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NEW RESULTS RELATED TO THE CONVEXITY OF THE BERNARDI INTEGRAL OPERATOR

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Abstract. In this paper we prove the convexity of the image of a close-to-convex function by the Bernardi integral operator given by

$$L_\gamma(f)(z) = F(z) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt, \quad z \in U. \tag{1}$$

This result extends the result obtained by N. Pascu in [9], where it has been shown that the Bernardi operator transforms a close-to-convex function into a close-to-convex function under certain conditions.

1. Introduction and preliminaries

Let U be the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathscr{H}(U)$ be the space of holomorphic functions in U. Also, let

$$A_n = \{f \in \mathscr{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$$

with $A_1 = A$ and

$$S = \{f \in A : f \text{ is univalent in } U\}.$$

Let

$$K = \left\{f \in A,\ \text{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U\right\},$$

denote the class of normalized convex functions in U,

$$S^* = \left\{f \in A,\ \text{Re}\,\frac{zf'(z)}{f(z)} > 0,\ z \in U\right\}$$

denote the class of starlike functions in U, and

$$C = \left\{f \in A : \exists\, \varphi \in K,\ \text{Re}\,\frac{f'(z)}{\varphi'(z)} > 0,\ z \in U\right\}$$

denote the class of close-to-convex functions.

In order to prove our original results, we use the following lemmas:

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LEMMA 1. [3], [4], [6, Theorem 2.3.i, p. 35] *Let* $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, *satisfy the condition*

$$\operatorname{Re} \psi(is,t;z) \leqslant 0, \quad z \in U,$$

for $s,t \in \mathbb{R}$, $t \leqslant -\frac{n}{2}(1+s^2)$.

If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ *satisfies*

$$\operatorname{Re} [p(z), zp'(z); z] > 0$$

then

$$\operatorname{Re} p(z) > 0, \quad z \in U.$$

More general forms of this lemma can be found in [6].

LEMMA 2. [7, Theorem 4.6.3, p. 84] *The function* $f \in A$, *with* $f'(z) \neq 0$, $z \in U$, *is close-to-convex if and only if*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)}\right] d\theta > -\pi, \quad z = re^{i\theta},$$

for all θ_1, θ_2 *with* $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ *and all* $r \in (0,1)$.

If $L_\gamma : A \to A$ is the integral operator defined by $L_\gamma[f] = F$, where F is given by

$$L_\gamma[f](z) = F(z) = \frac{\gamma+1}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt$$

and $\operatorname{Re} \gamma \geqslant 0$, $z \in U$, then it is well known that

(i) $L_\gamma[S^*] \subset S^*$,

(ii) $L_\gamma[K] \subset K$,

(iii) $L_\gamma[C] \subset C$.

These results are obtained in [2] and [9].

2. Main results

We determine conditions such that, for a function $f \in A_n$, the image under the Bernardi integral operator is convex.

THEOREM 1. *Let* $f \in A_n$, $\gamma \geqslant 1$, $n \geqslant 1$ *and*

$$L_\gamma(f)(z) = F(z) = \frac{\frac{\gamma}{n}+1}{z^{\frac{\gamma}{n}}} \int_0^z f(t) t^{\frac{\gamma}{n}-1} dt, \quad z \in U. \tag{2}$$

If

$$\operatorname{Re} \left[\frac{zf''(z)}{f'(z)} + 1\right] > -\frac{1}{2\gamma}, \quad z \in U, \tag{3}$$

then the function F given by (2) is convex.

Proof. Let $f \in A_n$, $f(z) = z + a_{n+1}z^{n+1} + \dots$, $z \in U$. Then, from (2), we have:

$$L_\gamma(f)(z) = F(z) = \frac{\frac{\gamma}{n}+1}{z^{\frac{\gamma}{n}}}\int_0^z (t + a_{n+1}t^{n+1} + \dots)t^{\frac{\gamma}{n}-1}dt \tag{4}$$
$$= \frac{\frac{\gamma}{n}+1}{z^{\frac{\gamma}{n}}}\left(\frac{z^{\frac{\gamma}{n}+1}}{\frac{\gamma}{n}+1} + a_{n+1}\frac{z^{n+\frac{\gamma}{n}+1}}{n+\frac{\gamma}{n}+1} + \dots\right) = z + b_{n+1}z^{n+1} + \dots,$$

hence $F \in A_n$.

