Journal of
Mathematical
Inequalities
Volume 7, Number 3 (2013), 399–412 doi:10.7153/jmi-07-35

A ONE–PARAMETER FAMILY OF BIVARIATE MEANS

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(*Communicated by G. Toader*)

Abstract. A one-parameter family of bivariate means is introduced. Members of the new family of means are derived from a bivariate symmetric mean. It is shown that new means are symmetric in their variables. Several inequalities involving parametric versions of two Seiffert means, the Neuman-Sándor mean, and the logarithmic means are obtained. It is shown that the last four means belong to the family of the Schwab-Borchardt means. Among inequalities established in this paper some provide generalizations of known results obtained recently by several researchers.

1. Introduction

The history of mean values is long and laden with detail. In recent years a significant progress has been made in theory of bivariate means with special emphasis on inequalities involving those mens. Among means of two variables the Schwab-Borchardt mean has attracted attention of several researchers. The interested reader is referred to [1], [2], [14], [16], [18], [19], [23] and to the references therein. Importance of this mean is justified by the fact that some known means such as Seiffert means P and T (see [20] and [21]), the logarithmic mean L and the mean M studied in [15], [18], [19], [11] can be represented as the Schwab-Borchardt means of other elementary bivariate means such as arithmetic mean, geometric mean, and the power mean of order 2. For a recent developments in the theory of inequalities for the Seiffert means see [3], [5], [6], [7] [8], [9], [10], [12], [16], [23], [24].

In this paper we introduce and study a one-parameter family of bivariate means which are obtained from an arbitrary bivariate symmetric mean by forming an arithmetic convex combinations of the variables of the underlying mean. In Section 2 we recall definitions of several known bivariate means. Also, some preliminary facts needed in the subsequent sections are included there. A one-parameter family of means is introduced in Section 3. Therein we give some elementary properties of those means. Several inequalities involving means under discussion are established in the next section of this paper.

Mathematics subject classification (2010): Primary: 26E60; Secondary: 26D07, 26D20.
Keywords and phrases: Bivariate means, Seiffert means, Neuman-Sándor mean, logarithmic mean, Schwab-Borchardt mean, inequalities, convex combinations.

Paper JMI-07-35

2. Definitions and preliminaries

Let $a,b>0$. In order to avoid trivialities we will always assume that $a\neq b$. The unweighted arithmetic mean of a and b is defined as

$$A=\frac{a+b}{2}.$$

Other unweighted bivariate means used in this paper are the harmonic mean H, geometric mean G, root-square mean Q and the contra-harmonic mean C which are defined as follows

$$H=\frac{2ab}{a+b},\quad G=\sqrt{ab},\quad Q=\sqrt{\frac{a^2+b^2}{2}},\quad C=\frac{a^2+b^2}{a+b}. \tag{2.1}$$

Let

$$v=\frac{a-b}{a+b}. \tag{2.2}$$

Clearly $0<|v|<1$. One can easily verify that the means defined in (2.1) all can be expressed in terms of A and v. We have

$$\begin{aligned} H&=A(1-v^2), \quad G=A\sqrt{1-v^2},\\ Q&=A\sqrt{1+v^2}, \quad C=A(1+v^2). \end{aligned} \tag{2.3}$$

Other bivariate means utilized in this paper include the first and the second Seiffert means, denoted by P and T, respectively, the Neuman-Sándor mean M, and the logarithmic mean L. Recall that

$$\begin{aligned} P&=A\frac{v}{\sin^{-1}v}, \quad T=A\frac{v}{\tan^{-1}v},\\ M&=A\frac{v}{\sinh^{-1}v}, \quad L=A\frac{v}{\tanh^{-1}v} \end{aligned} \tag{2.4}$$

(see [20], [21], [18]).

All the means mentioned above are comparable. It is known that

$$H<G<L<P<A<M<T<Q<C \tag{2.5}$$

(see, e.g., [18]).

