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ANOTHER APPROACH TO BECKNER'S INEQUALITY

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Abstract. A classical inequality which was proved by Beckner is an important tool for the study of Banach space geometry. In this note, we present another proof of that inequality.

In this note, we consider the following classical inequality which was proved by Beckner [1] (cf. [3, Lemma 1.e.14]).

THEOREM 1. *Let $1 < p \leqslant q < \infty$, and let $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then*

$$\left(\frac{|u+\gamma_{p,q}v|^q + |u-\gamma_{p,q}v|^q}{2}\right)^{\frac{1}{q}} \leqslant \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{\frac{1}{p}}$$

for all $u, v \in \mathbb{R}$.

It is known that $\gamma_{p,q}$ in Theorem 1 is the best constant, that is, if $a \geqslant 0$ and

$$\left(\frac{|u+av|^q + |u-av|^q}{2}\right)^{\frac{1}{q}} \leqslant \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{\frac{1}{p}}$$

for all $u, v \in \mathbb{R}$, then we have $a \leqslant \gamma_{p,q}$. We note that the case $0 \leqslant a \leqslant 1$ is essential in this direction. Indeed, letting $u = 0$ and $v = 1$ in the above inequality, we obtain $a \leqslant 1$. The proof of this fact can be found in the proof of [7, Theorem 6].

Our aim is to present an elementary proof of Theorem 1 and the above fact (cf. [2, 4, 5, 6]). It is needless to say that Theorem 1 is trivial if $p = q$. So we only consider the case $p \neq q$. Suppose that $1 < p < q < \infty$ and that $b \in [0,1]$. Let A_b be the linear operator from $(\mathbb{R}^2, \|\cdot\|_p)$ into $(\mathbb{R}^2, \|\cdot\|_q)$ defined by

$$A_b = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$$

and let $\|A_b\|_{p,q}$ denote the operator norm of A_b. Put $f_{p,q,b}$ be the real-valued function on $[0,1]$ defined by

$$f_{p,q,b}(t) = \left(\left(t^{\frac{1}{p}} + b(1-t)^{\frac{1}{p}}\right)^q + \left(bt^{\frac{1}{p}} + (1-t)^{\frac{1}{p}}\right)^q\right)^{\frac{1}{q}}.$$

First we prove the following two lemmas.

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LEMMA 1. $\|A_b\|_{p,q} = \max_{0\leqslant t\leqslant 1/2} f_{p,q,b}(t)$.

Proof. Take an arbitrary $(u,v)\in\mathbb{R}^2$ such that $\|(u,v)\|_p = 1$. Then we have $|v| = (1-|u|^p)^{1/p}$, and so

$$\begin{aligned}\left\|\begin{pmatrix}1 & b\\ b & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}\right\|_q &= (|u+bv|^q + |bu+v|^q)^{\frac{1}{q}}\\ &\leqslant ((|u|+b|v|)^q + (b|u|+|v|)^q)^{\frac{1}{q}}\\ &= \left(\left(|u| + b(1-|u|^p)^{\frac{1}{p}}\right)^q + \left(b|u| + (1-|u|^p)^{\frac{1}{p}}\right)^q\right)^{\frac{1}{q}}\\ &= f_{p,q,b}(|u|^p)\\ &\leqslant \max_{0\leqslant t\leqslant 1} f_{p,q,b}(t).\end{aligned}$$

Therefore, we have $\|A_b\|_{p,q} \leqslant \max_{0\leqslant t\leqslant 1} f_{p,q,b}(t)$. On the other hand, for each $t\in[0,1]$, putting $x_t = (t^{1/p},(1-t)^{1/p})$ then we obtain

$$\|A_b\|_{p,q} \geqslant \|A_b x_t\|_q = f_{p,q,b}(t).$$

Hence, we have $\|A_b\|_{p,q} = \max_{0\leqslant t\leqslant 1} f_{p,q,b}(t)$.

Since $f_{p,q,b}(t) = f_{p,q,b}(1-t)$ for all $t\in[0,1]$, it follow that $\max_{0\leqslant t\leqslant 1} f_{p,q,b}(t) = \max_{0\leqslant t\leqslant 1/2} f_{p,q,b}(t)$. Thus, we have $\|A_b\|_{p,q} = \max_{0\leqslant t\leqslant 1/2} f_{p,q,b}(t)$. □

LEMMA 2. *Let* $a\in[0,1]$ *and let* $b=(1-a)/(1+a)$. *Then, the following are equivalent:*

(i) *The inequality*

$$\left(\frac{|u+av|^q + |u-av|^q}{2}\right)^{\frac{1}{q}} \leqslant \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{\frac{1}{p}}$$

holds for all $u,v\in\mathbb{R}$.

