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A NOTE ON MATRIX VERSIONS OF KANTOROVICH–TYPE INEQUALITY

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Abstract. Some new matrix versions of Kantorovich-Type inequalities for Hermitian matrix are proposed in this paper. We consider what happens to these inequalities when the positive definite matrix is allowed to be positive semidefinite singular or indefinite.

1. Introduction and preliminaries

Let $M_{m,n}$ denote the space of $m\times n$ complex matrices and write $M_n \equiv M_{n,n}$. The symbol 0 will be used to denote a scalar, vector, or matrices of all zeros, and by I the identity matrix; their dimensions should be clear from the context. As usual, A^* denotes the conjugate transpose of matrix A. A matrix $A\in M_n$ is Hermitian if $A = A^*$. A Hermitian matrix A is said to be positive semidefinite, written as $A \geqslant 0$, if

$$x^*Ax \geqslant 0, \quad \forall x \in C^n. \tag{1}$$

A is further called positive definite, symbolized $A > 0$, if the inequality in (1) holds strictly for all nonzero $x\in C^n$.

Let A and B be two Hermitian matrices of the same size. If $A-B$ is semidefinite, then we write

$$A \geqslant B. \tag{2}$$

Since A is Hermitian matrix , its real eigenvalues are arranged in decreasing order,that is, $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$. Then, the well-known Kantorovich inequality for a positive definite Hermite matrix is

$$1 \leqslant x^*Axx^*A^{-1}x \leqslant \frac{(\lambda_1+\lambda_n)^2}{4\lambda_1\lambda_n}, \tag{3}$$

for any $x\in C^n, \|x\| = 1$ (see [1, 2]). Marshall and Olkin [6] extend (3) by replacing the vector x with an $n\times p$ matrix X and the usual scalar inequality with the L$\ddot{o}$wner partial ordering.

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Liu and Neudecker [3] presented two matrix Kantorovich-type inequalities. Liu [4], [5] proposed an improvement of the Kantorovich inequality of [3].

We establish in this paper some new Kantorovich-type inequalities, the classical Kantorovich-type inequalities are modified to apply not only to positive definite but also to invertible, semidefinite and indefinite Hermitian matrices.

Let M be a Hermitian matrix, i.e., $M = M^*$, with the spectral decomposition $M = Q\Lambda Q^* = \sum_{i=1}^n \lambda_i q_i q_i^*$, where Λ is a diagonal matrix with all the eigenvalues of M along its diagonal, Q is an unitary matrix, i.e., $QQ^* = I$, and each column q_i of Q is an eigenvector of M corresponding to the eigenvalue λ_i.

For a Hermitian matrix A and $X \in M_{n,k}$, $X^*X = I$, as in [7], we define the following transform

$$C(A,X) = X^*(\lambda_n I - A)(A - \lambda_1 I)X. \tag{4}$$

When A is invertible, if $\lambda_1\lambda_n > 0$, then,

$$C(A^{-1},X) = X^*\left(\frac{1}{\lambda_1}I - A^{-1}\right)\left(A^{-1} - \frac{1}{\lambda_n}I\right)X. \tag{5}$$

Otherwise, $\lambda_1\lambda_n < 0$, then,

$$C(A^{-1},X) = X^*\left(\frac{1}{\lambda_{k+1}}I - A^{-1}\right)\left(A^{-1} - \frac{1}{\lambda_k}I\right)X, \tag{6}$$

where

$$\lambda_1 \leqslant \ldots \leqslant \lambda_k < 0 < \lambda_{k+1} \leqslant \ldots \leqslant \lambda_n. \tag{7}$$

To simplify the proof, we first introduce some lemmas.

2. Lemmas

LEMMA 2.1. *Let $A \geqslant 0$, $X \in M_{n,k}$, $X^*X = I$, then*

$$X^*AX \geqslant 0. \tag{8}$$

LEMMA 2.2. *Let A and B are Hermitian matrices, $AB = BA$, then*

$$A^2 + B^2 \geqslant 2AB. \tag{9}$$

LEMMA 2.3. *Let $A \leqslant 0$, $X \in M_{n,k}$, $X^*X = I$, then*

$$X^*AX \leqslant 0. \tag{10}$$

LEMMA 2.4. *Let $A \geqslant 0$, $B \geqslant 0$, if $AB = BA$, then*

$$AB \geqslant 0. \tag{11}$$

LEMMA 2.5. (See, e.g., [3]) *Let $A > 0$, $X \in M_{n,k}$, $X^*X = I$, then*

$$X^*A^2X \leqslant \frac{(\lambda_n + \lambda_1)^2}{4\lambda_n\lambda_1}(X^*AX)^2, \tag{12}$$

$$X^*A^2X - (X^*AX)^2 \leqslant \frac{(\lambda_n - \lambda_1)^2}{4}I. \tag{13}$$

3. Main Results

LEMMA 3.1. *For a Hermitian matrix A and $X \in M_{n,k}$, $X^*X = I$, we have*

$$0 \leqslant (\lambda_n I - X^*AX)(X^*AX - \lambda_1 I) \leqslant \frac{1}{4}(\lambda_n - \lambda_1)^2 I. \tag{14}$$

