Journal of Mathematical Inequalities
Volume 7, Number 3 (2013), 377–387 doi:10.7153/jmi-07-33

A HILBERT INTEGRAL INEQUALITY WITH HURWITZ ZETA FUNCTION

HE LEPING AND GAO MINGZHE

(*Communicated by M. Krnić*)

Abstract. It is shown that an extension of the Hilbert integral inequality can be established by introducing two parameters and a proper logarithm function. The constant factor expressed by product of the gamma function and the Hurwitz Zeta function is proved to be the best possible. And base on it, some interesting special results are enumerated. As applications, some equivalent forms are given.

1. Introduction and Lemmas

Let $f(x), g(x) \in L^2(0,+\infty)$. Then

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\frac{x}{y}\right) f(x)g(y)}{x-y}dxdy \leqslant \pi^2\left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)\,dx\right\}^{\frac{1}{2}} \tag{1.1}$$

and

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leqslant \pi\left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)\,dx\right\}^{\frac{1}{2}}. \tag{1.2}$$

They are the famous Hilbert integral inequalities, where the constant factors π^2 and π are the best possible. And the equalities in (1.1) and (1.2) hold if and only if $f(x)=0$, or $g(x)=0$. These results can be found in papers [1] and [2]. Owing to the importance of the Hilbert inequality and the Hilbert type inequality in Mathematical analysis and applications, some mathematicians have been studying them. Recently, various refinements, extensions and generalizations of (1.2) appear in a great deal of the articles (such as [3]–[10] etc.). However, the research articles of (1.1) are few. The aim of the present paper is to build an inequality of the form

$$\int_0^\infty\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha f(x)g(y)}{\left|x^\lambda-y^\lambda\right|}dxdy \leqslant C(\alpha,\lambda)\left\{\int_0^\infty \omega(x)f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty \omega(x)g^2(x)\,dx\right\}^{\frac{1}{2}} \tag{1.3}$$

Mathematics subject classification (2010): 26D15, 33E20.
Keywords and phrases: Hilbert integral inequality, gamma function, Hurwitz Zeta function, Bernoulli number, the best constant.

where $\alpha, \lambda > 0$, to discuss the constant factor of which is related to Riemann Zeta function, to give some important and especial results, and then to study some equivalent forms.

In order to prove our main results, we need to introduce the Hurwitz Zeta functions and some lemmas.

Let $Rez > 0$ and $0 < q < 1$. Then the Hurwitz Zeta function is defined by

$$\zeta(z,q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}e^{-qt}}{1-e^{-t}} dt, \tag{1.4}$$

where $\Gamma(z)$ is the gamma function.

The Hurwitz Zeta function can be expressed by the series as follows:

$$\zeta(z,q) = \sum_{k=0}^\infty \frac{1}{(k+q)^z}, \tag{1.5}$$

where $Rez > 1$, $q \neq 0, -1, -2, \cdots$.

When $q = 1$, we obtain from (15) the Riemann Zeta function:

$$\zeta(z) = \sum_{k=1}^\infty \frac{1}{k^z}. \tag{1.6}$$

LEMMA 1.1. *Let α and λ be positive numbers. Then*

$$\int_0^\infty \frac{s^\alpha e^{-\frac{\lambda s}{2}}}{1-e^{-\lambda s}} ds = \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \zeta(\alpha+1, \tfrac{1}{2}), \tag{1.7}$$

where $\Gamma(z)$ is the gamma function and $\zeta(z,q)$ is the Hurwitz Zeta function, and that $Rez > 0$ and $0 < q < 1$.

Proof. By substitution $t = \lambda s$, we can obtain $\int_0^\infty \frac{s^\alpha e^{-\frac{\lambda s}{2}}}{1-e^{-\lambda s}} ds = \frac{1}{\lambda^{\alpha+1}} \int_0^\infty \frac{t^\alpha e^{-\frac{t}{2}}}{1-e^{-t}} dt$. It follows from (1.4) that the relation (1.7) holds. □

LEMMA 1.2. *Let α be a positive number. Then*

$$\zeta\left(\alpha+1, \tfrac{1}{2}\right) = \left(2^{\alpha+1} - 1\right) \zeta(\alpha+1) \tag{1.8}$$

where $\zeta(z) = \sum_{k=1}^\infty \frac{1}{k^z}$, $(Rez > 1)$ is the Riemann Zeta function.