(ii) $f_{p,q,b}(1/2) = \max_{0\leqslant t\leqslant 1/2} f_{p,q,b}(t)$.

Proof. First we note that $f_{p,q,b}(1/2) = 2^{1/q-1/p}(1+b)$. So, (ii) is equivalent to

$$\|A_b\|_{p,q} = 2^{\frac{1}{q}-\frac{1}{p}}(1+b)$$

by Lemma 1.

Suppose that (i) holds. Take arbitrary $u,v\in\mathbb{R}$. Applying the inequality for $u_1 = (u+v)/2$ and $v_1 = (u-v)/2$, we have

$$\left(\frac{|(1+a)u+(1-a)v|^q + |(1-a)u+(1+a)v|^q}{2^{q+1}}\right)^{\frac{1}{q}} \leqslant \left(\frac{|u|^p+|v|^p}{2}\right)^{\frac{1}{p}},$$

and therefore

$$\frac{1+a}{2}\left(\frac{|u+bv|^q+|bu+v|^q}{2}\right)^{\frac{1}{q}} \leqslant \left(\frac{|u|^p+|v|^p}{2}\right)^{\frac{1}{p}}.$$

Since $(1+a)(1+b)=2$, it follows that

$$\|A_b(u,v)\|_q = (|u+bv|^q+|bu+v|^q)^{\frac{1}{q}} \leqslant 2^{\frac{1}{q}-\frac{1}{p}}(1+b)\|(u,v)\|_p,$$

which implies $\|A_b\|_{p,q} = 2^{1/q-1/p}(1+b)$. This proves (i) $\Rightarrow$ (ii).

Conversely, we assume that (ii) holds. Let $u,v\in\mathbb{R}$. Put $u_2 = u+v$ and $v_2 = u-v$, respectively. Then, we have

$$\begin{aligned}\left(\frac{|u+av|^q+|u-av|^q}{2}\right)^{\frac{1}{q}} &= \frac{1+a}{2^{\frac{1}{q}+1}}(|u_2+bv_2|^q+|bu_2+v_2|^q)^{\frac{1}{q}}\\ &= \frac{2^{-\frac{1}{q}}}{(1+b)}\|A_b(u_2,v_2)\|_q\\ &\leqslant \frac{2^{-\frac{1}{q}}}{(1+b)}\|A_b\|_{p,q}\|(u_2,v_2)\|_p\\ &= 2^{-\frac{1}{p}}\|(u_2,v_2)\|_p\\ &= \left(\frac{|u+v|^p+|u-v|^p}{2}\right)^{\frac{1}{p}}.\end{aligned}$$

Thus we obtain (ii) $\Rightarrow$ (i). □

Now, let

$$\delta_{p,q} = \frac{1-\gamma_{p,q}}{1+\gamma_{p,q}} = \frac{\sqrt{q-1}-\sqrt{p-1}}{\sqrt{q-1}+\sqrt{p-1}} = \frac{p+q-2-2\sqrt{(p-1)(q-1)}}{q-p},$$

and let $\alpha = 1/p$ and $\beta = q-1$, respectively. We note that $0<\alpha<1$ and $\beta+1 > \alpha\beta-\alpha+1$. Henceforth, $\delta_{p,q}$ is simply denoted by δ.

LEMMA 3. *Let $b\in[0,\delta]$ and let $g_{1,b}$ be the real-valued function on $[0,1]$ defined by*

$$g_{1,b}(u) = -\beta bu^2+(\alpha\beta+\alpha-1)(1+b^2)u-(2\alpha-1)\beta b-2(1-\alpha)bu^{\frac{1}{1-\alpha}}.$$

(i) *If $1<p<2$, then there exists a real number $u_0\in(0,1)$ such that $g_{1,b}(u_0)=0$, $g_{1,b}(u)<0$ for all $u\in[0,u_0)$, and $g_{1,b}(u)>0$ for all $u\in(u_0,1)$.*

(ii) *If $2\leqslant p<\infty$, then $g_{1,b}(u)>0$ for all $u\in(0,1)$.*

Proof. Since $0<\alpha<1$, we have $(1-\alpha)^{-1}>1$. This implies that g_1 is strictly concave on $[0,1]$. We first note that

$$\begin{aligned} g_{1,b}(1) &= (\alpha\beta+\alpha-1)(1+b^2)-2(\alpha\beta-\alpha+1)b \\ &= \frac{1}{p}\left((q-p)b^2-2(p+q-2)b+q-p\right) \geqslant 0. \end{aligned}$$

