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INEQUALITIES RELATED TO HEINZ AND HERON MEANS

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Abstract. We present a matrix inequality related to Heinz and Heron means, and show that it is a refinement of some improved Heinz inequalities for matrices.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices and $\|\cdot\|$ stand for any unitarily invariant norm on M_n, i.e., $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. For $A = [a_{ij}] \in M_n$, the Hilbert-Schmidt norm of A is defined by

$$\|A\|_2 = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}.$$

It is known that the Hilbert-Schmidt norm is unitarily invariant.

In this paper, we always suppose that $A, B, X \in M_n$ with A and B positive semidefinite.

Let a and b be nonnegative real numbers. The Heinz means are defined as

$$H_v(a,b) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2}, \quad 0 \leqslant v \leqslant 1.$$

The Heron means [1] are defined as the linear interpolation between the geometric and the arithmetic means:

$$F_\alpha(a,b) = (1-\alpha)G(a,b) + \alpha A(a,b), \quad 0 \leqslant \alpha \leqslant 1,$$

where

$$G(a,b) = \sqrt{ab}, \quad A(a,b) = \frac{a+b}{2}$$

are the geometric mean and the arithmetic mean of a and b, respectively.

It is well known that

$$G(a,b) \leqslant H_v(a,b) \leqslant A(a,b), \quad 0 \leqslant v \leqslant 1 \tag{1.1}$$

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and

$$G(a,b) \leqslant F_\alpha(a,b) \leqslant A(a,b), \quad 0 \leqslant \alpha \leqslant 1. \tag{1.2}$$

After seeing the inequalities (1.1) and (1.2), it is hard not to be curious about the relationship between the Heinz and Heron means. Bhatia [1] proved that if $0 \leqslant v \leqslant 1$, then

$$H_v(a,b) \leqslant F_{\alpha(v)}(a,b), \tag{1.3}$$

where $\alpha(v) = 1 - 4\left(v - v^2\right)$.

It is natural to raise the question of whether there exists any matrix version for the inequality (1.3). A matrix version of the inequality (1.3) is

$$\left\|\frac{A^vXB^{1-v}+A^{1-v}XB^v}{2}\right\| \leqslant \left\|(1-\alpha(v))A^{1/2}XB^{1/2}+\alpha(v)\left(\frac{AX+XB}{2}\right)\right\|, \tag{1.4}$$

which was introduced by Bhatia [1]. Unfortunately, as pointed out in [1], the inequality (1.4) holds only in the trivial cases of v equal to 0, 1/2 and 1. Nevertheless, it is an interesting issue to know that whether it holds for some special unitarily invariant norms. This is a part of the motivation for the present paper.

In this paper, we prove that the inequality (1.4) holds at least for the Hilbert-Schmidt norm. Then we show that it is an improvement of some improved Heinz inequalities for matrices obtained by Kittaneh [6], Kittaneh and Manasrah [7], Zhan [9], and Zou [10], which are presented in Section 2.

2. Some improved Heinz inequalities

The matrix version of the inequality (1.1) was proved by Bhatia and Davis [2] says that if $0 \leqslant v \leqslant 1$, then

$$\left\|A^{1/2}XB^{1/2}\right\| \leqslant \left\|\frac{A^vXB^{1-v}+A^{1-v}XB^v}{2}\right\| \leqslant \left\|\frac{AX+XB}{2}\right\|. \tag{2.1}$$

The second part of the inequality (2.1) is known as Heinz inequality. Let $0 \leqslant v \leqslant 1$, $r_0 = \min\{v,\ 1-v\}$, Kittaneh [6] gave a refinement of the Heinz inequality as follows:

$$\left\|\frac{A^vXB^{1-v}+A^{1-v}XB^v}{2}\right\| \leqslant 2r_0\left\|A^{1/2}XB^{1/2}\right\| + (1-2r_0)\left\|\frac{AX+XB}{2}\right\|. \tag{2.2}$$