Proof. When $\alpha > 0$, it is known from (1.5) that

$$\zeta\left(\alpha+1, \tfrac{1}{2}\right) = \sum_{k=0}^\infty \frac{1}{\left(k+\frac{1}{2}\right)^{\alpha+1}} = \sum_{k=0}^\infty \frac{2^{\alpha+1}}{(2k+1)^{\alpha+1}}.$$

Based on (1.6), we have $\zeta(\alpha+1)=\sum_{k=0}^{\infty}\frac{1}{(k+1)^{\alpha+1}}$ and $\frac{1}{2^{\alpha+1}}\zeta(\alpha+1)=\sum_{k=1}^{\infty}\frac{1}{(2k)^{\alpha+1}}$.

It is obvious that

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{\alpha+1}}=\zeta(\alpha+1)-\frac{1}{2^{\alpha+1}}\zeta(\alpha)=\frac{2^{\alpha+1}-1}{2^{\alpha+1}}\zeta(\alpha+1).$$

It follows that the equality (1.8) holds. □

LEMMA 1.3. *With the assumptions as Lemma 1.1, and define $C(\alpha,\lambda)$ by*

$$C(\alpha,\lambda)=\frac{2\left(2^{\alpha+1}-1\right)}{\lambda^{\alpha+1}}\Gamma(\alpha+1)\zeta(\alpha+1). \tag{1.9}$$

Then

$$\int_0^{\infty}\frac{\left|\ln\frac{1}{u}\right|^{\alpha}}{\left|1-u^{\lambda}\right|}u^{\frac{\lambda}{2}-1}du=C(\alpha,\lambda). \tag{1.10}$$

Proof. It is easy to deduce that

$$\begin{aligned}\int_0^{\infty}\frac{\left|\ln\frac{1}{u}\right|^{\alpha}}{\left|1-u^{\lambda}\right|}u^{\frac{\lambda}{2}-1}du&=\int_0^{1}\frac{\left|\ln\frac{1}{u}\right|^{\alpha}}{1-u^{\lambda}}u^{\frac{\lambda}{2}-1}du+\int_1^{\infty}\frac{\left|\ln\frac{1}{u}\right|^{\alpha}}{u^{\lambda}-1}u^{\frac{\lambda}{2}-1}du\\&=\int_0^{1}\frac{\left(\ln\frac{1}{u}\right)^{\alpha}}{1-u^{\lambda}}u^{\frac{\lambda}{2}-1}du+\int_1^{\infty}\frac{(\ln u)^{\alpha}}{u^{\lambda}-1}u^{\frac{\lambda}{2}-1}du\\&=\int_0^{1}\frac{\left(\ln\frac{1}{u}\right)^{\alpha}}{1-u^{\lambda}}u^{\frac{\lambda}{2}-1}du+\int_0^{1}\frac{\left(\ln\frac{1}{v}\right)^{\alpha}}{1-v^{\lambda}}v^{\frac{\lambda}{2}-1}dv\\&=2\int_0^{1}\frac{\left(\ln\frac{1}{u}\right)^{\alpha}}{1-u^{\lambda}}u^{\frac{\lambda}{2}-1}du=2\int_0^{\infty}\frac{s^{\alpha}e^{-\frac{\lambda s}{2}}}{1-e^{-\lambda s}}ds.\end{aligned} \tag{1.11}$$

From (1.7), (1.8) and (1.11), it follows that the relation (1.10) holds. □

LEMMA 1.4. *With the assumptions as Lemma 1.1, define a function by*

$$\omega(\alpha,\lambda,x)=\int_0^{\infty}\frac{\left|\ln\left(\frac{x}{y}\right)\right|^{\alpha}}{\left|x^{\lambda}-y^{\lambda}\right|}\left(\frac{x}{y}\right)^{1-\frac{\lambda}{2}}dy.$$

Then

$$\omega(\alpha,\lambda,x)=C(\alpha,\lambda)x^{1-\lambda}, \tag{1.12}$$

where $C(\alpha,\lambda)$ is defined by (1.9).