The four means listed in (2.4) are special cases of the Schwab-Borchardt mean SB which is defined as follows

$$SB(a,b)\equiv SB=\begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} & \text{if } a<b,\\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & \text{if } b<a, \end{cases}$$

(see, e.g., [1], [2]). This mean has been studied recently in [14], [18], and [19]. It is well known that the mean SB is strict, nonsymmetric and homogeneous of degree one in its variables.

It has been pointed out in [18] that

$$\begin{aligned} P &= SB(G,A), \quad T = SB(A,Q), \\ M &= SB(Q,A), \quad L = SB(A,G). \end{aligned} \tag{2.6}$$

3. Definition and basic properties of the one-parameter means

The goal of this section is to introduce a family of bivariate which depend on the parameter p which satisfies $|p| \leqslant 1$. First we define two nonnegative numbers w_1 and w_2:

$$w_1 = \frac{1+p}{2}, \quad w_2 = \frac{1-p}{2}. \tag{3.1}$$

Clearly $w_1 + w_2 = 1$. We associate with the pair (a,b) a pair of positive numbers (x,y), where

$$x = w_1 a + w_2 b, \quad y = w_1 b + w_2 a. \tag{3.2}$$

Thus x and y are the convex combinations of a and b. One can easily verify that $a < x < y < b$ if $a < b$ or $b < y < x < a$ if $b < a$.

For the sake of presentation let N stand for a bivariate symmetric mean. We define a mean $N_p(a,b) \equiv N_p$ as follows

$$N_p(a,b) = N(x,y). \tag{3.3}$$

We will call the mean N_p the p-mean or the p-mean generated by N.

We will present now some elementary properties of the p-means. Using (3.3), (3.1), and (3.2) we see that

$$N_{-p}(a,b) = N(y,x) = N(x,y) = N_p(a,b).$$

Thus the function $p \to N_p$ is an even function. To this end we will assume that $0 \leqslant p \leqslant 1$. It follows from (3.1) and (3.2) that

$$N_0 = A, \quad N_1 = N. \tag{3.4}$$

Moreover, the function $p \to N_p$ is strictly decreasing if $N < A$, i.e.,

$$N_1 \leqslant N_p \leqslant N_0 \tag{3.5}$$

or is strictly increasing if $N > A$, i.e.,

$$N_0 \leqslant N_p \leqslant N_1. \tag{3.6}$$

We now present formulas for the p-means mentioned in Section 2. Let us begin with the case when $N = A$. We have

$$A_p = A_p(a,b) = A(x,y) = A.$$

Thus we shall always write A instead of A_p when no confusion would arise. To obtain the p-versions of the eight means listed in (2.1) and (2.3) let us introduce a quantity u, where

$$u = \frac{x-y}{x+y}. \tag{3.7}$$

Using (3.2) and (2.2) we obtain

$$u = pv. \tag{3.8}$$

Since $0 < |v| < 1$, $0 < |u| < p \leqslant 1$

Making use of (2.3) we obtain formulas for the p-means derived from means listed in (2.3)

$$\begin{aligned} H_p &= A(1-u^2), \quad G_p = A\sqrt{1-u^2}, \\ Q_p &= A\sqrt{1+u^2}, \quad C_p = A(1+u^2). \end{aligned} \tag{3.9}$$

Similarly, using (2.4) we obtain

$$\begin{aligned} P_p &= A\frac{u}{\sin^{-1} u}, \quad T_p = A\frac{u}{\tan^{-1} u}, \\ M_p &= A\frac{u}{\sinh^{-1} u}, \quad L_p = A\frac{u}{\tanh^{-1} u}. \end{aligned} \tag{3.10}$$

Another one-parameter generalizations of the Seiffert means P and T has been proposed by G. Toader (see [22]). For instance, a generalization of P reads as follows

$$P_{\mathscr{M},q} = \mathscr{M}\frac{u}{\sin^{-1} u},$$

where $\mathscr{M} = \mathscr{M}(a,b)$ is a mean of a and b,

$$u = \frac{a-b}{q\mathscr{M}},$$

and q is a positive number. With $\mathscr{M} = A$ and $q = 2/p$ mean $P_{\mathscr{M},q}$ becomes P_p. It is also worth mentioning that in [7] the authors have investigated first two means defined in (3.10) with $p = 2q$.