Meanwhile, Kittaneh and Manasrah [7] also obtained two refinements of the Heinz inequality for the Hilbert-Schmidt norm as follows:

$$\left\|\frac{A^vXB^{1-v}+A^{1-v}XB^v}{2}\right\|_2^2 \leqslant \left\|\frac{AX+XB}{2}\right\|_2^2 - 2r_0\left\|\frac{AX-XB}{2}\right\|_2^2, \tag{2.3}$$

$$\left\|\frac{A^vXB^{1-v}+A^{1-v}XB^v}{2}\right\|_2 \leqslant \left\|\frac{AX+XB}{2}\right\|_2 - r_0\left(\sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}\right)^2. \tag{2.4}$$

He, Zou and Qaisar [4] proved that

$$\left\| \frac{A^{v}XB^{1-v}+A^{1-v}XB^{v}}{2} \right\|^{2} \leqslant 2r_0 \left\| A^{1/2}XB^{1/2} \right\|^{2} + (1-2r_0) \left\| \frac{AX+XB}{2} \right\|^{2}.$$

It is weaker than the inequality (2.2) and it is equivalent to the inequality (2.3) for the Hilbert-Schmidt norm [4]. Zhan [9] proved that if $\frac{1}{4} \leqslant v \leqslant \frac{3}{4}$ and $-2 < t \leqslant 2$, then

$$\left\| \frac{A^{v}XB^{1-v}+A^{1-v}XB^{v}}{2} \right\| \leqslant \frac{1}{t+2} \left\| tA^{1/2}XB^{1/2} + AX + XB \right\|. \tag{2.5}$$

It is also a refinement of the Heinz inequality for matrices. Zou [10] proved that if $0 \leqslant v \leqslant 1$, then

$$\left\| \frac{A^{v}XB^{1-v}+A^{1-v}XB^{v}}{2} \right\|_2^{2} \leqslant \left\| \frac{AX+XB}{2} \right\|_2^{2} - 4v(1-v) \left\| \frac{AX-XB}{2} \right\|_2^{2}. \tag{2.6}$$

It is an improvement of (2.3).

3. Main results

In this section, we first present a matrix version of the inequality (1.3) for the Hilbert-Schmidt norm. And after that, we prove that it is stronger than the inequalities (2.2), (2.4), (2.5), and (2.6).

THEOREM 3.1. *Let* $\alpha(v) = 1 - 4\left(v - v^2\right)$. *If* $0 \leqslant v \leqslant 1$, *then*

$$\left\| \frac{A^{v}XB^{1-v}+A^{1-v}XB^{v}}{2} \right\|_2 \leqslant \left\| (1-\alpha(v)) A^{1/2}XB^{1/2} + \alpha(v) \frac{AX+XB}{2} \right\|_2. \tag{3.1}$$

Proof. Since A and B are positive semidefinite, it follows by the spectral theorem that there exist unitary matrices U, $V \in M_n$ such that

$$A = U\Lambda_1 U^* \text{ and } B = V\Lambda_2 V^*,$$

where

$$\Lambda_1 = diag(\lambda_1, \cdots, \lambda_n), \quad \Lambda_2 = diag(\mu_1, \cdots, \mu_n), \quad \lambda_i,\ \mu_i \geqslant 0, \quad i = 1, \cdots, n.$$

Let

$$Y = U^*XV = [y_{ij}],$$

then

$$\begin{aligned}
\frac{A^{v}XB^{1-v}+A^{1-v}XB^{v}}{2} &= \frac{(U\Lambda_1U^*)^{v} X (V\Lambda_2V^*)^{1-v} + (U\Lambda_1U^*)^{1-v} X (V\Lambda_2V^*)^{v}}{2} \\
&= \frac{(U\Lambda_1^{v}U^*) X \left(V\Lambda_2^{1-v}V^*\right) + \left(U\Lambda_1^{1-v}U^*\right) X (V\Lambda_2^{v}V^*)}{2} \\
&= \frac{U\Lambda_1^{v} (U^*XV) \Lambda_2^{1-v}V^* + U\Lambda_1^{1-v} (U^*XV) \Lambda_2^{v}V^*}{2} \\
&= U \left(\frac{\Lambda_1^{v}Y\Lambda_2^{1-v} + \Lambda_1^{1-v}Y\Lambda_2^{v}}{2} \right) V^*.
\end{aligned}$$