(i) Suppose that $1<p<2$. To prove (i), we first show that $g_{1,b}(u_1)>0$ for some $u_1\in(0,1]$. If $b<\delta$, then we have $g_{1,b}(1)>0$. For $b=\delta$, the derivative of $g_{1,\delta_{p,q}}$ is

$$g'_{1,\delta}(u) = -2\beta\delta u+(\alpha\beta+\alpha-1)(1+\delta^2)-2\delta u^{\frac{\alpha}{1-\alpha}}.$$

Since $\beta+1>\alpha\beta-\alpha+1$, we have

$$\begin{aligned} g'_{1,\delta}(1) &= (\alpha\beta+\alpha-1)(1+\delta^2)-2(\beta+1)\delta \\ &< (\alpha\beta+\alpha-1)(1+\delta^2)-2(\alpha\beta-\alpha+1)\delta = 0. \end{aligned}$$

Therefore $g_{1,\delta}$ is strictly decreasing on $(1-\varepsilon,1]$ for some $\varepsilon>0$, and so $g_{1,\delta}(1-\varepsilon)>g_{1,\delta}(1)=0$.

Thus, we obtain $g_{1,b}(u_1)>0$ for some $u_1\in(0,1]$. Since $g_{1,b}(0)=-(2\alpha-1)\beta b<0$, by the intermediate value theorem, there exists a real number $u_0\in(0,1)$ such that $g_{1,b}(u_0)=0$. By the strict concavity of $g_{1,b}$, we have (i).

(ii) If $2\leqslant p<\infty$, then we have $g_{1,b}(0)=-(2\alpha-1)\beta b\geqslant 0$. So we obtain $g_{1,b}(u)>0$ for all $u\in(0,1)$ since $g_{1,b}$ is strictly concave. This completes the proof. □

LEMMA 4. *Let $b\in[0,\delta]$ and let g_2 be the real-valued function on $[0,1]$ defined by*

$$g_{2,b}(s) = (\alpha\beta+\alpha-1)(1+b^2)s^\alpha-\alpha\beta b(s^{2\alpha-1}+s)-(1-\alpha)b(s^{2\alpha}+1).$$

(i) $g_{2,\delta}(s)\leqslant 0$ *for all* $s\in[0,1]$.

(ii) *If* $0\leqslant b<\delta$, *then there exists a real number* $s_0\in(0,1)$ *such that* $g_{2,b}(s_0)=0$, $g_{2,b}(s)<0$ *for all* $s\in[0,s_0)$, *and* $g_{2,b}(s)>0$ *for all* $s\in(s_0,1)$.

Proof. The derivative of g_2 is

$$g'_{2,b}(s) = \alpha s^{2\alpha-2}g_{1,b}(s^{1-\alpha}).$$

We note that $g_{2,b}(0)=-(1-\alpha)b<0$. So by Lemma 3, the behavior of $g_{2,b}$ is as follows: If $1<p<2$, putting

$$u_1 = u_0^{\frac{1}{1-\alpha}}$$

then $g_{2,b}$ is strictly decreasing on $[0,u_1]$ and strictly increasing on $[u_1,1]$.

s	0	$\cdots$	u_1	$\cdots$	1
$g'_{2,b}$		$-$	0	$+$	
$g_{2,b}$	$-$	↘		↗	

If $2 \leqslant p < \infty$, then $g_{2,b}$ is strictly increasing on $[0,1]$.

s	0	$\cdots$	1
$g'_{2,b}$		$+$	
$g_{2,b}$	$-$	$\nearrow$	

On the other hand, we have

$$g_{2,b}(1) = (\alpha\beta + \alpha - 1)(1 + b^2) - 2(\alpha\beta - \alpha + 1)b.$$

It follows that $g_{2,\delta}(1) = 0$ and $g_{2,b}(1) > 0$ for all $b < \delta$. Thus, we have $g_{2,\delta}(s) \leqslant 0$ for all $s \in [0,1]$. If $0 \leqslant b < \delta$, by the intermediate value theorem, there exists a real number $s_0 \in (0,1)$ such that $g_{2,b}(s) < 0$ for all $s \in [0, s_0)$ and $g_{2,b}(s) > 0$ for all $s \in (s_0, 1)$, as desired. $\square$

LEMMA 5. *Let $g_{3,b}$ be the real-valued function on $[0,1]$ defined by*

$$g_{3,b}(s) = (s^\alpha + b)^\beta (s^{\alpha-1} - b) + (bs^\alpha + 1)^\beta (bs^{\alpha-1} - 1).$$

(i) $g_{3,\delta}(s) \geqslant 0$ *for all* $s \in [0,1]$.

