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MINKOWSKI AND BECKENBACH—DRESHER INEQUALITIES AND FUNCTIONALS ON TIME SCALES

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Abstract. We obtain integral forms of the Minkowski inequality and Beckenbach–Dresher inequality on time scales. Also, we investigate a converse of Minkowski's inequality and several functionals arising from the Minkowski inequality and the Beckenbach–Dresher inequality.

1. Introduction and preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The theory of time scales was introduced by Stefan Hilger [6] in order to unify the theory of difference equations and the theory of differential equations. For an introduction to the theory of dynamic equations on time scales, we refer to [2, 7]. Martin Bohner and Gusein Sh. Guseinov [3, 4] defined the multiple Riemann and multiple Lebesgue integration on time scales and compared the Lebesgue Δ-integral with the Riemann Δ-integral.

Let $n \in \mathbb{N}$ be fixed. For each $i \in \{1, \ldots, n\}$, let $\mathbb{T}_i$ denote a time scale and

$$\Lambda^n = \mathbb{T}_1 \times \ldots \times \mathbb{T}_n = \{t = (t_1, \ldots, t_n) : t_i \in \mathbb{T}_i, \ 1 \leqslant i \leqslant n\}$$

an n-dimensional time scale. Let μ_Δ be the σ-additive Lebesgue Δ-measure on Λ^n and $\mathscr{F}$ be the family of Δ-measurable subsets of Λ^n. Let $E \in \mathscr{F}$ and $(E, \mathscr{F}, \mu_\Delta)$ be a time scale measure space. Then for a Δ-measurable function $f : E \to \mathbb{R}$, the corresponding Δ-integral of f over E will be denoted according to [4, (3.18)] by

$$\int_E f(t_1, \ldots, t_n)\Delta_1 t_1 \ldots \Delta_n t_n, \quad \int_E f(t)\Delta t, \quad \int_E f \mathrm{d}\mu_\Delta, \quad \text{or} \quad \int_E f(t)\mathrm{d}\mu_\Delta(t).$$

By [4, Section 3], all theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue Δ-integrals on Λ^n. Here we state Fubini's theorem for integrals on time scales. It is used in the proofs of our main results.

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THEOREM 1.1. (Fubini's theorem) *Let* $(X,\mathscr{M},\mu_\Delta)$ *and* $(Y,\mathscr{L},\nu_\Delta)$ *be two finite-dimensional time scale measure spaces. If* $f: X\times Y\to\mathbb{R}$ *is a* Δ*-integrable function and if we define the functions*

$$\varphi(y)=\int_X f(x,y)\mathrm{d}\mu_\Delta(x) \quad \textit{for a.e.} \quad y\in Y$$

and

$$\psi(x)=\int_Y f(x,y)\mathrm{d}\nu_\Delta(y) \quad \textit{for a.e.} \quad x\in X,$$

then φ *is* Δ*-integrable on* Y *and* ψ *is* Δ*-integrable on* X *and*

$$\int_X \mathrm{d}\mu_\Delta(x)\int_Y f(x,y)\mathrm{d}\nu_\Delta(y)=\int_Y \mathrm{d}\nu_\Delta(y)\int_X f(x,y)\mathrm{d}\mu_\Delta(x). \tag{1.1}$$

Some classical inequalities, including Jensen's inequality, Hölder's inequality, Minkowski's inequality and their converses for multiple integration on time scales were investigated in [1]. These inequalities hold for both Riemann integrals and Lebesgue integrals on time scales. For completeness, let us recall these inequalities from [1].

THEOREM 1.2. (Jensen's inequality [1, Theorem 4.2]) *Assume* $\Phi\in \mathrm{C}(I,\mathbb{R})$ *is convex, where* $I\subseteq\mathbb{R}$ *is an interval. Let* $(E,\mathscr{F},\mu_\Delta)$ *be a time scale measure space and suppose* f *is* Δ*-integrable on* E *such that* $f(E)=I$*. Moreover, let* $h:E\to\mathbb{R}$ *be nonnegative* Δ*-integrable such that* $\int_E h\mathrm{d}\mu_\Delta>0$*. Then*

$$\Phi\left(\frac{\int_E f(t)h(t)\mathrm{d}\mu_\Delta(t)}{\int_E h(t)\mathrm{d}\mu_\Delta(t)}\right)\leqslant\frac{\int_E \Phi(f(t))h(t)\mathrm{d}\mu_\Delta(t)}{\int_E h(t)\mathrm{d}\mu_\Delta(t)}. \tag{1.2}$$