Therefore,

$$\left\|\frac{A^{v}XB^{1-v}+A^{1-v}XB^{v}}{2}\right\|_2^2=\left\|\frac{\Lambda_1^{v}Y\Lambda_2^{1-v}+\Lambda_1^{1-v}Y\Lambda_2^{v}}{2}\right\|_2^2=\sum_{i,j=1}^{n}\left(\frac{\lambda_i^{v}\mu_j^{1-v}+\lambda_i^{1-v}\mu_j^{v}}{2}\right)^2\left|y_{ij}\right|^2. \tag{3.2}$$

Similarly, we have

$$\left\|(1-\alpha(v))A^{1/2}XB^{1/2}+\alpha(v)\frac{AX+XB}{2}\right\|_2^2=\sum_{i,j=1}^{n}\left((1-\alpha(v))\sqrt{\lambda_i\mu_j}+\alpha(v)\frac{\lambda_i+\mu_j}{2}\right)^2\left|y_{ij}\right|^2.$$

It follows from the inequality (1.3) that

$$\sum_{i,j=1}^{n}\left(\frac{\lambda_i^{v}\mu_j^{1-v}+\lambda_i^{1-v}\mu_j^{v}}{2}\right)^2\left|y_{ij}\right|^2\leqslant\sum_{i,j=1}^{n}\left((1-\alpha(v))\sqrt{\lambda_i\mu_j}+\alpha(v)\frac{\lambda_i+\mu_j}{2}\right)^2\left|y_{ij}\right|^2.$$

This completes the proof. □

The following result implies that for the Hilbert-Schmidt norm, the inequality (3.1) is a refinement of the inequality (2.2).

THEOREM 3.2. *Let* $\alpha(v)=1-4\left(v-v^2\right)$, $r_0=\min\{v,\ 1-v\}$. *If* $0\leqslant v\leqslant 1$, *then*

$$(1-\alpha(v))\left\|A^{1/2}XB^{1/2}\right\|_2+\alpha(v)\left\|\frac{AX+XB}{2}\right\|_2\leqslant 2r_0\left\|A^{1/2}XB^{1/2}\right\|_2+(1-2r_0)\left\|\frac{AX+XB}{2}\right\|_2.$$

Proof. Let

$$l_1=(1-\alpha(v))\left\|A^{1/2}XB^{1/2}\right\|_2+\alpha(v)\left\|\frac{AX+XB}{2}\right\|_2$$

and

$$l_2=2r_0\left\|A^{1/2}XB^{1/2}\right\|_2+(1-2r_0)\left\|\frac{AX+XB}{2}\right\|_2.$$

By a small calculation, we have

$$l_2-l_1=\begin{cases}2v(1-2v)\left(\left\|\frac{AX+XB}{2}\right\|_2-\left\|A^{1/2}XB^{1/2}\right\|_2\right), & 0\leqslant v\leqslant\frac{1}{2}\\ 2(2v-1)(1-v)\left(\left\|\frac{AX+XB}{2}\right\|_2-\left\|A^{1/2}XB^{1/2}\right\|_2\right), & \frac{1}{2}\leqslant v\leqslant 1\end{cases}.$$

So, by the inequality (2.1), we have

$$l_2 - l_1 \geqslant 0.$$

This completes the proof. □

In what follows, we present an inequality, by which it follows from Theorem 3.2 that the inequality (3.1) refines the inequality (2.4).