Proof. It is easy to deduce that

$$\omega(\alpha,\lambda,x) = \int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha}{|x^\lambda - y^\lambda|}\left(\frac{x}{y}\right)^{1-\frac{\lambda}{2}} dy = \int_0^\infty \frac{\left|\ln\left(\frac{x}{y}\right)\right|^\alpha}{x^\lambda\left|1-\frac{y^\lambda}{x^\lambda}\right|}\left(\frac{x}{y}\right)^{1-\frac{\lambda}{2}} dy$$
$$= x^{1-\lambda}\int_0^\infty \frac{\left|\ln\frac{1}{u}\right|^\alpha}{|1-u^\lambda|} u^{\frac{\lambda}{2}-1} du.$$

It follows from (1.10) that the equality (1.12) holds. □

2. Main Results

In this section, we will prove our assertions by using the above Lemmas.

THEOREM 2.1. *Let α and λ be two positive numbers, f and g be two real functions. If $\int_0^\infty x^{1-\lambda} f^2(x)dx < +\infty$ and $\int_0^\infty x^{1-\lambda} g^2(x)dx < +\infty$, then*

$$\int_0^\infty\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha f(x)g(y)}{|x^\lambda - y^\lambda|} dxdy \leqslant C(\alpha,\lambda)\left\{\int_0^\infty x^{1-\lambda} f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty x^{1-\lambda} g^2(x)\,dx\right\}^{\frac{1}{2}}, \tag{2.1}$$

where $C(\alpha,\lambda)$ is defined by (1.9), and $C(\alpha,\lambda)$ in (2.1) is the best possible. And the equality in (2.1) holds if and only if $f(x) = 0$, or $g(x) = 0$.

Proof. We may apply the Cauchy inequality to estimate the left-hand side of (2.1) as follows:

$$\int_0^\infty\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha f(x)g(y)}{|x^\lambda - y^\lambda|} dxdy$$
$$= \int_0^\infty\int_0^\infty \left(\frac{\left|\ln\frac{x}{y}\right|^\alpha}{|x^\lambda - y^\lambda|}\right)^{\frac{1}{2}}\left(\frac{x}{y}\right)^{\frac{2-\lambda}{4}} f(x)\left(\frac{\left|\ln\frac{x}{y}\right|^\alpha}{|x^\lambda - y^\lambda|}\right)^{\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{2-\lambda}{4}} g(y)dxdy$$
$$\leqslant \left(\int_0^\infty \omega(\alpha,\lambda,x) f^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty \omega(\alpha,\lambda,x) g^2(x)dx\right)^{\frac{1}{2}}, \tag{2.2}$$

where $\omega(\alpha,\lambda,x) = \int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha}{|x^\lambda - y^\lambda|}\left(\frac{x}{y}\right)^{1-\frac{\lambda}{2}} dy$.

It follows from (1.12) and (2.2) that the inequality (2.1) is valid. □

REMARK 1. The paper [11] provides a unified treatment of a general Hilbert-type inequality with a special emphasis to a homogeneous kernel, hence the relation (2.1) can be deduced by it. In fact, according to Corollary 4 in the paper [11], we can put $A_1 = A_2 = \frac{2-\lambda}{4}$, $\lambda \in (0,2]$, $p = q = 2$, $K(x,y) = \frac{\left|\ln \frac{x}{y}\right|^{\alpha}}{\left|x^{\lambda} - y^{\lambda}\right|}$, and by substitution $u = \frac{y}{x}$, we obtain

$$L = k(pA_2)^{\frac{1}{p}} k(2-\lambda-qA_1)^{\frac{1}{q}} = k(1-\tfrac{\lambda}{2}) = \int_0^{\infty} K(1,u) u^{\frac{\lambda}{2}-1} du$$
$$= \int_0^{\infty} \frac{\left|\ln \frac{1}{u}\right|^{\alpha}}{\left|1-u^{\lambda}\right|} u^{\frac{\lambda}{2}-1} du = C(\alpha,\lambda).$$

Therefore, the relation (2.1) is obtained immediately by the relation (29) in the paper [11].