If Φ *is concave, then* (1.2) *is reversed.*

THEOREM 1.3. (Hölder's inequality [1, Theorem 6.2]) *For* $p\neq 1$*, define* $q=p/(p-1)$*. Let* $(E,\mathscr{F},\mu_\Delta)$ *be a time scale measure space. Assume* w*,* f*,* g *are nonnegative functions such that* wf^p*,* wg^q*,* wfg *are* Δ*-integrable on* E*. If* $p>1$*, then*

$$\int_E w(t)f(t)g(t)\mathrm{d}\mu_\Delta(t)\leqslant\left(\int_E w(t)f^p(t)\mathrm{d}\mu_\Delta(t)\right)^{1/p}\left(\int_E w(t)g^q(t)\mathrm{d}\mu_\Delta(t)\right)^{1/q}. \tag{1.3}$$

If $0<p<1$ *and* $\int_E wg^q\mathrm{d}\mu_\Delta>0$*, or if* $p<0$ *and* $\int_E wf^p\mathrm{d}\mu_\Delta>0$*, then* (1.3) *is reversed.*

THEOREM 1.4. (Minkowski's inequality [1, Theorem 7.2]) *Let* $(E,\mathscr{F},\mu_\Delta)$ *be a time scale measure space. For* $p\in\mathbb{R}$*, assume* w*,* f*,* g*, are nonnegative functions such that* wf^p*,* wg^p*,* $w(f+g)^p$ *are* Δ*-integrable on* E*. If* $p\geqslant 1$*, then*

$$\left(\int_E w(t)(f(t)+g(t))^p\mathrm{d}\mu_\Delta(t)\right)^{\frac{1}{p}} \leqslant\left(\int_E w(t)f^p(t)\mathrm{d}\mu_\Delta(t)\right)^{1/p}+\left(\int_E w(t)g^p(t)\mathrm{d}\mu_\Delta(t)\right)^{1/p}. \tag{1.4}$$

If $0 < p < 1$ *or* $p < 0$, *then* (1.4) *is reversed provided each of the two terms on the right-hand side is positive.*

THEOREM 1.5. (Converse of Hölder's inequality [1, Theorem 11.3]) *For* $p \neq 1$, *define* $q = p/(p-1)$. *Let* $(E, \mathscr{F}, \mu_\Delta)$ *be a time scale measure space. Assume* w, f, g *are nonnegative functions such that* wf^p, wg^q, wfg *are* Δ*-integrable on* E. *Suppose*

$$0 < m \leqslant f(t)g^{-q/p}(t) \leqslant M \quad \text{for all} \quad t \in E.$$

If $p > 1$, *then*

$$\int_E w(t)f(t)g(t)\mathrm{d}\mu_\Delta(t) \geqslant K(p,m,M)\left(\int_E w(t)f^p(t)\mathrm{d}\mu_\Delta(t)\right)^{1/p} \times \left(\int_E w(t)g^q(t)\mathrm{d}\mu_\Delta(t)\right)^{1/q}, \tag{1.5}$$

where

$$K(p,m,M) = |p|^{1/p}|q|^{1/q}\frac{(M-m)^{1/p}|mM^p - Mm^p|^{1/q}}{|M^p - m^p|}. \tag{1.6}$$

If $0 < p < 1$ *or* $p < 0$, *then* (1.5) *is reversed provided either* $\int_E wg^q \mathrm{d}\mu_\Delta > 0$ *or* $\int_E wf^p \mathrm{d}\mu_\Delta > 0$.

2. Minkowski inequalities

Theorem 1.4 also holds if we have a finite number of functions. The next theorem gives an inequality of Minkowski type for infinitely many functions. In the sequel, we assume that all occurring integrals are finite.

THEOREM 2.1. (Integral Minkowski inequality) *Let* $(X, \mathscr{M}, \mu_\Delta)$ *and* $(Y, \mathscr{L}, \nu_\Delta)$ *be time scale measure spaces and let* u, v, *and* f *be nonnegative functions on* X, Y, *and* $X \times Y$, *respectively. If* $p \geqslant 1$, *then*

$$\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}} \leqslant \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \tag{2.1}$$

holds provided all integrals in (2.1) *exists. If* $0 < p < 1$ *and*

$$\int_X \left(\int_Y fv\mathrm{d}\nu_\Delta\right)^p u\mathrm{d}\mu_\Delta > 0, \quad \int_Y fv\mathrm{d}\nu_\Delta > 0 \tag{2.2}$$