For the later use let us record the fact that the means P_p, T_p, M_p, and L_p can be represented as the Schwab-Borchardt means. Making use of (2.6) we obtain

$$\begin{aligned} P_p &= SB(G_p, A), \quad T_p = SB(A, Q_p), \\ M_p &= SB(Q_p, A), \quad L_p = SB(A, G_p). \end{aligned} \tag{3.11}$$

For this reason we call (G_p, A), (A, Q_p), (Q_p, A) and (A, G_p) the pairs of generating means.

We close this section with the following remarks. The idea of using the p-means was motivated by a recent development in theory of means. Let R and S be bivariate symmetric means and let $0 \leqslant \lambda \leqslant 1$. Many researchers (see, e.g., [3], [5], [6], [8], [9], [10], [12], [24]) have studied problems of finding all values of λ for which inequality $R(\lambda r + (1-\lambda)s, \lambda s + (1-\lambda)r) < (>) S(r,s)$ is satisfied for all positive numbers r and

s. Let us note that with $\lambda = (1+p)/2 = w_1$ we have $1-\lambda = (1-p)/2 = w_2$. Thus the inequality in question can be written as $R_p(r,s) < (>) S(r,s)$. With the parameter λ used instead of p formula (3.8) should be changed $u = (2\lambda - 1)v$, which is a little bit more cumbersome in analytic computations than (3.8) is.

4. Inequalities involving the p-means

The goal of this section is to establish several inequalities involving the p-means defined in Section 3. Recall that in what follows the parameter p is assumed to be such that $0 < p \leqslant 1$.

Our first result reads as follows.

THEOREM 4.1. *The following inequalities*

$$H_p < G_p < L_p < P_p < M_p < T_p < Q_p < C_p, \tag{4.1}$$

$$(G_pA^2)^{1/3} < (AL_p)^{1/2} < P_p < \frac{A+L_p}{2} < \frac{G_p+2A}{3}, \tag{4.2}$$

$$(AQ_p{}^2)^{1/3} < (Q_pM_p)^{1/2} < T_p < \frac{Q_p+M_p}{2} < \frac{A+2Q_p}{3}, \tag{4.3}$$

$$(Q_pA^2)^{1/3} < (AT_p)^{1/2} < M_p < \frac{A+T_p}{2} < \frac{Q_p+2A}{3}, \tag{4.4}$$

$$(AG_p{}^2)^{1/3} < (G_pP_p)^{1/2} < L_p < \frac{G_p+P_p}{2} < \frac{A+2G_p}{3} \tag{4.5}$$

are satisfied.

Proof. For the proof of (4.1) let us notice that the inequalities (2.5) are valid provided the parameter v of all means listed in (2.3) and (2.4) is such that $0 < |v| < 1$. Therefore, they are also satisfied if v is replaced by u. The assertion now follows using (2.5), (3.9), and (3.10). For the proof of the inequalities (4.2)–(4.5) we shall utilize the following inequalities for the Schwab-Borchardt mean [19], [17]:

$$(rs^2)^{1/3} < \big(sSB(s,r)\big)^{1/2} < SB(r,s) < \frac{s+SB(s,r)}{2} < \frac{r+2s}{3}, \tag{4.6}$$

where $r,s > 0, r \neq s$. For the proof of (4.2) we use (4.6) with $r = G_p$ and $s = A$ and next apply formulas listed in (3.11). In a similar way one can establish inequalities (4.3) using (4.6) with $r = A$ and $s = Q_p$. Again we appeal to (3.11) to obtain the asserted result. The remaining chains of inequalities (4.4) and (4.5) can be established in an analogous manner. We omit further details. □

In the next theorem we shall establish optimal lower and upper bounds for the Seiffert mean P in terms of the p-harmonic means and also in terms of the p-geometric means. Another pair of double inequalities provide optimal bounds for the second Seiffert mean T in terms of the p-root-square means and in terms of the p-contraharmonic means as well.

