequality (35) is the best possible in the sense that the constant

$$\frac{1}{2^n(n+1)!}\sqrt{\frac{2n^3-n^2-(4n^4-5n^2+1)\theta+(2n^5+n^4-4n^3-2n^2+2n+1)\theta^2}{4n^2-1}}$$

cannot be replaced by a smaller one.

Proof. From (7), (9), (11) and (30),we can easily get

$$\Big|\int_a^b f(x)\,dx-\frac{b-a}{2}\Big[\theta f(a)+2(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta f(b)\Big]-G_n-\frac{[1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^n}D_n\Big|$$
$$=\Big|\int_a^b K_n(x,\theta)f^{(n)}(x)\,dx-\frac{1}{b-a}\int_a^b K_n(x,\theta)\,dx\int_a^b f^{(n)}(x)\,dx\Big|$$
$$=\frac{1}{2(b-a)}\Big|\int_a^b\int_a^b [K_n(x,\theta)-K_n(t,\theta)][f^{(n)}(x)-f^{(n)}(t)]\,dx\,dt\Big|$$
$$\leqslant \frac{1}{2(b-a)}\Big\{\int_a^b\int_a^b [K_n(x,\theta)-K_n(t,\theta)]^2\,dx\,dt\Big\}^{\frac12}\Big\{\int_a^b\int_a^b [f^{(n)}(x)-f^{(n)}(t)]^2\,dx\,dt\Big\}^{\frac12}$$
$$=\Big\{\int_a^b K_n^2(x,\theta)\,dx-\frac{1}{b-a}\Big[\int_a^b K_n(x,\theta)\,dx\Big]^2\Big\}^{\frac12}$$
$$\times\Big\{\int_a^b [f^{(n)}(x)]^2\,dx-\frac{1}{b-a}\Big[\int_a^b f^{(n)}(x)\,dx\Big]^2\Big\}^{\frac12}$$
$$=\Big\{\frac{[2n^3-n^2-(4n^4-5n^2+1)\theta+(2n^5+n^4-4n^3-2n^2+2n+1)\theta^2](b-a)^{2n+1}}{(4n^2-1)[(n+1)!]^2 2^{2n}}\Big\}^{\frac12}$$
$$\times\Big\{\|f^{(n)}\|_2^2-\frac{[f^{(n-1)}(b)-f^{(n-1)}(a)]^2}{b-a}\Big\}^{\frac12}$$
$$=\frac{(b-a)^{n+\frac12}}{2^n(n+1)!}\sqrt{\frac{2n^3-n^2-(4n^4-5n^2+1)\theta+(2n^5+n^4-4n^3-2n^2+2n+1)\theta^2}{4n^2-1}}\sqrt{\sigma(f^{(n)})}.$$

We now suppose that (35) holds with a constant $E > 0$ as

$$\begin{aligned} \Big| \int_a^b f(x)\,dx - \frac{b-a}{2}\Big[\theta f(a) + 2(1-\theta) f\Big(\frac{a+b}{2}\Big) + \theta f(b)\Big] - G_n \\ - \frac{[1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^n} D_n \Big| \leqslant E(b-a)^{n+\frac{1}{2}} \sqrt{\sigma(f^{(n)})}. \end{aligned} \tag{36}$$

We may find a function $f : [a,b] \to \mathbf{R}$ such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ as

$$f^{(n-1)}(x) = \begin{cases} \dfrac{(x-a)^{n+1}}{(n+1)!} - \dfrac{\theta(b-a)(x-a)^n}{2n!} - \dfrac{[1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+1}} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \in \Big[a, \dfrac{a+b}{2}\Big], \\ \dfrac{(x-b)^{n+1}}{(n+1)!} + \dfrac{\theta(b-a)(x-b)^n}{2n!} + \dfrac{[1-(n+1)\theta](b-a)^{n+1}}{(n+1)!2^{n+1}} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \in \Big(\dfrac{a+b}{2}, b\Big]. \end{cases}$$

It follows that

$$f^{(n)}(x) = \begin{cases} \dfrac{(x-a)^n}{n!} - \dfrac{\theta(b-a)(x-a)^{n-1}}{2(n-1)!} & \text{if } x \in \Big[a, \dfrac{a+b}{2}\Big], \\ \dfrac{(x-b)^n}{n!} + \dfrac{\theta(b-a)(x-b)^{n-1}}{2(n-1)!} & \text{if } x \in \Big(\dfrac{a+b}{2}, b\Big] \end{cases}$$

which is just the function defined in (8). Thus by (7), (8), (9) and (11), it is not difficult to find that the left-hand side of the inequality (36) becomes

$$\begin{aligned} &\text{L.H.S. } (36) \\ &= \frac{(2n^3 - n^2 - (4n^4 - 5n^2 + 1)\theta + (2n^5 + n^4 - 4n^3 - 2n^2 + 2n + 1)\theta^2)(b-a)^{2n+1}}{(4n^2-1)[(n+1)!]^2 2^{2n}}, \end{aligned} \tag{37}$$

and the right-hand side of the inequality (36) is

$$\begin{aligned} &\text{R.H.S. } (36) \\ &= \frac{E(b-a)^{2n+1}}{2^n(n+1)!} \sqrt{\frac{2n^3 - n^2 - (4n^4 - 5n^2 + 1)\theta + (2n^5 + n^4 - 4n^3 - 2n^2 + 2n + 1)\theta^2}{4n^2-1}}. \end{aligned} \tag{38}$$

From (36), (37) and (38), we find that

$$E \geqslant \frac{1}{2^n(n+1)!} \sqrt{\frac{2n^3 - n^2 - (4n^4 - 5n^2 + 1)\theta + (2n^5 + n^4 - 4n^3 - 2n^2 + 2n + 1)\theta^2}{4n^2-1}},$$

proving that the constant

$$\frac{1}{2^n(n+1)!} \sqrt{\frac{2n^3 - n^2 - (4n^4 - 5n^2 + 1)\theta + (2n^5 + n^4 - 4n^3 - 2n^2 + 2n + 1)\theta^2}{4n^2-1}}$$

is the best possible in (35).

REMARK 8. For $\theta = 0, 1, \frac{1}{3}, \frac{1}{2}$ in (31) and (35) we get inequalities related to the midpoint, trapezoid, Simpson and averaged midpoint-trapezoid quadrature formulae, respectively.

Acknowledgement. The author wishes to thank the referee for his helpful comments and suggestions.

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(Received April 6, 2012)

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