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MILLOUX INEQUALITY OF MEROMORPHIC FUNCTION IN ANNULI

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Abstract. The purpose of this paper is to establish the Milloux inequality of meromorphic function in annuli. We obtain a general form of Milloux inequality of meromorphic function in annuli when the multiple values are considered.

1. Introduction and preliminaries

In 2005, Khrystiyanyn and Kondratyuk [6, 7] gave an extension of the Nevalinna value distribution theory for meromorphic functions in annuli. In their extension the main characteristics of meromorphic function are one-parameter and posses the same properties as in the classical case of a simply connected domain. In [6] and [7], we can get the analogues of the Jensen's formula, the first fundamental theorem, the lemma on logarithmic derivative and the second fundamental theorem of the Nevanlinna theory for meromorphic functions on annuli. After [6, 7], Fernàndez [5] study the value distribution of meromorphic functions in the punctured plane, Cao, Deng, Yi and Xu [1]–[3] study the uniqueness of the meromorphic functions in annuli, Chen and Wu [4] study the exceptional values for meromorphic function and its derivatives in annuli. In the following, we introduce the definitions, notations and results of [4] and [6, 7] which will be used in this paper.

Let $f(z)$ be a meromorphic function in the annuli

$$A(R_0) = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\},$$

where $1 < R_0 \leqslant +\infty$. Denote

$$m\left(R, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(Re^{i\theta}) - a|} d\theta,$$

$$m(R,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(Re^{i\theta})| d\theta,$$

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where $a \in \mathbb{C}$ and $\frac{1}{R_0} < R < R_0$. Let

$$m_0\left(R, \frac{1}{f-a}\right) = m\left(R, \frac{1}{f-a}\right) + m\left(\frac{1}{R}, \frac{1}{f-a}\right), \quad 1 < R < R_0$$

and

$$m_0(R, f) = m(R, f) + m\left(\frac{1}{R}, f\right), \quad 1 < R < R_0.$$

Put

$$N_1\left(R, \frac{1}{f-a}\right) = \int_{\frac{1}{R}}^{1} \frac{n_1(t, \frac{1}{f-a})}{t} dt, \quad N_2\left(R, \frac{1}{f-a}\right) = \int_{1}^{R} \frac{n_2(t, \frac{1}{f-a})}{t} dt,$$

where $1 < R < R_0$, $n_1(t, \frac{1}{f-a})$ is the counting function of poles of the function $\frac{1}{f-a}$ in $\{z : t < |z| \leqslant 1\}$ and $n_2(t, \frac{1}{f-a})$ is the counting function of poles of the function $\frac{1}{f-a}$ in $\{z : 1 < |z| \leqslant t\}$. Denote also

$$N_1(R, f) = \int_{\frac{1}{R}}^{1} \frac{n_1(t, f)}{t} dt, \quad N_2(R, f) = \int_{1}^{R} \frac{n_2(t, f)}{t} dt,$$

where $1 < R < R_0$, $n_1(t, f)$ is the counting function of poles of the function f in $\{z : t < |z| \leqslant 1\}$ and $n_2(t, f)$ is the counting function of poles of the function f in $\{z : 1 < |z| \leqslant t\}$. Let

$$N_0(R, a, f) = N_0\left(R, \frac{1}{f-a}\right) = N_1\left(R, \frac{1}{f-a}\right) + N_2\left(R, \frac{1}{f-a}\right),$$

$$N_0(R, \infty, f) = N_0(R, f) = N_1(R, f) + N_2(R, f).$$

Finally, we define the Nevanlinna characteristic of f in $A(R_0)$ by

$$T_0(R, f) = m_0(R, f) - 2m(1, f) + N_0(R, f), \quad 1 < R < R_0,$$

where $R_0 \leqslant +\infty$. Suppose that f, g are two meromorphic functions in $A(R_0)$, where $R_0 \leqslant +\infty$. Then

$$m_0(R, fg) \leqslant m_0(R, f) + m_0(R, g) + O(1). \tag{1.1}$$

DEFINITION 1.1. Let f be a nonconstant meromorphic function on in $A(R_0)$, where $1 < R_0 \leqslant +\infty$. We call f admissible if

$$\limsup_{R \to +\infty} \frac{T_0(R, f)}{\log R} = \infty, \quad 1 < R < R_0 = +\infty;$$

or

$$\limsup_{R \to R_0} \frac{T_0(R, f)}{-\log(R_0 - R)} = \infty, \quad 1 < R < R_0 < +\infty.$$