When A is invertible, if $\lambda_1\lambda_n > 0$, then,

$$0 \leqslant \left(\frac{1}{\lambda_1}I - X^*A^{-1}X\right)\left(X^*A^{-1}X - \frac{1}{\lambda_n}I\right) \leqslant \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_1\lambda_n)^2}I. \tag{15}$$

Otherwise, $\lambda_1\lambda_n < 0$, then,

$$0 \leqslant \left(\frac{1}{\lambda_{k+1}}I - X^*A^{-1}X\right)\left(X^*A^{-1}X - \frac{1}{\lambda_k}I\right) \leqslant \frac{(\lambda_{k+1} - \lambda_k)^2}{4(\lambda_k\lambda_{k+1})^2}I. \tag{16}$$

Proof. To prove (14), let $X^*AX = U\Lambda U^*$, $UU^* = I$, where Λ is a diagonal matrix with all the eigenvalues of X^*AX along its diagonal. It is easy to see that $\lambda_1 I \leqslant A \leqslant \lambda_n I$, then $\lambda_1 I \leqslant X^*AX \leqslant \lambda_n I$. Considering the following inequality

$$0 \leqslant (\lambda_n - \lambda_i^{'})(\lambda_i^{'} - \lambda_1) \leqslant \frac{(\lambda_n - \lambda_1)^2}{4},$$

where $\lambda_i^{'}$ is the ith eigenvalue of X^*AX, $i = 1, \dots, k$. Then,

$$0 \leqslant (\lambda_n I - \Lambda)(\Lambda - \lambda_1 I) \leqslant \frac{(\lambda_n - \lambda_1)^2}{4}I.$$

Multiply U and U^* on the both sides of it, we can get the inequality. The other inequalities can be obtained similarly, the proof is completed. □

LEMMA 3.2. *For a Hermitian matrix A and $X \in M_{n,k}$, $X^*X = I$, we have*

$$0 \leqslant C(A,X) \leqslant \frac{1}{4}(\lambda_n - \lambda_1)^2 I. \tag{17}$$

When A is invertible, if $\lambda_1\lambda_n > 0$,

$$0 \leqslant C(A^{-1},X) \leqslant \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_1\lambda_n)^2}I. \tag{18}$$

Otherwise, $\lambda_1\lambda_n < 0$, then

$$0 \leqslant C(A^{-1},X) \leqslant \frac{(\lambda_{k+1} - \lambda_k)^2}{4(\lambda_k\lambda_{k+1})^2}I. \tag{19}$$

Proof. From (9) and (11), we can obtain when $A \geqslant 0$ and $B \geqslant 0$, $AB = BA$, then,

$$0 \leqslant AB \leqslant \frac{(A+B)^2}{4},$$

so

$$0 \leqslant (\lambda_n I - A)(A - \lambda_1 I) \leqslant \frac{(\lambda_n - \lambda_1)^2}{4} I.$$

The other inequalities can be obtained similarly, we complete the proof. □

THEOREM 3.3. *Suppose that A is a Hermitian matrix, for any $X \in M_{n,k}$, $X^*X = I$, then*

$$4\lambda_n\lambda_1 X^*A^2X \leqslant (\lambda_n + \lambda_1)^2 (X^*AX)^2 \tag{20}$$

Proof. When $\lambda_1\lambda_n \leqslant 0$, the conclusion is straightforward. We consider $\lambda_1\lambda_n > 0$. If $\lambda_n \geqslant \lambda_1 > 0$, that is, $A > 0$. Using (12), the inequality is hold.

Otherwise, $\lambda_1 \leqslant \lambda_n < 0$, that is, $A < 0$. Inequality (20) follows from (12) applied to the matrix $-A$. □

REMARK 3.4. It is easy to see that if $A > 0$, our result coincides with the inequality (12). So we conclude that our result gives an improvement of the Kantorovich inequality (12), that applies all Hermite matrices.