According to Lemma 2 we obtain

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right]d\theta \geqslant \int_{\theta_1}^{\theta_2} -\frac{1}{2\gamma}d\theta = -\frac{1}{2\gamma}\int_{\theta_1}^{\theta_2} d\theta \tag{5}$$
$$= -\frac{1}{2\gamma}(\theta_2 - \theta_1) = -\frac{2\pi}{2\gamma} = -\frac{\pi}{\gamma} > -\pi, \quad \gamma \geqslant 1.$$

From (5) we have $f \in C$, hence it is univalent. If $f \in C$, then from (iii) we have $L_\gamma[f] = F \in C$, hence F is univalent.

From (2), we have

$$z^{\frac{\gamma}{n}}F(z) = \left(1 + \frac{\gamma}{n}\right)\int_0^z f(t)t^{\frac{\gamma}{n}-1}dt, \quad z \in U. \tag{6}$$

By differentiating (6), we obtain

$$\frac{\gamma}{n}z^{\frac{\gamma}{n}-1}F(z) + z^{\frac{\gamma}{n}}F'(z) = \left(\frac{\gamma}{n}+1\right)f(z)\cdot z^{\frac{\gamma}{n}-1}, \quad z \in U, \tag{7}$$

and by a simple calculation, we have

$$\frac{\gamma}{n}F(z) + zF'(z) = \left(\frac{\gamma}{n}+1\right)f(z), \quad z \in U. \tag{8}$$

By differentiating (8) and by a simple calculation, we obtain

$$\frac{\gamma}{n}F'(z) + F'(z)\left[1 + \frac{zF''(z)}{F'(z)}\right] = \left(\frac{\gamma}{n}+1\right)f'(z), \quad z \in U. \tag{9}$$

Let

$$1 + \frac{zF''(z)}{F'(z)} = p(z), \quad z \in U, \quad p(0) = 1, \quad p(z) = 1 + p_n z^n + \dots, \tag{10}$$

then (9) is equivalent to

$$F'(z)[\frac{\gamma}{n} + p(z)] = \left(\frac{\gamma}{n}+1\right)f'(z), \quad z \in U. \tag{11}$$

Since $F'(z) \neq 0$, $p(z)+\gamma \neq 0$, $f \in C$, we have $f'(z) \neq 0$, $z \in U$, and by differentiating (11), we obtain

$$1+\frac{zF''(z)}{F'(z)}+\frac{zp'(z)}{p(z)+\frac{\gamma}{n}}=\frac{zf''(z)}{f'(z)}+1, \quad z \in U. \tag{12}$$

Using (10), we have

$$p(z)+\frac{zp'(z)}{p(z)+\frac{\gamma}{n}}=1+\frac{zf''(z)}{f'(z)}, \quad z \in U. \tag{13}$$

Using (3), we obtain

$$\operatorname{Re}\left[p(z)+\frac{zp'(z)}{p(z)+\frac{\gamma}{n}}\right] > -\frac{1}{2\gamma}, \quad z \in U, \quad \gamma \geqslant 1, \tag{14}$$

which is equivalent to

$$\operatorname{Re}\left[p(z)+\frac{zp'(z)}{p(z)+\frac{\gamma}{n}}+\frac{1}{2\gamma}\right] > 0, \quad z \in U, \quad \gamma \geqslant 1. \tag{15}$$

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$,

$$\psi(p(z), zp'(z); z) = p(z)+\frac{zp'(z)}{p(z)+\frac{\gamma}{n}}+\frac{1}{2\gamma}, \quad z \in U, \quad \gamma \geqslant 1. \tag{16}$$