We have the following.

THEOREM 4.2. *In order for the inequalities*

$$H_p < P < H_q \tag{4.7}$$

to be satisfied it is necessary and sufficient that

$$\sqrt{1-\frac{2}{\pi}} \leqslant p < 1 \ \text{ and } 0 < q \leqslant \frac{1}{\sqrt{6}}. \tag{4.8}$$

Similarly, the inequalities

$$G_r < P < G_s \tag{4.9}$$

are satisfied if and only if

$$\sqrt{1-\frac{4}{\pi^2}} \leqslant r < 1 \ \text{ and } 0 < s \leqslant \frac{1}{\sqrt{3}}. \tag{4.10}$$

Also,

$$Q_p < T < Q_q \tag{4.11}$$

if and only if

$$0 < p \leqslant \sqrt{\frac{16}{\pi^2}-1} \ \text{ and } \sqrt{\frac{2}{3}} \leqslant q < 1. \tag{4.12}$$

Finally, the two-sided inequality

$$C_r < T < C_s \tag{4.13}$$

is satisfied if and only if

$$0 < r \leqslant \sqrt{\frac{4}{\pi}-1} \ \text{ and } \frac{1}{\sqrt{3}} \leqslant s < 1. \tag{4.14}$$

Proof. We shall prove first the left-hand side inequality of (4.7). Using (3.8), (3.9), and (2.4) we see that the inequality in question can be written as

$$A(1-p^2v^2) < A\frac{v}{\sin^{-1} v}.$$

Hence

$$p^2 > \frac{1}{v^2} - \frac{1}{v\sin^{-1} v}.$$

Letting $v = \sin t, 0 < t < \frac{\pi}{2}$, we get

$$p > \sqrt{\frac{t-\sin t}{t\sin^2 t}} =: \phi(t). \tag{4.15}$$

It follows from [8] that the function $\phi(t)$ is strictly increasing on $(0, \pi/2)$. Hence

$$\phi(0+) \leqslant \phi(t) \leqslant \phi(\pi/2)$$

or

$$\frac{1}{\sqrt{6}} \leqslant \phi(t) \leqslant \sqrt{1-\frac{2}{\pi}}. \tag{4.16}$$

The second inequality in (4.16) yields the lower bound for p in (4.8). To obtain the upper bond for p we follow the lines introduced above to obtain

$$q < \phi(t).$$

Combining this with the first inequality in (4.16) we obtain the desired result.

We shall prove now the inequalities (4.9) are satisfied if and only if r and s satisfy conditions (4.10). It follows from (2.4) and (3.9) that the left inequality in (4.9) can be written as

$$1-\left(\frac{v}{\sin^{-1} v}\right)^2 < r^2v^2.$$

Hence

$$r^2 > \frac{1}{v^2} - \frac{1}{(\sin^{-1} v)^2} = \frac{1}{\sin^2 t} - \frac{1}{t^2} =: \phi(t),$$

where $\sin(t) = v$ $(0 < t \leqslant \pi/2)$. Differentiation of $\phi(t)$ yields

$$\phi'(t) = \frac{2}{t^3}\left(1-\left(\frac{t}{\sin t}\right)^3\cos t\right)$$

To prove that $\phi'(t) > 0$ on the stated domain we employ inequality of Adamović and Mitrinović [13]:

$$\cos t < \left(\frac{\sin t}{t}\right)^3$$

which is the same as $(\frac{t}{\sin t})^3 \cos t < 1$. This yields $\phi'(t) > 0$ and in consequence that the function $\phi(t)$ is strictly increasing on $(0, \pi/2)$. Hence

$$\frac{1}{3} = \phi(0+) \leqslant \phi(t) \leqslant \phi\left(\frac{\pi}{2}\right) = 1 - \frac{4}{\pi^2}$$

We appeal now to the definition of the function $\phi(t)$ to obtain the first inequalities in (4.9). The second ones can be obtained in an analogous manner. We omit further details.