It is noticeable that the relation (2.1) can be extended for non-negative conjugate parameters p and q by the relation (29) in the paper [11].

If (2.2) takes the form of the equality, then there exist a pair of non-zero constants c_1 and c_2 such that $c_1 \frac{\left|\ln \frac{x}{y}\right|^{\alpha}}{\left|x^{\lambda}-y^{\lambda}\right|} f^2(x) \left(\frac{x}{y}\right)^{1-\frac{\lambda}{2}} = c_2 \frac{\left|\ln \frac{x}{y}\right|^{\alpha}}{\left|x^{\lambda}-y^{\lambda}\right|} g^2(y) \left(\frac{y}{x}\right)^{1-\frac{\lambda}{2}}$, a.e. on $(0,+\infty) \times (0,+\infty)$.

Thereupon we have $c_1 x^{2-\lambda} f^2(x) = c_2 y^{2-\lambda} g^2(y) = C_0$ (constant) a.e. on $(0,+\infty) \times (0,+\infty)$.

Without losing the generality, we can suppose that $c_1 \neq 0$, then

$$\int_0^{\infty} x^{1-\lambda} f^2(x)\, dx = \frac{C_0}{c_1} \int_0^{\infty} x^{-1} dx.$$

This contradicts that $0 < \int_0^{\infty} x^{1-\lambda} f^2(x)\, dx < +\infty$. Hence it is impossible to take the equality in (2.2). It shows that the equality in (2.1) holds if and only if $f(x) = 0$, or $g(x) = 0$. So the inequality (2.1) is valid.

It remains to need only to show that $C(\alpha,\lambda)$ in (2.1) is the best possible, $\forall 0 < \varepsilon < \lambda$.

Define two functions by

$$\tilde{f}(x) = \begin{cases} 0 & x \in (0,1) \\ x^{-\frac{2-\lambda+\varepsilon}{2}} & x \in [1,\infty) \end{cases} \quad \text{and} \quad \tilde{g}(y) = \begin{cases} 0 & y \in (0,1) \\ y^{-\frac{2-\lambda+\varepsilon}{2}} & y \in [1,\infty) \end{cases}$$

It is easy to deduce that

$$\int_0^{+\infty} x^{1-\lambda} \tilde{f}^2(x)\, dx = \int_0^{+\infty} y^{1-\lambda} \tilde{g}^2(y)\, dy = \frac{1}{\varepsilon}.$$

If $C(\alpha,\lambda)$ is not the best possible, then there exists $K > 0$, such that

$$H(\alpha,\lambda) = \int_0^{\infty}\int_0^{\infty} \frac{\left|\ln \frac{x}{y}\right|^{\alpha} \tilde{f}(x)\tilde{g}(y)}{\left|x^{\lambda}-y^{\lambda}\right|} dxdy \leqslant K \left(\int_1^{\infty} x^{1-\lambda} \tilde{f}^2(x)\, dx\right)^{\frac{1}{2}} \left(\int_1^{\infty} y^{1-\lambda} \tilde{g}^2(y)\, dy\right)^{\frac{1}{2}}$$
$$= \frac{K}{\varepsilon} < \frac{C(\alpha,\lambda)}{\varepsilon}. \tag{2.3}$$