holds, then (2.1) *is reversed. If* $p < 0$ *and* (2.2) *and*

$$\int_X f^p u\mathrm{d}\mu_\Delta > 0, \tag{2.3}$$

hold, then (2.1) *is reversed as well.*

Proof. Let $p \geqslant 1$. Put

$$H(x) = \int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y).$$

Now, by using Fubini's theorem (Theorem 1.1) and Hölder's inequality (Theorem 1.3) on time scales, we have

$$\begin{aligned}
&\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x) = \int_X H(x)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\\
&= \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right) H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\\
&= \int_Y \left(\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\right) v(y)\mathrm{d}\nu_\Delta(y)\\
&\leqslant \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} \left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\\
&= \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}}
\end{aligned}$$

and hence

$$\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} \leqslant \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y).$$

For $p < 0$ and $0 < p < 1$, the corresponding results can be obtained similarly. □

THEOREM 2.2. (Converse of integral Minkowski inequality) *Let $(X, \mathscr{M}, \mu_\Delta)$ and $(Y, \mathscr{L}, \nu_\Delta)$ be time scale measure spaces and let u, v, and f be nonnegative functions on X, Y, and $X \times Y$, respectively. Suppose*

$$0 < m \leqslant \frac{f(x,y)}{\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)} \leqslant M \quad \textit{for all} \quad x \in X,\ y \in Y.$$

If $p \geqslant 1$, then

$$\begin{aligned}
&\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}\\
&\qquad \geqslant K(p,m,M) \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) \qquad (2.4)
\end{aligned}$$

provided all integrals in (2.4) exist, where $K(p,m,M)$ is defined by (1.6). If $0 < p < 1$ and (2.2) holds, then (2.4) is reversed. If $p < 0$ and (2.2) and (2.3) hold, then (2.4) is reversed as well.

Proof. Let $p \geqslant 1$. Put

$$H(x) = \int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y).$$

Then by using Fubini's theorem (Theorem 1.1) and the converse Hölder inequality (Theorem 1.5) on time scales, we get

$$\begin{aligned}
\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x) &= \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right) H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x) \\
&= \int_Y \left(\int_X f(x,y)H^{p-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\right) v(y)\mathrm{d}\nu_\Delta(y) \\
&\geqslant K(p,m,M)\int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{1/p} \\
&\quad \times \left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}} v(y)\mathrm{d}\nu_\Delta(y).
\end{aligned}$$

Dividing both sides by $\left(\int_X H^p(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{p-1}{p}}$, we obtain (2.4). For $0 < p < 1$ and $p < 0$, the corresponding results can be obtained similarly. □

Let the functions f, u, v be defined as in Theorem 2.1. Now we define the rth power mean $M^{[r]}(f,\mu_\Delta)$ of the function f with respect to the measure μ_Δ by

$$M^{[r]}(f,\mu_\Delta) = \begin{cases} \left(\dfrac{\int_X f^r(x,y)u(x)\mathrm{d}\mu_\Delta(x)}{\int_X u(x)\mathrm{d}\mu_\Delta(x)}\right)^{\frac{1}{r}} & \text{if} \quad r \neq 0, \\ \exp\left(\dfrac{\int_X \log f(x,y)u(x)\mathrm{d}\mu_\Delta(x)}{\int_X u(x)\mathrm{d}\mu_\Delta(x)}\right) & \text{if} \quad r = 0, \end{cases} \tag{2.5}$$

where $\int_X u\mathrm{d}\mu_\Delta > 0$.

COROLLARY 2.3. *Let* $0 < s \leqslant r$. *Then*

$$M^{[r]}(M^{[s]}(f,\mathrm{d}\nu_\Delta),\mathrm{d}\mu_\Delta) \geqslant K\left(\frac{r}{s},m,M\right) M^{[s]}(M^{[r]}(f,\mathrm{d}\mu_\Delta),\mathrm{d}\nu_\Delta).$$

Proof. By putting $p = r/s$ and replacing f by f^s in (2.4), raising to the power of $\frac{1}{s}$ and dividing by

$$\left(\int_X u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{r}} \left(\int_Y v(y)\mathrm{d}\nu_\Delta(y)\right)^{\frac{1}{s}},$$

we get the above result. □

3. Minkowski functionals

In this section, we will consider some functionals which arise from the Minkowski inequality. Similar results (but not for time scales measure spaces) can be found in [8].