THEOREM 3.3. *Let* $r_0 = \min\{v,\ 1-v\}$. *If* $0 \leqslant v \leqslant 1$, *then*

$$2r_0 \left\| A^{1/2} X B^{1/2} \right\|_2 + (1-2r_0) \left\| \frac{AX+XB}{2} \right\|_2 \leqslant \left\| \frac{AX+XB}{2} \right\|_2 - r_0 \left(\sqrt{\|AX\|_2} - \sqrt{\|XB\|_2} \right)^2.$$

Proof. Let

$$l_1 = 2r_0 \left\| A^{1/2} X B^{1/2} \right\|_2 + (1-2r_0) \left\| \frac{AX+XB}{2} \right\|_2$$

and

$$l_2 = \left\| \frac{AX+XB}{2} \right\|_2 - r_0 \left(\sqrt{\|AX\|_2} - \sqrt{\|XB\|_2} \right)^2.$$

By a small calculation, we have

$$l_2 - l_1 = 2r_0 \left(\left\| \frac{AX+XB}{2} \right\|_2 - \left\| A^{1/2} X B^{1/2} \right\|_2 - \frac{1}{2} \left(\sqrt{\|AX\|_2} - \sqrt{\|XB\|_2} \right)^2 \right).$$

It is known [7, 8] that

$$\left\| \frac{AX+XB}{2} \right\|_2 - \left\| A^{1/2} X B^{1/2} \right\|_2 - \frac{1}{2} \left(\sqrt{\|AX\|_2} - \sqrt{\|XB\|_2} \right)^2 \geqslant 0.$$

So, we have

$$l_2 - l_1 \geqslant 0.$$

This completes the proof. □

The next theorem shows that the inequality (3.1) is a refinement of the inequality (2.5) for the Hilbert-Schmidt norm.

THEOREM 3.4. *Let* $\alpha(v) = 1 - 4\left(v - v^2\right)$. *If* $\frac{2-\sqrt{2}}{4} \leqslant v \leqslant \frac{2+\sqrt{2}}{4}$ *and* $-2 < t \leqslant 2$, *then*

$$\left\| (1-\alpha(v)) A^{1/2} X B^{1/2} + \alpha(v) \frac{AX+XB}{2} \right\|_2 \leqslant \frac{1}{t+2} \left\| t A^{1/2} X B^{1/2} + AX + XB \right\|_2.$$

Proof. Let

$$l_1 = \left\| (1-\alpha(v)) A^{1/2} X B^{1/2} + \alpha(v) \frac{AX+XB}{2} \right\|_2^2,$$

$$l_2 = \left\| \frac{t}{t+2} A^{1/2} X B^{1/2} + \frac{2}{t+2} \left(\frac{AX+XB}{2} \right) \right\|_2^2 .$$

In a manner similar to the steps used to obtain (3.2), we have

$$l_1 = \sum_{i,j=1}^{n} \left((1-\alpha(v)) \sqrt{\lambda_i \mu_j} + \alpha(v) \frac{\lambda_i + \mu_j}{2} \right)^2 |y_{ij}|^2 ,$$

$$l_2 = \sum_{i,j=1}^{n} \left(\frac{t\sqrt{\lambda_i \mu_j}}{t+2} + \frac{2}{t+2} \left(\frac{\lambda_i + \mu_j}{2} \right) \right)^2 |y_{ij}|^2 .$$