On the other hand, we have

$$\begin{aligned}
H(\alpha,\lambda) &= \int_0^\infty\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha \tilde{f}(x)\tilde{g}(y)}{\left|x^\lambda - y^\lambda\right|}dxdy \\
&= \int_1^\infty\int_1^\infty \frac{\left\{x^{-\frac{2-\lambda+\varepsilon}{2}}\right\}\left\{\left|\ln\frac{x}{y}\right|^\alpha y^{-\frac{2-\lambda+\varepsilon}{2}}\right\}}{\left|x^\lambda - y^\lambda\right|}dxdy \\
&= \int_1^\infty \left\{\int_1^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha y^{-\frac{2-\lambda+\varepsilon}{2}}}{x^\lambda\left|1-\left(\frac{y}{x}\right)^\lambda\right|}dy\right\}\left\{x^{-\frac{2-\lambda+\varepsilon}{2}}\right\}dx \\
&= \int_1^\infty \left\{\int_{1/x}^\infty \frac{\left|\ln\frac{1}{u}\right|^\alpha u^{-\frac{2-\lambda+\varepsilon}{2}}}{\left|1-u^\lambda\right|}du\right\}\left\{x^{-1-\varepsilon}\right\}dx \\
&= \int_1^\infty \left\{\int_{1/x}^1 \frac{\left|\ln\frac{1}{u}\right|^\alpha u^{-\frac{2-\lambda+\varepsilon}{2}}}{\left|1-u^\lambda\right|}du\right\}\left\{x^{-1-\varepsilon}\right\}dx \\
&\quad + \int_1^\infty \left\{\int_1^\infty \frac{\left|\ln\frac{1}{u}\right|^\alpha u^{-\frac{2-\lambda+\varepsilon}{2}}}{\left|1-u^\lambda\right|}du\right\}\left\{x^{-1-\varepsilon}\right\}dx \\
&= \int_0^1 \left\{\int_{1/u}^\infty x^{-1-\varepsilon}dx\right\}\frac{\left|\ln\frac{1}{u}\right|^\alpha u^{-\frac{2-\lambda+\varepsilon}{2}}}{\left|1-u^\lambda\right|}du \\
&\quad + \int_1^\infty \left\{\int_1^\infty \frac{\left|\ln\frac{1}{u}\right|^\alpha u^{-\frac{2-\lambda+\varepsilon}{2}}}{\left|1-u^\lambda\right|}du\right\}\left\{x^{-1-\varepsilon}\right\}dx \\
&= \frac{1}{\varepsilon}\int_0^1 \frac{\left|\ln\frac{1}{u}\right|^\alpha u^{-\frac{2-\lambda-\varepsilon}{2}}}{\left|1-u^\lambda\right|}du + \frac{1}{\varepsilon}\int_1^\infty \frac{\left|\ln\frac{1}{u}\right|^\alpha u^{-\frac{2-\lambda+\varepsilon}{2}}}{\left|1-u^\lambda\right|}du. \qquad (2.4)
\end{aligned}$$

When ε is sufficiently small, we obtain from (2.4) that

$$\begin{aligned}
H(\alpha,\lambda) &= \frac{1}{\varepsilon}\left(\int_0^1 \frac{\left|\ln\frac{1}{u}\right|^\alpha u^{-\frac{2-\lambda}{2}}}{\left|1-u^\lambda\right|}du + \circ_1(1)\right) + \frac{1}{\varepsilon}\left(\int_1^\infty \frac{\left|\ln\frac{1}{u}\right|^\alpha u^{-\frac{2-\lambda}{2}}}{\left|1-u^\lambda\right|}du + \circ_2(1)\right) \\
&= \frac{1}{\varepsilon}\left(\int_0^\infty \frac{\left|\ln\frac{1}{u}\right|^\alpha u^{\frac{\lambda}{2}-1}}{\left|1-u^\lambda\right|}du + \circ(1)\right) \quad (\varepsilon \to 0)
\end{aligned}$$

It follows from (1.10) that

$$H(\alpha,\lambda) = \frac{1}{\varepsilon}\left(C(\alpha,\lambda) + \circ(1)\right) \quad (\varepsilon \to 0). \qquad (2.5)$$

Evidently, the inequality (2.5) is in contradiction with (2.3). Therefore, the constant factor $C(\alpha,\lambda)$ in (2.1) is the best possible. Thus the proof of Theorem is completed.

REMARK 2. The papers [12] and [13] deal with the best possible constant factors in Hilbert-type inequalities, where the paper [12] deals with a particular homogeneous kernel $K(x,y) = (x+y)^{-s}$.

It proves that if the parameters A_1 and A_2 satisfy condition $pA_2 + qA_1 = 2-\lambda$, then the constant L is the best possible. The constant factor $C(\alpha,\lambda)$ in relation (2.1) is obviously the best possible, because the above-mentioned condition is satisfied (see Remark 1). The paper [13] adopts Fubini's theorem to prove the best possible constant in Hilbert-type inequalities for a wide class of homogeneous functions, this method is recommendable, and its technique is worthy.

Let $x, y > 0$ and λ be a positive number. If α is odd, then we have the following result.