Let f and v be fixed functions satisfying the assumptions of Theorem 2.1. Let us consider the functional M_1 defined by

$$\mathsf{M}_1(u) = \left[\int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p - \int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x),$$

where u is a nonnegative function on X such that all occurring integrals exist. Also, if we fix the functions f and u, then we can consider the functional

$$\mathsf{M}_2(v) = \int_Y \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) - \left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}},$$

where v is a nonnegative function on Y such that all occurring integrals exist.

REMARK 3.1. (i) It is obvious that M_1 and M_2 are positive homogeneous, i.e., $\mathsf{M}_1(au) = a\mathsf{M}_1(u)$, and $\mathsf{M}_2(av) = a\mathsf{M}_2(v)$, for any $a > 0$.

(ii) If $p \geqslant 1$ or $p < 0$, then $\mathsf{M}_1(u) \geqslant 0$, and if $0 < p < 1$, then $\mathsf{M}_1(u) \leqslant 0$.

(iii) If $p \geqslant 1$, then $\mathsf{M}_2(v) \geqslant 0$, and if $p < 1$ and $p \neq 0$, then $\mathsf{M}_2(v) \leqslant 0$.

THEOREM 3.2. (i) *If $p \geqslant 1$ or $p < 0$, then M_1 is superadditive. If $0 < p < 1$, then M_1 is subadditive.*

(ii) *If $p \geqslant 1$, then M_2 is superadditive. If $p < 1$ and $p \neq 0$, then M_2 is subadditive.*

(iii) *Suppose u_1 and u_2 are nonnegative functions such that $u_2 \geqslant u_1$. If $p \geqslant 1$ or $p < 0$, then*

$$0 \leqslant \mathsf{M}_1(u_1) \leqslant \mathsf{M}_1(u_2), \tag{3.1}$$

and if $0 < p < 1$, then (3.1) is reversed.

(iv) *Suppose v_1 and v_2 are nonnegative functions such that $v_2 \geqslant v_1$. If $p \geqslant 1$, then*

$$0 \leqslant \mathsf{M}_2(v_1) \leqslant \mathsf{M}_2(v_2), \tag{3.2}$$

and if $p < 1$ and $p \neq 0$, then (3.2) is reversed.

Proof. First we show (i). We have

$$
\begin{aligned}
&\mathsf{M}_1(u_1+u_2)-\mathsf{M}_1(u_1)-\mathsf{M}_1(u_2)\\
&=\left[\int_Y\left(\int_X f^p(x,y)(u_1+u_2)(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad-\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p(u_1+u_2)(x)\mathrm{d}\mu_\Delta(x)\\
&\quad-\left[\int_Y\left(\int_X f^p(x,y)u_1(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad+\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u_1(x)\mathrm{d}\mu_\Delta(x)\\
&\quad-\left[\int_Y\left(\int_X f^p(x,y)u_2(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad+\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u_2(x)\mathrm{d}\mu_\Delta(x)\\
&=\left[\int_Y\left(\int_X f^p(x,y)(u_1+u_2)(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad-\left[\int_Y\left(\int_X f^p(x,y)u_1(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p\\
&\quad-\left[\int_Y\left(\int_X f^p(x,y)u_2(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}}v(y)\mathrm{d}\nu_\Delta(y)\right]^p.
\end{aligned}
$$

Using the Minkowski inequality (1.4) for integrals (Theorem 1.4) with p replaced by $1/p$, we have

$$
\mathsf{M}_1(u_1+u_2)-\mathsf{M}_1(u_1)-\mathsf{M}_1(u_2)\begin{cases}\geqslant 0 & \text{if}\quad p\geqslant 1 \text{ or } p<0,\\ \leqslant 0 & \text{if}\quad 0<p\leqslant 1.\end{cases}\tag{3.3}
$$

So, M_1 is superadditive for $p\geqslant 1$ or $p<0$, and it is subadditive for $0<p\leqslant 1$. The proof of (ii) is similar: After a simple calculation, we have

$$
\begin{aligned}
&\mathsf{M}_2(v_1+v_2)-\mathsf{M}_2(v_1)-\mathsf{M}_2(v_2)\\
&=\left[\int_X\left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}\\
&\quad+\left[\int_X\left(\int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}\\
&\quad-\left[\int_X\left(\int_Y f(x,y)(v_1+v_2)(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}}.
\end{aligned}
$$