In order to obtain (4.11)–(4.12) we use (2.4) and (3.9) again to write the left-hand side inequality in (4.11) as

$$\sqrt{1+p^2v^2} < \frac{v}{\tan^{-1} v}$$

Letting above $v = \tan t$ $(0 < t \leqslant \pi/4)$ we obtain

$$p^2 < \frac{1}{t^2} - \frac{1}{\tan^2 t} =: \psi(t).$$

Differentiation yields

$$\psi'(t) = \frac{2}{t^3}\left(\left(\frac{t}{\sin t}\right)^3 \cos t - 1\right).$$

Comparison with $\phi'(t)$, where $\phi(t)$ is the same as defined earlier in this proof, yields $\psi'(t) < 0$ on $(0, \pi/4)$. Thus the function $\psi(t)$ is strictly decreasing on the stated domain. This in turn yields

$$\frac{16}{\pi^2} - 1 = \psi(\pi/4) \leqslant \psi(t) \leqslant \psi(0+) = \frac{2}{3}.$$

The asserted conditions (4.12) now follow. Finally, for the proof of (4.13)–(4.14) we use (2.4) and (3.9) to obtain

$$1 + r^2 v^2 < \frac{v}{\tan^{-1} v} < 1 + s^2 v^2.$$

Letting $v = \tan t$ $(0 < t < \pi/4)$ and extracting the square roots we obtain

$$r < \sqrt{\psi(t)} < s, \tag{4.17}$$

where

$$\psi(t) = \frac{1}{t \tan t} - \frac{1}{\tan^2 t}.$$

Differentiation yields

$$(\tan^3 t \cos^2 t)\psi'(t) = 2 - \frac{\tan t}{t} - \left(\frac{\sin t}{t}\right)^2. \tag{4.18}$$

We shall demonstrate now that $\psi(t)$ is a strictly decreasing function on $(0, \pi/4)$. Using inequality of Adamović and Mitrinović we obtain

$$-\frac{\tan t}{t} < -(\cos t)^{-2/3} \text{ and } -\left(\frac{\sin t}{t}\right)^2 < -(\cos t)^{2/3}.$$

This in conjunction with (4.18) yields

$$(\tan^3 t \cos^2 t)\psi'(t) < 2 - \frac{1}{c} - c,$$

where $c = (\cos t)^{2/3}$. Hence

$$(\tan^3 t \cos^2 t)\psi'(t) < -\frac{(c-1)^2}{c} < 0.$$

Thus the function $\psi(t)$ is strictly decreasing on $(0, \pi/4)$. Elementary computations give

$$\sqrt{\frac{4}{\pi} - 1} = \sqrt{\psi(\pi/4)} \leqslant \sqrt{\psi(t)} \leqslant \sqrt{\psi(0+)} = \frac{1}{\sqrt{3}}. \tag{4.19}$$

Combining (4.17) and (4.19) we obtain the desired result. □

Inequalities (4.7)–(4.8) have been established in [8] by use of different means. Also, inequalities (4.11) together with (4.12) have been obtained, by a different method, in [4].

In the next two theorems we give optimal lower and upper bounds for the four p-means P_p, T_p, M_p, and L_p. The bounding quantities are either the algebraic or geometric convex combinations of two means listed in (3.11). Recall that for $r, s > 0$ and $0 \leqslant \alpha \leqslant 1$ quantities $\alpha r + (1-\alpha)s$ and $r^{\alpha}s^{1-\alpha}$ are called, respectively, the algebraic or the geometric, convex combinations of r and s.

We are in a position to prove the following.