THEOREM 2.2. *Let n be a positive integer and $\lambda > 0$. If $\int_0^\infty x^{1-\lambda} f^2(x)dx < +\infty$ and $\int_0^\infty x^{1-\lambda} g^2(x)dx < +\infty$, then*

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n-1} f(x)g(y)}{x^\lambda - y^\lambda} dxdy \leqslant C(2n-1,\lambda)\left\{\int_0^\infty x^{1-\lambda} f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty x^{1-\lambda} g^2(x)\,dx\right\}^{\frac{1}{2}}, \tag{2.6}$$

where $C(2n-1,\lambda) = \frac{2^{2n-1}\left(2^{2n}-1\right)}{n}\left(\frac{\pi}{\lambda}\right)^{2n} B_n$, and the $B_{n's}$ are the Bernoulli numbers, viz. $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, etc., and the constant factor $C(2n-1,\lambda)$ is the best possible. And then the equality in (2.6) holds if and only if $f(x) = 0$, or $g(x) = 0$.

Proof. We need only to verify the constant factor $C(2n-1,\lambda)$ in (2.6). When $\alpha = 2n-1$, it is known from (2.1) and (1.9) that

$$C(2n-1,\lambda) = \frac{2\left(2^{2n}-1\right)}{\lambda^{2n}}\Gamma(2n)\,\zeta(2n). = \frac{2\left(2^{2n}-1\right)}{\lambda^{2n}}(2n-1)!\sum_{k=1}^{\infty}\frac{1}{k^{2n}}.$$

It is known from the paper [14] that

$$\sum_{k=1}^{\infty}\frac{1}{k^{2n}} = \frac{2^{2n-1}\pi^{2n}}{(2n)!}B_n,$$

where the $B_{n's}$ are the Bernoulli numbers, viz. $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, etc.

It follows that the constant factor $C(2n-1,\lambda)$ in (2.6) is correct. □

Based on (2.6), we can list some important and especial results.

When $n = 1$, according to (2.6) we obtain the following result:

COROLLARY 2.1. *If* $\int_0^\infty x^{1-\lambda} f^2(x)dx < +\infty$ *and* $\int_0^\infty x^{1-\lambda} g^2(x)dx < +\infty$*, then*

$$\int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right) f(x)g(y)}{x^\lambda - y^\lambda} dxdy \leqslant \left(\frac{\pi}{\lambda}\right)^2 \left\{\int_0^\infty x^{1-\lambda} f^2(x)\,dx\right\}^{\frac{1}{2}} \left\{\int_0^\infty x^{1-\lambda} g^2(x)\,dx\right\}^{\frac{1}{2}} \tag{2.7}$$

where the constant factor $\left(\frac{\pi}{\lambda}\right)^2$ *is the best possible. And the equality in (2.7) holds if and only if* $f(x) = 0$*, or* $g(x) = 0$*.*

In particular, when $\lambda = 1$, the inequality (2.7) is reduced to (1.1). It follows that inequalities (2.1), (2.6) and (2.7) are extensions of (1.1).

When $n = 2$ and $\lambda = 1$, the following result is gotten by (2.6).

COROLLARY 2.2. *If* $\int_0^\infty f^2(x)dx < +\infty$ *and* $\int_0^\infty g^2(x)dx < +\infty$*, then*

$$\int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right)^3 f(x)g(y)}{x-y} dxdy \leqslant 2\pi^4 \left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}} \left\{\int_0^\infty g^2(x)\,dx\right\}^{\frac{1}{2}}, \tag{2.8}$$

where the constant factor $2\pi^4$ *is the best possible. And the equality in (2.8) holds if and only if* $f(x) = 0$*, or* $g(x) = 0$*.*

When $n = \lambda = 2$*, the following result is obtained by (2.6).*

COROLLARY 2.3. *If* $\int_0^\infty x^{-1} f^2(x)dx < +\infty$ *and* $\int_0^\infty x^{-1} g^2(x)dx < +\infty$*, then*

$$\int_0^\infty \int_0^\infty \frac{\left(\ln \frac{x}{y}\right)^3 f(x)g(y)}{x^2-y^2} dxdy \leqslant \frac{\pi^4}{8} \left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}} \left\{\int_0^\infty g^2(x)\,dx\right\}^{\frac{1}{2}}, \tag{2.9}$$

where the constant factor $\frac{\pi^4}{8}$ *in (2.9) is the best possible. And the equality in (2.9) holds if and only if* $f(x) = 0$*, or* $g(x) = 0$*.*

Similarly, we can also establish a great deal of new inequalities, which are omitted here.