Using the Minkowski inequality (2.1) for integrals (Theorem 2.1), we have that this is nonnegative for $p \geqslant 1$ and nonpositive for $p < 1$ and $p \neq 0$. Now we show (iii). If $p \geqslant 1$ or $p < 0$, then using superadditivity and positivity of M_1, $u_2 \geqslant u_1$ implies

$$\mathsf{M}_1(u_2) = \mathsf{M}_1(u_1 + (u_2 - u_1)) \geqslant \mathsf{M}_1(u_1) + \mathsf{M}_1(u_2 - u_1) \geqslant \mathsf{M}_1(u_1),$$

and the proof of (3.1) is established. If $0 < p < 1$, then using subadditivity and negativity of M_1, $u_2 \geqslant u_1$ implies

$$\mathsf{M}_1(u_2) \leqslant \mathsf{M}_1(u_1) + \mathsf{M}_1(u_2 - u_1) \leqslant \mathsf{M}_1(u_1).$$

The proof of (iv) is similar. □

REMARK 3.3. From Theorem 3.2, we obtain a refinement of the discrete Minkowski inequality given in [8]. Namely, put $X, Y \subseteq \mathbb{N}$ and let u be Δ-measurable on X and v_1 and v_2 be Δ-measurable on Y such that $u(i) = u_i \geqslant 0$, $i \in X$, $v_1(j) = n_j \geqslant 0$, $v_2(j) = p_j \geqslant 0$, $j \in Y$. Then, for fixed f and u, the function M_2 has the form

$$\mathsf{M}_2(v_1) = \sum_{j\in Y} n_j \left(\sum_{i\in X} u_i a_{ij}^p\right)^{1/p} - \left(\sum_{i\in X} u_i \left(\sum_{j\in Y} n_j a_{ij}\right)^p\right)^{1/p},$$

where $f(i,j) = a_{ij} \geqslant 0$. If $p \geqslant 1$, then the mapping M_2 is superadditive, and $p_j \geqslant n_j$ for all $j \in Y$ implies

$$\begin{aligned} 0 &\leqslant \sum_{j\in Y} n_j \left(\sum_{i\in X} u_i a_{ij}^p\right)^{1/p} - \left(\sum_{i\in X} u_i \left(\sum_{j\in Y} n_j a_{ij}\right)^p\right)^{1/p} \\ &\leqslant \sum_{j\in Y} p_j \left(\sum_{i\in X} u_i a_{ij}^p\right)^{1/p} - \left(\sum_{i\in X} u_i \left(\sum_{j\in Y} p_j a_{ij}\right)^p\right)^{1/p} \end{aligned}$$

provided all occurring sums are finite.

COROLLARY 3.4. (i) *Suppose u_1 and u_2 are nonnegative functions such that $Cu_2 \geqslant u_1 \geqslant cu_2$, where $c, C \geqslant 0$. If $p \geqslant 1$ or $p < 0$, then*

$$c\mathsf{M}_1(u_2) \leqslant \mathsf{M}_1(u_1) \leqslant C\mathsf{M}_1(u_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

(ii) *Suppose v_1 and v_2 are nonnegative functions such that $Cv_2 \geqslant v_1 \geqslant cv_2$, where $c, C \geqslant 0$. If $p \geqslant 1$, then*

$$c\mathsf{M}_2(v_2) \leqslant \mathsf{M}_2(v_1) \leqslant C\mathsf{M}_2(v_2),$$

and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

COROLLARY 3.5. *If v_1 and v_2 are nonnegative functions such that $v_2 \geqslant v_1$, then*

$$M^{[0]}\left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y),\mu_\Delta\right)-\int_Y M^{[0]}(f,\mu_\Delta)v_1(y)\mathrm{d}\nu_\Delta(y)$$
$$\leqslant M^{[0]}\left(\int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y),\mu_\Delta\right)-\int_Y M^{[0]}(f,\mu_\Delta)v_2(y)\mathrm{d}\nu_\Delta(y), \quad (3.4)$$

where $M^{[0]}(f,\mu_\Delta)$ is defined in (2.5).