THEOREM 4.3. *The two-sided inequality*

$$\alpha_1 A + (1-\alpha_1)G_p < P_p < \beta_1 A + (1-\beta_1)G_p \tag{4.20}$$

is valid if and only if $0 \leqslant \alpha_1 \leqslant \frac{2}{\pi}$ *and* $\frac{2}{3} \leqslant \beta_1 \leqslant 1$. *Similarly,*

$$\alpha_2 Q_p + (1-\alpha_2)A < T_p < \beta_2 Q_p + (1-\beta_2)A \tag{4.21}$$

if and only if $0 \leqslant \alpha_2 \leqslant \dfrac{4-\pi}{\pi(\sqrt{2}-1)}$ *and* $\frac{2}{3} \leqslant \beta_2 \leqslant 1$. *Also,*

$$\alpha_3 Q_p + (1-\alpha_3)A < M_p < \beta_3 Q_p + (1-\beta_3)A \tag{4.22}$$

if and only if $0 \leqslant \alpha_3 \leqslant \dfrac{1-\gamma}{(\sqrt{2}-1)\gamma}$ *and* $\frac{1}{3} \leqslant \beta_3 \leqslant 1$, *where* $\gamma = \sinh^{-1}(1) = \ln(\sqrt{2}+1)$. *Finally,*

$$\alpha_4 A + (1-\alpha_4)G_p < L_p < \beta_4 A + (1-\beta_4)G_p \tag{4.23}$$

if and only if $\alpha_4 = 0$ *and* $\frac{1}{3} \leqslant \beta_4 \leqslant 1$.

Proof. Inequalities (4.20) and (4.21) together with the associated bounds for the $\alpha's$ and the $\beta's$ follow from Theorem 4.1 and Corollary 4.1 in [23].

For the proof of (4.22) let us write this two-sided inequality as

$$\alpha_3 < \frac{M_p - A}{Q_p - A} < \beta_3.$$

Taking into account that $M_p - A = A\left(\dfrac{u}{\sinh^{-1}u} - 1\right)$ and $Q_p - A = A(\sqrt{1+u^2}-1)$ (see (3.10) and (3.9)) we can write the last double inequality as follows

$$\alpha_3 < \frac{\dfrac{u}{\sinh^{-1}u} - 1}{\sqrt{1+u^2}-1} < \beta_3.$$

Letting above $u = \sinh t$ $(0 < t \leqslant \gamma)$ we get

$$\alpha_3 < \phi(t) < \beta_3 \tag{4.24}$$

where

$$\phi(t) = \frac{\sinh t - t}{t\cosh t - t}. \tag{4.25}$$

It follows from the proof of Theorem 3.1 in [15] that the function $\phi(t)$ is strictly decreasing on $[0,\gamma]$. Thus

$$\phi(\gamma) \leqslant \phi(t) \leqslant \phi(0+).$$

Making use of

$$\phi(\gamma) = \frac{1-\gamma}{(\sqrt{2}-1)\gamma} \quad \text{and} \quad \phi(0+) = \frac{1}{3}$$

(see [15]) and (4.24) we obtain bounds for α_3 and β_3.

In order to establish inequalities (4.23) with conditions of validity as stated above, we put the two-sided inequality (4.23) in the form

$$\alpha_4 < \frac{L_p - G_p}{A - G_p} < \beta_4.$$

Using (3.10) and (3.9), with $u = \tanh t \ \ (t > 0)$, we can write the last double inequality as

$$\alpha_4 < \phi(t) < \beta_4, \tag{4.26}$$

where $\phi(t)$ is defined in (4.25). It follows from the proof of Theorem 3.1 in [15] that the function $\phi(t)$ is also strictly decreasing on the positive semiaxis. Since $\lim_{t\to\infty}\phi(t) = 0$, we conclude that

$$0 \leqslant \phi(t) \leqslant \frac{1}{3}.$$

This in conjunction with (4.26) yields $\alpha_4 = 0$ and $\frac{1}{3} \leqslant \beta_4 \leqslant 1$. The proof is complete.

A special case of (4.21), when $p = 1$, has been established in [6] and in [23] while inequalities (4.22), with $p = 1$, have been obtained in [15]. □

A counterpart of Theorem 4.3 with bounding terms in the form of geometric convex combinations reads as follows.