3. Some equivalent forms

As applications, we will build some new inequalities.

THEOREM 3.1. *Let* α *and* λ *be two positive numbers, and* f *be a real function. If* $\int_0^\infty x^{1-\lambda} f^2(x)dx < +\infty$*, then*

$$\int_0^\infty y^{\lambda-1} \left\{\int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\alpha}{\left|x^\lambda - y^\lambda\right|} f(x)\,dx\right\}^2 dy \leqslant (C(\alpha,\lambda))^2 \int_0^\infty x^{1-\lambda} f^2(x)\,dx, \tag{3.1}$$

where $C(\alpha,\lambda)$ is defined by (1.9), and $C(\alpha,\lambda)$ in (3.1) is the best possible, and the equality in (3.1) holds if and only if $f(x)=0$. And the inequalities (3.1) and (2.1) are equivalent.

Proof. First, we assume that the inequality (2.1) is valid. Set a real function $g(y)$ as

$$g(y)=y^{\lambda-1}\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha}{|x^\lambda-y^\lambda|}f(x)\,dx,\quad y\in(0,+\infty)$$

and using (2.1), we have

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha}{|x^\lambda-y^\lambda|}f(x)\,dx\right\}^2 dy=\int_0^\infty\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha}{|x^\lambda-y^\lambda|}f(x)\,g(y)dxdy$$

$$\leqslant C(\alpha,\lambda)\left\{\int_0^\infty x^{1-\lambda}f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{1-\lambda}g^2(y)dy\right\}^{\frac{1}{2}}$$

$$=C(\alpha,\lambda)\left\{\int_0^\infty x^{1-\lambda}f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha}{|x^\lambda-y^\lambda|}f(x)dx\right)^2 dy\right\}^{\frac{1}{2}}\qquad(3.2)$$

It follows from (3.2) that the inequality (3.1) is valid after some simplifications.

On the other hand, assume that the inequality (3.1) keeps valid. Applying Cauchy's inequality and (3.1) in turn, we have

$$\int_0^\infty\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha}{|x^\lambda-y^\lambda|}f(x)\,g(y)\,dxdy$$

$$=\int_0^\infty y^{\frac{\lambda-1}{2}}\left\{\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha}{|x^\lambda-y^\lambda|}f(x)\,dx\right\}y^{\frac{1-\lambda}{2}}g(y)\,dy$$

$$\leqslant\left\{\int_0^\infty y^{\lambda-1}\left(\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^\alpha}{|x^\lambda-y^\lambda|}f(x)\,dx\right)^2 dy\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{1-\lambda}g^2(y)\,dy\right\}^{\frac{1}{2}}$$

$$\leqslant\left\{(C(\alpha,\lambda))^2\int_0^\infty x^{1-\lambda}f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{1-\lambda}g^2(y)\,dy\right\}^{\frac{1}{2}}$$

$$=(C(\alpha,\lambda))^2\left\{\int_0^\infty x^{1-\lambda}f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty y^{1-\lambda}g^2(y)\,dy\right\}^{\frac{1}{2}}\qquad(3.3)$$

If the constant factor $(C(\alpha,\lambda))^2$ in (3.1) is not the best possible, then it is known from (3.3) that the constant factor $C(\alpha,\lambda)$ in (2.1) is also not the best possible. This is a contradiction. It is obvious that the equality in (3.1) holds if and only if $f(x)=0$. Therefore, the inequalities (3.1) and (2.1) are equivalent. The proof of Theorem is completed. □

REMARK 3. The relations (3.1) can be also deduced and extended for non-negative conjugate parameters p and q by the relation (30) in the paper [11] (see Remark 1.).