REMARK 3.6. If the measures are discrete, then from Corollary 3.5, we get the following result: Let $u_j, v_i, w_i, a_{ij} > 0$ for all $i = 1,\dots,n$ and all $j = 1,\dots,k$. Put $U = \sum_{j=1}^k u_j$. If $v_i \leqslant w_i$ for all $i = 1,\dots,n$, then

$$\prod_{j=1}^k\left(\sum_{i=1}^n v_i a_{ij}\right)^{\frac{u_j}{U}} - \sum_{i=1}^n v_i\left(\prod_{j=1}^k a_{ij}^{\frac{u_j}{U}}\right) \leqslant \prod_{j=1}^k\left(\sum_{i=1}^n w_i a_{ij}\right)^{\frac{u_j}{U}} - \sum_{i=1}^n w_i\left(\prod_{j=1}^k a_{ij}^{\frac{u_j}{U}}\right).$$

This inequality is a refinement of the discrete Hölder inequality

$$\prod_{j=1}^k\left(\sum_{i=1}^n w_i a_{ij}\right)^{\frac{u_j}{U}} \geqslant \sum_{i=1}^n w_i\left(\prod_{j=1}^k a_{ij}^{\frac{u_j}{U}}\right).$$

The next result gives another property of M_1, but a similar result can also be stated for M_2.

THEOREM 3.7. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a concave function. Suppose u_1 and u_2 are nonnegative functions such that*

$$\varphi\circ u_1, \quad \varphi\circ u_2, \quad \varphi\circ(\alpha u_1+(1-\alpha)u_2)$$

are Δ-integrable for $\alpha \in [0,1]$. If $p \geqslant 1$, then

$$\mathsf{M}_1(\varphi\circ(\alpha u_1+(1-\alpha)u_2)) \geqslant \alpha\mathsf{M}_1(\varphi\circ u_1)+(1-\alpha)\mathsf{M}_1(\varphi\circ u_2),$$

and if $0 < p < 1$, then the above inequality is reversed.

Proof. We show this only for $p \geqslant 1$ as the other case follows similarly. Since φ is concave, we have

$$\varphi(\alpha u_1+(1-\alpha)u_2)) \geqslant \alpha\varphi(u_1)+(1-\alpha)\varphi(u_2).$$

Now, from (3.1) and (3.3), we have

$$\begin{aligned}\mathsf{M}_1(\varphi\circ(\alpha u_1+(1-\alpha)u_2)) &\geqslant \mathsf{M}_1(\alpha(\varphi\circ u_1)+(1-\alpha)(\varphi\circ u_2))\\ &\geqslant \mathsf{M}_1(\alpha(\varphi\circ u_1))+\mathsf{M}_1((1-\alpha)(\varphi\circ u_2))\\ &\geqslant \alpha\mathsf{M}_1(\varphi\circ u_1)+(1-\alpha)\mathsf{M}_1(\varphi\circ u_2),\end{aligned}$$

and the proof is established. □

Let f, u and v be fixed functions satisfying the assumptions of Theorem 2.1. Let us define functionals M_3 and M_4 by

$$\mathsf{M}_3(A) = \left[\int_Y \left(\int_A f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^p - \int_A \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)$$

and

$$\mathsf{M}_4(B) = \int_B \left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y) - \left[\int_X \left(\int_B f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{p}},$$

where $A \subseteq X$ and $B \subseteq Y$.

The following theorem establishes superadditivity and monotonicity of the mappings M_3 and M_4.

THEOREM 3.8. (i) *Suppose* $A_1, A_2 \subseteq X$ *and* $A_1 \cap A_2 = \emptyset$. *If* $p \geqslant 1$ *or* $p < 0$, *then*

$$\mathsf{M}_3(A_1 \cup A_2) \geqslant \mathsf{M}_3(A_1) + \mathsf{M}_3(A_2),$$

and if $0 < p < 1$, *then the above inequality is reversed.*

(ii) *Suppose* $A_1, A_2 \subseteq X$ *and* $A_1 \subseteq A_2$. *If* $p \geqslant 1$ *or* $p < 0$, *then*

$$\mathsf{M}_3(A_1) \leqslant \mathsf{M}_3(A_2),$$

and if $0 < p < 1$, *then the above inequality is reversed.*

(iii) *Suppose* $B_1, B_2 \subseteq Y$ *and* $B_1 \cap B_2 = \emptyset$. *If* $p \geqslant 1$, *then*

$$\mathsf{M}_4(B_1 \cup B_2) \geqslant \mathsf{M}_4(B_1) + \mathsf{M}_4(B_2),$$

and if $p < 1$ *and* $p \neq 0$, *then the above inequality is reversed.*

(iv) *Suppose* $B_1, B_2 \subseteq Y$ *and* $B_1 \subseteq B_2$. *If* $p \geqslant 1$, *then*

$$\mathsf{M}_4(B_1) \leqslant \mathsf{M}_4(B_2),$$

and if $p < 1$ *and* $p \neq 0$, *then the above inequality is reversed.*

The proof of Theorem 3.8 is omitted as it is similar to the proof of Theorem 3.2.