(ii) *If* $0 \leqslant b < \delta$, *then there exists a real number* $s_1 \in (0,1)$ *such that* $g_{3,b}(s_1) = 0$, $g_{3,b}(s) > 0$ *for all* $s \in [0, s_1)$, *and* $g_{3,b}(s) < 0$ *for all* $s \in (s_1, 1)$.

Proof. We put

$$b_0 = b^{\frac{1}{1-\alpha}}.$$

Since $s^{\alpha-1} \geqslant 1$ for each $s \in [0,1]$, we have $s^{\alpha-1} - b > 0$. If $0 \leqslant s \leqslant b_0$, we obtain $bs^{\alpha-1} - 1 \geqslant 0$, and so $g_{3,b}(s) > 0$. Let $g_{4,b}$ be the real-valued function on $(b_0, 1]$ defined by

$$g_{4,b}(s) = \log(s^\alpha + b)^\beta (s^{\alpha-1} - b) - \log(bs^\alpha + 1)^\beta (1 - bs^{\alpha-1}).$$

Then, it is clear that $g_{3,b}(s) \geqslant 0$ if and only if $g_{4,b}(s) \geqslant 0$. The derivative of $g_{4,b}$ is

$$g'_{4,b}(s) = \frac{(1-b^2)s^{\alpha-2} g_{2,b}(s)}{(s^\alpha + b)(s^{\alpha-1} - b)(bs^\alpha + 1)(1 - bs^{\alpha-1})}.$$

Thus, we have $g'_{4,\delta}(s) \leqslant 0$ for all $s \in (b_0, 1]$ by Lemma 4 (i). Since the function g_4 is decreasing on $(b_0, 1]$, we have $g_{4,\delta}(s) \geqslant g_{4,\delta}(1) = 0$ for all $s \in (b_0, 1]$.

If $0 \leqslant b < \delta$, the behavior of $g_{4,b}$ is as follows by Lemma 4 (ii):

s	b_0	$\cdots$	s_0	$\cdots$	1
$g'_{4,b}$		$-$	0	$+$	
$g_{4,b}$	∞	$\searrow$	$-$	$\nearrow$	0

Hence, by the intermediate value theorem, there exists a real number $s_1 \in (0,1)$ such that $g_{4,b}(s) > 0$ for all $s \in [0,s_1)$ and $g_{4,b}(s) < 0$ for all $s \in (s_1,1)$. $\square$

Proof of Theorem. Suppose that $b \in [0,\delta]$. Let g_b be the real-valued function on $[0,1/2]$ defined by

$$g_b(t) = \left(f_{p,q,b}(t)\right)^q = (t^{\frac{1}{p}} + b(1-t)^{\frac{1}{p}})^q + (bt^{\frac{1}{p}} + (1-t)^{\frac{1}{p}})^q.$$

The derivative of g_b is

$$g_b'(t) = \frac{q}{p}(1-t)^{\frac{q}{p}-1} g_{3,b}\left(\frac{t}{1-t}\right).$$

By Lemma 5 (i), we have $g_\delta'(t) \geqslant 0$ for all $t \in [0,1/2]$. Thus the function g_δ is nondecreasing on $[0,1/2]$, and hence we obtain $g_\delta(1/2) = \max_{0 \leqslant t \leqslant 1/2} g(t)$. This means that $f_{p,q,\delta}(1/2) = \max_{0 \leqslant t \leqslant 1/2} f_{p,q,\delta}(t)$. Thus, by Lemma 2, we have

$$\left(\frac{|u+\gamma_{p,q}v|^q + |u-\gamma_{p,q}v|^q}{2}\right)^{\frac{1}{q}} \leqslant \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{\frac{1}{p}}$$

for all $u,v \in \mathbb{R}$. This proves Theorem 1.

Finally, we show that $\gamma_{p,q}$ is the best constant for Beckner's inequality. Suppose that $\gamma_{p,q} < a \leqslant 1$. Let $b = (1-a)/(1+a)$. By Lemma 2, it is enough to prove that $f_{p,q,b}(1/2) < \max_{0 \leqslant t \leqslant 1/2} f_{p,q,b}(t)$. To this end, we remark that $0 \leqslant b < \delta$. By Lemma 5 (ii), g_b is strictly increasing on $[0,s_2]$ and strictly decreasing on $[s_2,1/2]$, where $s_2 = s_1/(1+s_1)$.

t	0	$\cdots$	s_2	$\cdots$	$1/2$
g_b'		$+$	0	$-$	
g_b		$\nearrow$		$\searrow$	

From this fact, we have $f_{p,q,b}(1/2) < f_{p,q,b}(s_2) = \max_{0 \leqslant t \leqslant 1/2} f_{p,q,b}(t)$. The proof is complete. $\square$

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