THEOREM 3.1. *Let f be a real function, and n be a positive integer and λ be a positive number.*

If $\int_0^\infty x^{1-\lambda} f^2(x)dx < +\infty$, then

$$\int_0^\infty y^{\lambda-1}\left\{\int_0^\infty \frac{(\ln(\frac{x}{y}))^{2n-1}}{x^\lambda - y^\lambda} f(x)\,dx\right\}^2 dy \leqslant (C(2n-1,\lambda))^2 \int_0^\infty x^{1-\lambda} f^2(x)\,dx, \tag{3.4}$$

where $C(2n-1,\lambda) = \frac{2^{2n-1}\left(2^{2n}-1\right)}{n}\left(\frac{\pi}{\lambda}\right)^{2n} B_n$, and the $B_{n's}$ are the Bernoulli numbers, viz. $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, $B_5 = \frac{5}{66}$, etc., and the constant factor $C(2n-1,\lambda)$ in (3.4) is the best possible, and then the equality in (3.4) holds if and only if $f(x)=0$. And the inequalities (3.4) and (2.6) are equivalent.

Its proof is similar to one of Theorem 3.1. Hence it is omitted.

Similarly, we can also establish some new inequalities which are respectively equivalent to the inequalities (2.7), (2.8) and (2.9). And they are omitted here.

Acknowledgement.

Authors would like to express their thanks to the Editors and the referees for their valuable comments and suggestions.

REFERENCES

[1] G. H. HARDY, J. E. LITTLEWOOD, G. POLYA, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

[2] KUANG JICHANG, *Applied Inequalities*, 3nd. Ed, Shandong Science and Technology Press, Jinan, 2004.

[3] YANG BICHENG, *On a Generalization of a Hilbert's Type Integral Inequality and Its Applications*, Math. Appl., Vol 16, 2 (2003), 82–86.

[4] ZHOU YU, GAO XUEMEI AND GAO MINGZHE, *An Extension of the Hilbert's Integral Inequality*, Journal of Inequalities and Applications, Volume 2009, Article ID 158690, 12 pages, 2009.

[5] GAO MINGZHE, HSU LIZHI, *A Survey of Various Refinements and Generalizations of Hilbert's Inequalities*, J. Math. Res. & Exp., Vol. 25, 2 (2005), 227–243.

[6] YANG BICHENG, *On a Basic Hilbert-type Integral Inequality and Extensions*, College Mathematics, Vol. 24, 1 (2008), 87–92.

[7] HONG YONG, *All-sided Generalization about Hardy-Hilbert's Integral Inequalities*, Acta Mathematica Sinica, Vol. 44, 4 (2001), 619–626.

[8] HU KE, *On Hilbert's Inequality*, Chin. Ann. Math., Ser. B, Vol. 13, 1 (1992), 35–39.

[9] HE LEPING, GAO MINGZHE, ZHOU YU, *On New Extensions of Hilbert's Integral Inequality*, International Journal of Mathematics and Mathematical Science, Volume 2008, Article ID 297508, 8 pages, 2008.

[10] YANG BICHENG, *A New Hilbert Type Integral Inequality and Its Generalization*, J. Jilin Univ. (Sci. Ed.), Vol. 43, 5 (2005), 580–584.

[11] M. KRNIĆ, J. PEČARIĆ, *General Hilbert's and Hardy's inequalities*, Math. Inequal. Appl. 8 (2005), 29–52.

[12] M. KRNIĆ, G. MINGZHE, J. PEČARIĆ, G. XUEMEI, *On the best constant in Hilbert's inequality*, Math. Inequal. Appl. 8 (2005), 317–329.

[13] I. PERIĆ, P. VUKOVIĆ, *Hardy-Hilbert's inequalities with a general homogeneous kernel*, Math. Inequal. Appl. 12 (2009), 525–536.

[14] WANG LIANXIANG, FANG DEXHI, *Mathematical Handbook*, People's Education Press, Beijing, 1979.

(Received September 3, 2012)

He Leping
College of Mathematics and Statistics
Jishou University
Hunan Jishou, 416000
People's Republic of China

Gao Mingzhe
Department of Mathematics and Computer Science
Normal College of Jishou University
Hunan Jishou, 416000
People's Republic of China
e-mail: mingzhegao@163.com