THEOREM 4.4. *Let γ be the same as defined in Theorem 4.3. The simultaneous inequality*

$$A^{\alpha_1} G_p^{1-\alpha_1} < P_p < A^{\beta_1} G_p^{1-\beta_1} \tag{4.27}$$

holds true if and only if $0 \leqslant \alpha_1 \leqslant \frac{2}{3}$ *and* $\beta_1 = 1$. *Also,*

$$Q_p^{\alpha_2} A^{1-\alpha_2} < T_p < Q_p^{\beta_2} A^{1-\beta_2} \tag{4.28}$$

if and only if $0 \leqslant \alpha_2 \leqslant \frac{2}{3}$ *and* $\frac{\ln(16/\pi^2)}{\ln 2} \leqslant \beta_2 \leqslant 1$. *Also, the following inequalities*

$$Q_p^{\alpha_3} A^{1-\alpha_3} < M_p < Q_p^{\beta_3} A^{1-\beta_3} \tag{4.29}$$

hold true if and only if $0 \leqslant \alpha_3 \leqslant \frac{1}{3}$ *and* $\frac{-\ln\gamma}{\ln(\cosh\gamma)} \leqslant \beta_3 \leqslant 1$. *Finally*

$$A^{\alpha_4} G_p^{1-\alpha_4} < L_p < A^{\beta_4} G_p^{1-\beta_4} \tag{4.30}$$

if and only if $0 \leqslant \alpha_4 \leqslant \frac{1}{3}$ *and* $\beta_4 = 1$.

Proof. Inequalities (4.27) and (4.28), with domains for the $\alpha's$ and the $\beta's$ as stated above, follow from Theorem 5.1 and Corollary 5.1 in [23]. For the proof of (4.29) we begin with the equivalent inequality

$$\alpha_3 < \frac{\ln(M_p/A)}{\ln(Q_p/A)} < \beta_3.$$

Using (3.10) and (3.9) with $u = \sinh t$, where $t \in (0,\gamma)$, we obtain

$$\alpha_3 < \phi(t) < \beta_3, \tag{4.31}$$

where now

$$\phi(t) = \frac{\ln\left(\frac{\sinh t}{t}\right)}{\ln(\cosh t)}. \tag{4.32}$$

In [24] the authors have pointed out that the function $\phi(t)$ defined in (4.32) is strictly increasing on $(0,\gamma)$. Thus

$$\phi(0+) \leqslant \phi(t) \leqslant \phi(\gamma)$$

or what is the same that

$$\frac{1}{3} \leqslant \phi(t) \leqslant \frac{-\ln\gamma}{\ln(\cosh\gamma)}.$$

This in conjunction with (4.31) yields the asserted result. For the proof of (4.30) we write this inequality in the equivalent form

$$\alpha_4 < \frac{\ln(L_p/G_p)}{\ln(A/G_p)} < \beta_4.$$

Utilizing appropriate parts of (3.10) and (3.9) we obtain, after a little algebra, that

$$\alpha_4 < \phi(t) < \beta_4, \tag{4.33}$$

where now $t > 0$ and $\phi(t)$ is defined in (4.32). Using the well-known inequality $\sinh t/t < \cosh t$ we see that

$$\frac{1}{3} \leqslant \phi(t) < 1.$$

This and (4.33) give the desired domains for α_4 and β_4. The proof is complete. □

In the last theorem of this section we give two double inequalities providing bounds for the Seiffert mean P. The bounding quantities are the arithmetic convex combinations of two p-means.

THEOREM 4.5. *Let* $0 < p,q \leqslant 1$. *In order for the inequalities*

$$\alpha A + (1-\alpha)H_p < P < \beta A + (1-\beta)H_q \tag{4.34}$$

to be satisfied it is necessary and sufficient that

$$p\sqrt{1-\alpha} > \sqrt{1-\frac{2}{\pi}} \tag{4.35}$$

and

$$q\sqrt{1-\beta} < \frac{1}{\sqrt{6}}, \tag{4.36}$$

where $0 < \alpha, \beta < 1$. *The two-sided inequality*

$$\alpha C_p + (1-\alpha)H_p < P < \beta C_q + (1-\beta)H_q \tag{4.37}$$

holds true if and only if

$$p\sqrt{1-2\alpha} > \sqrt{1-\frac{2}{\pi}} \tag{4.38}$$

and

$$q\sqrt{1-2\beta} < \frac{1}{\sqrt{6}}, \tag{4.39}$$

where $0 < \alpha, \beta < 1/2$.