THEOREM 3.5. *Suppose that A is a Hermitian matrix, for any $X \in M_{n,k}$, $X^*X = I$, then*

$$X^*A^2X - (X^*AX)^2 \leqslant \frac{(\lambda_n - \lambda_1)^2}{4} I - C(A,X). \tag{21}$$

When A is invertible, then, if $\lambda_1\lambda_n > 0$,

$$X^*(A^{-1})^2X - (X^*A^{-1}X)^2 \leqslant \frac{(\lambda_n - \lambda_1)^2}{4(\lambda_1\lambda_n)^2} I - C(A^{-1},X), \tag{22}$$

if $\lambda_1\lambda_n < 0$,

$$X^*(A^{-1})^2X - (X^*A^{-1}X)^2 \leqslant \frac{(\lambda_{k+1} - \lambda_k)^2}{4(\lambda_{k+1}\lambda_k)^2} I - C(A^{-1},X). \tag{23}$$

Proof. Thus,

$$\begin{aligned}
X^*A^2X - (X^*AX)^2 &= (\lambda_n I - X^*AX)(X^*AX - \lambda_1 I) + X^*[A^2 - (\lambda_1 + \lambda_n)A + \lambda_1\lambda_n I]X \\
&= (\lambda_n I - X^*AX)(X^*AX - \lambda_1 I) + X^*(A - \lambda_1 I)(A - \lambda_n I)X \\
&= (\lambda_n I - X^*AX)(X^*AX - \lambda_1 I) - X^*(A - \lambda_1 I)(\lambda_n I - A)X \\
&\leqslant \frac{(\lambda_n - \lambda_1)^2}{4} I - C(A,X).
\end{aligned}$$

Then we get the equality (21). The other inequalities can be obtained similarly, we complete the proof. □

REMARK 3.6. It is not difficult to see that if

$$X^*A^2X-(X^*AX)^2 \leqslant \frac{(\lambda_n-\lambda_1)^2}{4}I - C(A,X) \leqslant \frac{(\lambda_n-\lambda_1)^2}{4}I,$$

our result not only improves the Kantorovich inequality (13), but also gives an improvement of the Kantorovich inequality [4] that applies all Hermite matrices.

EXAMPLE 3.7. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of A are: $\lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = 4$, that is, $A \geqslant 0$. By easily calculating, we have

$$X^*A^2X = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix},$$

$$(X^*AX)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix},$$

$$C(A,X) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$X^*A^2X-(X^*AX)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \leqslant \frac{(\lambda_3-\lambda_1)^2}{4}I - C(A,X) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}.$$

EXAMPLE 3.8. Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 2 \\ 0 & 2 & -4 \end{bmatrix},$$

$$X = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix}.$$

The eigenvalues of A are: $\lambda_1 = \frac{-7-\sqrt{17}}{2}$, $\lambda_2 = \frac{-7+\sqrt{17}}{2}$, $\lambda_3 = 2$, that is, A is indefinite. By easily calculating, we have

$$X^*A^2X = \begin{bmatrix} 8.5000 & -9.8995 \\ -9.8995 & 20.0000 \end{bmatrix},$$

$$(X^*AX)^2 = \begin{bmatrix} 2.2500 & -6.3640 \\ -6.3640 & 18.0000 \end{bmatrix},$$

$$C(A,X) = \begin{bmatrix} 4.4040 & 4.8626 \\ 4.8626 & 5.3696 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} X^*A^2X - (X^*AX)^2 = \begin{bmatrix} 6.2500 & -3.5355 \\ -3.5355 & 2.0000 \end{bmatrix} &\leqslant \frac{(\lambda_3 - \lambda_1)^2}{4} I - C(A,X) \\ &= \begin{bmatrix} 9.8904 & -4.8626 \\ -4.8626 & 8.9248 \end{bmatrix}, \end{aligned}$$

we get a sharpen upper bound.

4. Conclusion

In this paper, we introduce some new Kantorovich-Type Inequalities for Hermitian matrices. In Theorem 3.3 and 3.4, if $\lambda_1 > 0$, $\lambda_n > 0$, the results is well-known Kantorovich-Type Inequalities [3], [4]. Moreover, it holds for negative definite Hermitian matrices, even for any Hermitian matrix, there exists a similar inequality.

REFERENCES

[1] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1985.

[2] A. S. HOUSEHOLDER, *The Theory of Matrices in Numerical Analysis*, Blaisdell, New York, NY, USA, 1964.

[3] S. LIU AND H. NEUDECKER, *Several Matrix Kantorovich-Type Inequalities*, Journal of Mathematical Analysis and Applications, vol. 197, no. 1, pp. 23–26, 1996.

[4] ZHIBING LIU, LINZHANG LU, AND KANMIN WANG, *On several matrix Kantorovich-Type inequalities*, Journal of Inequalities and Applications 2010, 2010:571629.

[5] ZHIBING LIU, KANMIN WANG AND CHENGFENG XU, *A note on Kantorovich inequality for Hermite matrices*, Journal of Inequalities and Applications 2011, 2011:245767.

[6] A. W. MARSHALL AND I. OLKIN, *Matrix versions of the Cauchy and Kantorovich inequalities*, Aequationes Mathematicae, vol. 40, no. 1, pp. 89–93, 1990.

[7] S. S. DRAGOMIR, *New inequalities of the Kantorovich type for bounded linear operators in Hilbert spaces*, Linear Algebra and Its Applications, vol. 428, no. 11–12, pp. 2750–2760, 2008.

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