Thus,

$$\begin{aligned} & l_2 - l_1 \\ &= \sum_{i,j=1}^{n} \left(\left(\frac{t\sqrt{\lambda_i \mu_j}}{t+2} + \frac{2}{t+2} \left(\frac{\lambda_i+\mu_j}{2} \right) \right)^2 - \left((1-\alpha(v)) \sqrt{\lambda_i \mu_j} + \alpha(v) \frac{\lambda_i+\mu_j}{2} \right)^2 \right) |y_{ij}|^2 \\ &= \sum_{i,j=1}^{n} \left(\begin{array}{l} \left(\left(\frac{t}{t+2} - 1 + \alpha(v) \right) \sqrt{\lambda_i \mu_j} + \left(\frac{2}{t+2} - \alpha(v) \right) \frac{\lambda_i + \mu_j}{2} \right) \\ \times \left(\left(\frac{t}{t+2} + 1 - \alpha(v) \right) \sqrt{\lambda_i \mu_j} + \left(\frac{2}{t+2} + \alpha(v) \right) \frac{\lambda_i + \mu_j}{2} \right) \end{array} \right) |y_{ij}|^2 \\ &= \sum_{i,j=1}^{n} \left(\begin{array}{l} \left(\left(-\frac{2}{t+2} + \alpha(v) \right) \sqrt{\lambda_i \mu_j} + \left(\frac{2}{t+2} - \alpha(v) \right) \frac{\lambda_i + \mu_j}{2} \right) \\ \times \left(\left(2 - \left(\frac{2}{t+2} + \alpha(v) \right) \right) \sqrt{\lambda_i \mu_j} + \left(\frac{2}{t+2} + \alpha(v) \right) \frac{\lambda_i + \mu_j}{2} \right) \end{array} \right) |y_{ij}|^2 \\ &= \sum_{i,j=1}^{n} \left(\begin{array}{l} \left(\left(\frac{2}{t+2} - \alpha(v) \right) \left(\frac{\lambda_i + \mu_j}{2} - \sqrt{\lambda_i \mu_j} \right) \right) \\ \times \left(2\sqrt{\lambda_i \mu_j} + \left(\frac{2}{t+2} + \alpha(v) \right) \left(\frac{\lambda_i + \mu_j}{2} - \sqrt{\lambda_i \mu_j} \right) \right) \end{array} \right) |y_{ij}|^2 . \end{aligned}$$

Since $\frac{2-\sqrt{2}}{4} \leqslant v \leqslant \frac{2+\sqrt{2}}{4}$ and $-2 < t \leqslant 2$, we have

$$\frac{2}{t+2} - \alpha(v) \geqslant \frac{2}{2+2} - \alpha(v) = -4v^2 + 4v - \frac{1}{2} \geqslant 0.$$

So,

$$l_2 - l_1 \geqslant 0.$$

This completes the proof. $\square$

Combining the triangle inequality and next result, we know that the inequality (3.1) is a refinement of the inequality (2.6).

THEOREM 3.5. *Let* $\alpha(v) = 1 - 4(v - v^2)$. *If* $0 \leqslant v \leqslant 1$, *then*

$$\left\| \frac{AX+XB}{2} \right\|_2^2 - 4v(1-v) \left\| \frac{AX-XB}{2} \right\|_2^2 = (1-\alpha(v)) \left\| A^{1/2} X B^{1/2} \right\|_2^2 + \alpha(v) \left\| \frac{AX+XB}{2} \right\|_2^2 .$$

Proof. Let

$$l = \left\|\frac{AX+XB}{2}\right\|_2^2 - 4v(1-v)\left\|\frac{AX-XB}{2}\right\|_2^2.$$

Note that

$$\left\|\frac{AX+XB}{2}\right\|_2^2 = \frac{1}{4}\left(\|AX\|_2^2 + \|XB\|_2^2\right) + \frac{1}{2}\left\|A^{1/2}XB^{1/2}\right\|_2^2 \tag{3.3}$$

and

$$\left\|\frac{AX-XB}{2}\right\|_2^2 = \frac{1}{4}\left(\|AX\|_2^2 + \|XB\|_2^2\right) - \frac{1}{2}\left\|A^{1/2}XB^{1/2}\right\|_2^2. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$l = (1-\alpha(v))\left\|A^{1/2}XB^{1/2}\right\|_2^2 + \alpha(v)\left\|\frac{AX+XB}{2}\right\|_2^2.$$

This completes the proof. □

4. Remarks

REMARK 4.1. Drissi [3] gave another refinement of the Heinz inequality, which says that if $\frac{1}{4} \leqslant v \leqslant \frac{3}{4}$, then

$$\left\|\frac{A^vXB^{1-v} + A^{1-v}XB^v}{2}\right\| \leqslant \left\|\int_0^1 A^xXB^{1-x}dx\right\|. \tag{4.1}$$

In view of the inequalities (3.1) and (4.1), we want to know the relationship between them. It should be noticed that neither (3.1) nor (4.1) is uniformly better than the other for the Hilbert-Schmidt norm. Here, we give two examples:

EXAMPLE 4.1. Let

$$A = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad v = \frac{1}{4}.$$

We have

$$\left\|\int_0^1 A^xXB^{1-x}dx\right\|_2 = 4.0345$$

and

$$\left\|\frac{3}{4}A^{1/2}XB^{1/2} + \frac{1}{4}\left(\frac{AX+XB}{2}\right)\right\|_2 = 3.8779.$$

EXAMPLE 4.2. Let

$$A = \begin{bmatrix} 1000 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad v = \frac{1}{4}.$$

We have

$$\left\| \int_0^1 A^x X B^{1-x} dx \right\|_2 = 144.6235$$

and

$$\left\| \frac{3}{4} A^{1/2} X B^{1/2} + \frac{1}{4} \left(\frac{AX + XB}{2} \right) \right\|_2 = 148.8454.$$

REMARK 4.2. An inequality weaker than (1.4) is

$$\left\| \frac{A^v X B^{1-v} + A^{1-v} X B^v}{2} \right\| \leqslant (1 - \alpha(v)) \left\| A^{1/2} X B^{1/2} \right\| + \alpha(v) \left\| \frac{AX + XB}{2} \right\|. \tag{4.2}$$

It is natural to raise the question of whether the inequality (4.2) holds for all $v \in [0, 1]$. Numerical experiments on computer show that this inequality is true. Therefore, we pose the following:

CONJECTURE. The inequality (4.2) holds for all unitarily invariant norms.

REMARK 4.3. Let

$$r_0 = \min\{v, 1 - v\}, \ a = \sqrt{\|AX\|_2} - \sqrt{\|XB\|_2}.$$

Recently, Hu [5] proved that

$$\left\| \frac{A^v X B^{1-v} + A^{1-v} X B^v}{2} \right\|_2^2 + 2 r_0 a^2 \left\| A^{1/2} X B^{1/2} \right\|_2 + \frac{1}{2} r_0 a^4 \leqslant \left\| \frac{AX + XB}{2} \right\|_2^2. \tag{4.3}$$

It is also an improvement of Heinz inequality. It follows from the following inequality [5]:

$$4 \left\| A^{1/2} X B^{1/2} \right\|_2^2 + 4a \left\| A^{1/2} X B^{1/2} \right\|_2 + a^4 \leqslant \left\| \frac{AX + XB}{2} \right\|_2^2$$

that the inequality (3.1) is a refinement of (4.3).

REMARK 4.4. It is easy to see that

$$\frac{a^v b^{1-v} + a^{1-v} b^v}{2} \leqslant 2 r_0 \sqrt{ab} + (1 - 2r_0) \frac{a + b}{2}, \quad 0 \leqslant v \leqslant 1, \quad r_0 = \min\{v, 1 - v\}.$$

At this stage it is natural to raise the following question: Is it true that

$$\left\| \frac{A^v X B^{1-v} + A^{1-v} X B^v}{2} \right\| \leqslant \left\| 2 r_0 A^{1/2} X B^{1/2} + (1 - 2r_0) \frac{AX + XB}{2} \right\| ?$$

This would be a strengthening of the inequality (2.2). To answer this we have to decide whether the function

$$f(x) = \frac{\cosh \beta x}{1 - \beta + \beta \cosh x}, \quad 0 \leqslant \beta \leqslant 1$$

is positive definite.

REMARK 4.5. Let $\alpha(v) = 1 - 4\left(v - v^2\right)$ and $r_0 = \min\{v, 1-v\}$. It is known that if $0 \leqslant v \leqslant 1$, then

$$(1-\alpha(v))\sqrt{ab} + \alpha(v)\frac{a+b}{2} \leqslant 2r_0\sqrt{ab} + (1-2r_0)\frac{a+b}{2}.$$

A matrix version of this inequality is

$$\left\|(1-\alpha(v))A^{1/2}XB^{1/2} + \alpha(v)\frac{AX+XB}{2}\right\| \leqslant \left\|2r_0A^{1/2}XB^{1/2} + (1-2r_0)\frac{AX+XB}{2}\right\|. \tag{4.4}$$

By the inequality (23) of [1], we know that if $v \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$, the inequality (4.4) is true. This restriction on v is necessary.

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