Proof. First we write the left-hand side inequality of (4.34) in the form

$$\alpha < \frac{P-H_p}{A-H_p}. \tag{4.40}$$

With $H_p = A(1-p^2v^2)$ and $P = A\dfrac{v}{\sin^{-1} v}$ (see (3.9), (3.10), and (3.8)) we see that (4.40) can be written as

$$\alpha < \frac{\dfrac{v}{\sin^{-1} v} - 1 + p^2v^2}{p^2v^2}. \tag{4.41}$$

Letting $v = \sin t$ $(0 < t < \pi/2)$ we write (4.41) as follows

$$p^2(1-\alpha) > \frac{t-\sin t}{t\sin^2 t} =: \phi(t) \tag{4.42}$$

We shall prove now that the function $\phi(t)$ is strictly increasing on $(0, \pi/2)$. Differentiation yields

$$\frac{\sin^3 t}{\cos t}\phi'(t) = \frac{\sin t}{t} + \frac{\sin t}{t}\frac{\tan t}{t} - 2 =: g(t).$$

To obtain the assertion it suffices to show that $g(t) > 0$ for all $t \in (0, \pi/2)$. To this aim we utilize the inequality of Adamović and Mitrinović which is equivalent to

$$1 < \frac{\sin t}{t}\left(\frac{\sin t}{t}\frac{\tan t}{t}\right).$$

Extracting the square roots on both sides and next applying the inequality of arithmetic and geometric means we obtain

$$1 < \sqrt{\frac{\sin t}{t}\left(\frac{\sin t}{t}\frac{\tan t}{t}\right)} < \frac{1}{2}\left(\frac{\sin t}{t} + \frac{\sin t}{t}\frac{\tan t}{t}\right).$$

Thus the function $g(t)$ is positive on $(0,\pi/2)$. This in turn implies that the function $\phi(t)$ is strictly increasing on the same domain. Moreover, $\phi(0+)=1/6$ and $\phi(\pi/2)=1-2/\pi$. This in conjunction with (4.42) yields (4.35). In a similar manner we can prove, using the right-hand side inequality in (4.38), that

$$q^2(1-\beta)<\phi(t)$$

which yields (4.36).

Inequalities (4.37) together with the associated conditions (4.38) and (4.39) can be established in an analogous manner. First we write the first inequality in (4.37) as

$$\alpha<\frac{P-H_p}{C_p-H_p}.$$

Using appropriate formulas listed in (3.9) and (3.10) we can write the last inequality as follows

$$\alpha<\frac{\dfrac{v}{\sin^{-1}v}-1+p^2v^2}{2p^2v^2}.$$

Substituting $v=\sin t\ \ (0<t<\pi/2)$ we obtain

$$\phi(t)<p^2(1-2\alpha), \tag{4.43}$$

where $\phi(t)$ is defined in (4.42). We already know that

$$\frac{1}{6}\leqslant\phi(t)\leqslant 1-\frac{2}{\pi}. \tag{4.44}$$

This in conjunction with (4.43) yields (4.38). In a similar fashion one can prove that the second inequality in (4.37) is equivalent to

$$q^2(1-2\beta)<\phi(t).$$

Combining this with the first inequality in (4.44) yields (4.39). □

We close this section with the remark that in the case when $p=q=1$ the first part of Theorem 4.5 has been established in [5] while the second one appears in [12].

Acknowledgements.

The author is indebted to an anonymous referee for constructive remarks on the first draft of this paper.

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(Received October 26, 2012)

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