Then (15) is equivalent to

$$\operatorname{Re} \psi(p(z), zp'(z); z) > 0, \quad z \in U. \tag{17}$$

In order to prove Theorem 1, we use Lemma 1. For that we calculate

$$\begin{aligned}
\operatorname{Re} \psi(is, t; z) &= \operatorname{Re}\left[is+\frac{t}{is+\frac{\gamma}{n}}+\frac{1}{2\gamma}\right] = \operatorname{Re}\left[is+\frac{1}{2\gamma}+\frac{t\left(\frac{\gamma}{n}-is\right)}{\frac{\gamma^2}{n^2}+s^2}\right] \\
&= \frac{1}{2\gamma}+\frac{t\frac{\gamma}{n}}{\frac{\gamma^2}{n^2}+s^2} \leqslant \frac{1}{2\gamma}-\frac{\frac{\gamma}{n}(1+s^2)\frac{n}{2}}{\frac{\gamma^2}{n^2}+s^2} = \frac{1}{2\gamma}-\frac{\gamma(1+s^2)}{2\left(\frac{\gamma^2}{n^2}+s^2\right)} \\
&= \frac{\frac{\gamma^2}{n^2}+s^2-\gamma^2-\gamma^2 s^2}{2\gamma\left(\frac{\gamma^2}{n^2}+s^2\right)} = \frac{\gamma^2\left(\frac{1}{n^2}-1\right)+s^2(1-\gamma^2)}{2\gamma\left(\frac{\gamma^2}{n^2}+s^2\right)} \leqslant 0,
\end{aligned}$$

since $n \geqslant 1$ and $\gamma \geqslant 1$.

Now, using Lemma 1 we get that $\operatorname{Re} p(z) > 0$, $z \in U$, i.e.

$$\operatorname{Re} \frac{zF''(z)}{F'(z)} + 1 > 0, \quad z \in U,$$

hence $F \in K$. $\square$

REMARK 1. For $n = 1$ we obtain the results from [8].

We determine conditions such that, for a function $f \in \mathscr{H}[1,1]$, the image under the Bernardi integral operator is convex.

THEOREM 2. *Let* $f \in \mathscr{H}[1,1]$, $\gamma \geqslant 1$, *and*

$$L_\gamma[f](z) = F(z) = \frac{\gamma}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt, \quad z \in U. \tag{18}$$

If

$$\operatorname{Re} \left[\frac{zf''(z)}{f'(z)} + 1 \right] > -\frac{1}{2\gamma}, \quad z \in U, \tag{19}$$

then the function F *given by (18) is convex.*

Proof. Let $f \in \mathscr{H}[1,1]$, $f(z) = 1 + a_1 z + a_2 z^2 + \ldots$, $z \in U$. Then, from (18), we have

$$\begin{aligned} L_\gamma[f](z) &= F(z) = \frac{\gamma}{z^\gamma} \int_0^z (1 + a_1 t + a_2 t^2 + \ldots) t^{\gamma-1} dt \\ &= \frac{\gamma}{z^\gamma} \left[\frac{z^\gamma}{\gamma} + a_1 \frac{z^{\gamma+1}}{\gamma+1} + a_2 \frac{z^{\gamma+2}}{\gamma+2} + \ldots \right] = 1 + b_1 z + b_2 z + \ldots, \end{aligned} \tag{20}$$

hence $f \in \mathscr{H}[1,1]$.

According to Lemma 2 we obtain

$$\begin{aligned} \int_{\theta_1}^{\theta_2} \operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] d\theta &\geqslant \int_{\theta_1}^{\theta_2} -\frac{1}{2\gamma} d\theta = -\frac{1}{2\gamma} \int_{\theta_1}^{\theta_2} d\theta \\ &= -\frac{1}{2\gamma}(\theta_2 - \theta_1) = -\frac{2\pi}{2\gamma} = -\frac{\pi}{\gamma} > -\pi, \quad \gamma \geqslant 1. \end{aligned} \tag{21}$$

From (21) we have $f \in C$, hence it is univalent. If $f \in C$, then from (iii) we have $L_\gamma[f] = F \in C$, hence F is univalent, $F'(z) \neq 0$, $z \in U$.