REMARK 3.9. For $p \geqslant 1$, if S_m is a subset of Y with m elements and if $S_m \supseteq S_{m-1} \supseteq \ldots \supseteq S_2$, then we have

$$\mathsf{M}_4(S_m) \geqslant \mathsf{M}_4(S_{m-1}) \geqslant \ldots \geqslant \mathsf{M}_4(S_2) \geqslant 0$$

and $\mathsf{M}_4(S_m) \geqslant \max\{\mathsf{M}_4(S_2) : S_2$ is any subset of S_m with 2 elements$\}$.

4. Beckenbach–Dresher inequalities

THEOREM 4.1. *Let $(X,\mathscr{M},\mu_\Delta)$, $(X,\mathscr{M},\lambda_\Delta)$ and $(Y,\mathscr{L},\nu_\Delta)$ be time scale measure spaces. Suppose u and w are nonnegative functions on X, v is a nonnegative function on Y, f is a nonnegative function on $X\times Y$ with respect to the measure $(\mu_\Delta\times\nu_\Delta)$, and g is a nonnegative function on $X\times Y$ with respect to the measure $(\lambda_\Delta\times\nu_\Delta)$. If*

$$s\geqslant 1,\quad q\leqslant 1\leqslant p,\quad \text{and}\quad q\neq 0 \tag{4.1}$$

or

$$s<0,\quad p\leqslant 1\leqslant q,\quad \text{and}\quad p\neq 0, \tag{4.2}$$

then

$$\frac{\left[\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X\left(\int_Y g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \leqslant \int_Y \frac{\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}}v(y)\mathrm{d}\nu_\Delta(y) \tag{4.3}$$

provided all occurring integrals in (4.3) exist. If

$$0<s\leqslant 1,\quad p\leqslant 1,\quad q\leqslant 1,\quad \text{and}\quad q\neq 0, \tag{4.4}$$

then (4.3) is reversed.

Proof. Assume (4.1) or (4.2). By using the integral Minkowski inequality (2.1) and Hölder's inequality (1.3), we have

$$\begin{aligned}
&\frac{\left[\int_X\left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X\left(\int_Y g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}}\\
&\leqslant \frac{\left[\int_Y\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)\mathrm{d}\nu_\Delta(y)\right]^s}{\left[\int_Y\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1}{q}} v(y)\mathrm{d}\nu_\Delta(y)\right]^{s-1}}\\
&=\left[\int_Y\left(\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}\right)^{\frac{1}{s}} v(y)\mathrm{d}\nu_\Delta(y)\right]^s\\
&\quad\times\left[\int_Y\left(\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1-s}{q}}\right)^{\frac{1}{1-s}} v(y)\mathrm{d}\nu_\Delta(y)\right]^{1-s}\\
&\leqslant \int_Y\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{1-s}{q}} v(y)\mathrm{d}\nu_\Delta(y).
\end{aligned}$$

If (4.4) holds, then the reversed inequality in (4.3) can be proved in a similar way. □

REMARK 4.2. If $X,Y \subseteq \mathbb{R}^n$, then Theorem 4.1 is a generalization of the well-known Beckenbach–Dresher inequality which states that for nonnegative real functions f, g and for $p \geqslant 1 \geqslant q \geqslant 0$, we have

$$\left(\frac{\int_E (f+g)^p \mathrm{d}\varphi}{\int_E (f+g)^q \mathrm{d}\varphi}\right)^{\frac{1}{p-q}} \leqslant \left(\frac{\int_E f^p \mathrm{d}\varphi}{\int_E f^q \mathrm{d}\varphi}\right)^{\frac{1}{p-q}} + \left(\frac{\int_E g^p \mathrm{d}\varphi}{\int_E g^q \mathrm{d}\varphi}\right)^{\frac{1}{p-q}}. \tag{4.5}$$

Some historical facts about (4.5) and new results which generalize (4.5) are given in [5, 9]. For a time scale analogue of (4.5), see [1, Theorem 8.2].

5. Beckenbach–Dresher functionals

Let f, g, u, w be fixed functions satisfying the assumptions of Theorem 4.1. We define the Beckenbach–Dresher functional $\mathsf{BD}(v)$ by

$$\mathsf{BD}(v) = \int_Y \frac{\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}} v(y)\mathrm{d}\nu_\Delta(y) - \frac{\left[\int_X \left(\int_Y f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}},$$

where we suppose that all occurring integrals exist.