From (18), we have

$$z^\gamma F(z) = \gamma \int_0^z f(t) t^{\gamma-1} dt, \quad z \in U. \tag{22}$$

By differentiating (22) and by a simple calculation, we obtain

$$\gamma F'(z) + F'(z) \left[1 + \frac{zF''(z)}{F'(z)} \right] = \gamma f'(z), \quad z \in U. \tag{23}$$

Let

$$1+\frac{zF''(z)}{F'(z)} = p(z), \quad z\in U, p(0)=1, p(z)=1+p_1z+p_2z^2+\dots. \tag{24}$$

Then (23) is equivalent to

$$F'(z)[p(z)+\gamma] = \gamma f'(z), \quad z\in U. \tag{25}$$

Since $F'(z)\neq 0$, $p(z)+\gamma\neq 0$, $f\in C$, we have $f'(z)\neq 0$, $z\in U$, and by differentiating (25), we obtain

$$1+\frac{zF''(z)}{F'(z)}+\frac{zp'(z)}{p(z)+\gamma} = \frac{zf''(z)}{f'(z)}+1, \quad z\in U. \tag{26}$$

Using (24), we have

$$p(z)+\frac{zp'(z)}{p(z)+\gamma} = 1+\frac{zf''(z)}{f'(z)}, \quad z\in U. \tag{27}$$

Using (19), we obtain

$$\mathrm{Re}\left[p(z)+\frac{zp'(z)}{p(z)+\gamma}\right] > -\frac{1}{2\gamma}, \quad z\in U, \quad \gamma\geqslant 1, \tag{28}$$

which is equivalent to

$$\mathrm{Re}\left[p(z)+\frac{zp'(z)}{p(z)+\gamma}+\frac{1}{2\gamma}\right] > 0, \quad z\in U, \quad \gamma\geqslant 1. \tag{29}$$

Let $\psi:\mathbb{C}^2\times U\to\mathbb{C}$,

$$\psi(p(z),zp'(z);z) = p(z)+\frac{zp'(z)}{p(z)+\gamma}+\frac{1}{2\gamma}, \quad z\in U, \quad \gamma\geqslant 1. \tag{30}$$

Then (29) is equivalent to

$$\mathrm{Re}\ \psi(p(z),zp'(z);z) > 0, \quad z\in U. \tag{31}$$

In order to prove Theorem 2, we use Lemma 1. For that we calculate

$$\begin{aligned}\mathrm{Re}\ \psi(is,t;z) &= \mathrm{Re}\left[is+\frac{1}{2\gamma}+\frac{t}{is+\gamma}\right] = \mathrm{Re}\left[is+\frac{1}{2\gamma}+\frac{t(\gamma-is)}{\gamma^2+s^2}\right] \\ &= \frac{1}{2\gamma}+\frac{t\gamma}{\gamma^2+s^2} \leqslant \frac{1}{2\gamma}-\frac{\gamma(1+s^2)}{2(\gamma^2+s^2)} \\ &= \frac{\gamma^2+s^2-\gamma^2-\gamma^2s^2}{2\gamma(\gamma^2+s^2)} = \frac{s^2(1-\gamma^2)}{2\gamma(\gamma^2+s^2)} \leqslant 0,\end{aligned}$$

since $\gamma\geqslant 1$, $n\geqslant 1$.

Now, using Lemma 1, we get that $\mathrm{Re}\ p(z)>0$, $z\in U$, i.e.

$$\mathrm{Re}\,\frac{zF''(z)}{F'(z)}+1>0, \quad z\in U,$$

hence $F\in K$. $\square$

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