THEOREM 5.1. *If* (4.1) *or* (4.2) *holds, then*

$$\mathsf{BD}(v_1+v_2) \geqslant \mathsf{BD}(v_1) + \mathsf{BD}(v_2). \tag{5.1}$$

If $v_2 \geqslant v_1$, *then*

$$\mathsf{BD}(v_1) \leqslant \mathsf{BD}(v_2). \tag{5.2}$$

If $C, c \geqslant 0$ *and* $Cv_2 \geqslant v_1 \geqslant cv_2$, *then*

$$C\mathsf{BD}(v_2) \geqslant \mathsf{BD}(v_1) \geqslant c\mathsf{BD}(v_1). \tag{5.3}$$

If (4.4) *holds, then* (5.1), (5.2) *and* (5.3) *are reversed.*

Proof. Assume (4.1) or (4.2). Then we have

$$\begin{aligned}
&\mathsf{BD}(v_1+v_2) - \mathsf{BD}(v_1) - \mathsf{BD}(v_2) \\
&= \frac{\left[\int_X \left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v_1(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}} \\
&\quad + \frac{\left[\int_X \left(\int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}}
\end{aligned}$$

$$-\frac{\left[\int_X \left(\int_Y f(x,y)v_1(y)\mathrm{d}\nu_\Delta(y)+\int_Y f(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_Y g(x,y)v_1(y)\mathrm{d}\nu_\Delta(y)+\int_Y g(x,y)v_2(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}}$$
$$\geqslant 0,$$

where in the last inequality we used (4.3) from Theorem 4.1. Using Theorem 4.1 again, $v_2 \geqslant v_1$ implies

$$\mathsf{BD}(v_2)=\mathsf{BD}(v_1+(v_2-v_1))\geqslant \mathsf{BD}(v_1)+\mathsf{BD}(v_2-v_1)\geqslant \mathsf{BD}(v_1).$$

The proof of (5.3) is similar. If (4.4) holds, then the reversed inequalities of (5.1), (5.2) and (5.3) can be proved in a similar way. □

Let f, g, u, v, w be fixed functions. We define a functional BD_1 by

$$\mathsf{BD}_1(A)=\int_A \frac{\left(\int_X f^p(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{s}{p}}}{\left(\int_X g^q(x,y)w(x)\mathrm{d}\lambda_\Delta(x)\right)^{\frac{s-1}{q}}}v(y)\mathrm{d}\nu_\Delta(y)$$
$$-\frac{\left[\int_X \left(\int_A f(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^p u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{s}{p}}}{\left[\int_X \left(\int_A g(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^q w(x)\mathrm{d}\lambda_\Delta(x)\right]^{\frac{s-1}{q}}},$$

where $A\subseteq Y$.

For BD_1, the following result holds.

THEOREM 5.2. (i) *Suppose* $A_1,A_2\subseteq Y$ *and* $A_1\cap A_2=\emptyset$. *If* (4.1) *or* (4.2) *holds, then*

$$\mathsf{BD}_1(A_1\cup A_2)\geqslant \mathsf{BD}_1(A_1)+\mathsf{BD}_1(A_2),$$

and if (4.4) *holds, then the above inequality is reversed.*

(ii) *Suppose* $A_1,A_2\subseteq Y$ *and* $A_1\subseteq A_2$. *If* (4.1) *or* (4.2) *holds, then*

$$\mathsf{BD}_1(A_1)\leqslant \mathsf{BD}_1(A_2),$$

and if (4.4) *holds, then the above inequality is reversed.*

The proof of Theorem 5.2 is omitted as it is similar to the proof of Theorem 5.1.

REMARK 5.3. If $S_k\subseteq Y$ has k elements and if $S_m\supseteq S_{m-1}\supseteq\ldots\supseteq S_2$, then (4.1) or (4.2) implies

$$\mathsf{BD}_1(S_m)\geqslant \mathsf{BD}_1(S_{m-1})\geqslant\cdots\geqslant \mathsf{BD}_1(S_2)\geqslant 0$$

and $\mathsf{BD}_1(S_m)\geqslant \max\{\mathsf{BD}_1(S_2): S_2$ is any subset of S_m with 2 elements$\}$, while (4.4) implies the reversed inequalities with $\max$ replaced by $\min$.

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