DEFINITION 1.2. Let f be a nonconstant meromorphic function on in $A(R_0)$, where $1 < R_0 \leqslant +\infty$. Then the order of $f(z)$ is defined by

$$\lambda(f) = \limsup_{R\to+\infty} \frac{\log T_0(R,f)}{\log R}, \quad 1 < R < R_0 = +\infty;$$

or

$$\lambda(f) = \limsup_{R\to R_0} \frac{\log T_0(R,f)}{-\log(R_0 - R)}, \quad 1 < R < R_0 < +\infty.$$

THEOREM A. (The First Fundamental Theorem, see [6, Theorem 2]) *Let f be a nonconstant meromorphic function in $A(R_0)$, where $1 < R_0 \leqslant +\infty$. Let $T_0(R,f)$ be its Nevanlinna characteristic. Then*

$$T_0\left(R, \frac{1}{f-a}\right) = T_0(R,f) + O(1), \quad 1 < R < R_0,$$

for every fixed $a \in \mathbb{C}$.

THEOREM B. (Lemma on the logarithmic derivative, see [7, Theorem 1]) *Let f be a nonconstant meromorphic function in $A(R_0)$, where $1 < R_0 \leqslant +\infty$, and let $\lambda \geqslant 0$. Then*

1. *in the case $R_0 = +\infty$,*

$$m_0\left(R, \frac{f'}{f}\right) = O(\log(RT_0(R,f)))$$

for $R \in (1,+\infty)$ except for the set $\triangle_R$ such that $\int_{\triangle_R} R^{\lambda-1} dR < +\infty$;

2. *in the case $R_0 < +\infty$,*

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log\left(\frac{T_0(R,f)}{R_0 - R}\right)\right)$$

for $R \in (1,R_0)$ except for the set $\triangle'_R$ such that $\int_{\triangle'_R} \frac{dR}{(R_0-R)^{\lambda-1}} < +\infty$.

THEOREM C. (The Second Fundamental Theorem, see [7, Theorem 2]) *Let f be a nonconstant meromorphic function in $A(R_0)$, where $1 < R_0 \leqslant +\infty$. Let $a_1, a_2, \cdots, a_p$ be p distinct finite complex numbers and $\lambda \geqslant 0$. Then*

$$m_0(R,f) + \sum_{\nu=1}^{p} m_0\left(R, \frac{1}{f-a_\nu}\right) \leqslant 2T_0(R,f) - N_0^{(1)}(R,f) + S(R,f),$$

where

$$N_0^{(1)}(R,f) = N_0\left(R, \frac{1}{f'}\right) + 2N_0(R,f) - N_0(R,f'),$$

and

1. in the case $R_0 = +\infty$,

$$S(R,f) = O(\log(RT_0(R,f)))$$

for $R \in (1,+\infty)$ *except for the set* $\triangle_R$ *such that* $\int_{\triangle_R} R^{\lambda-1}dR < +\infty$;

2. in the case $R_0 < +\infty$,

$$S(R,f) = O\left(\log\left(\frac{T_0(R,f)}{R_0 - R}\right)\right)$$

for $R \in (1,R_0)$ *except for the set* $\triangle'_R$ *such that* $\int_{\triangle'_R} \frac{dR}{(R_0-R)^{\lambda-1}} < +\infty$.

2. Milloux inequality meromorphic function in annuli

In this section, we shall establish the Milloux inequality of meromorphic function in annuli and prove the following theorems.

THEOREM 2.1. (Milloux inequality) *Suppose that* $f(z)$ *is an admissible meromorphic function in* $A(R_0)$, *where* $1 < R_0 \leqslant +\infty$. *Let* a,b *be two distinct finite complex number and* $b \neq 0$. *Then for any* $0 < R < R_0$, *we have*

$$T_0(R,f) \leqslant \overline{N}_0(R,f) + (k+1)\overline{N}_0\left(R,\frac{1}{f-a}\right) + \overline{N}_0\left(R,\frac{1}{f^{(k)}-b}\right) + S(R,f).$$

In order to prove Theorem 2.1, we shall prove the following general form of Milloux inequality of meromorphic function in annuli when the multiple values are considered. Firstly, we give the following notations (see [4]).

Let f be a meromorphic function of order ρ in $A(R_0)$, where $1 < R_0 \leqslant +\infty$, and let $a \in \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. We denote by $\overline{n}_1(t,f,a)$ the number of distinct zeros of $f-a$ in $\{z : t < |z| \leqslant 1\}$ (ignoring multiplicity) and by $\overline{n}_2(t,f,a)$ the number of distinct zeros $f-a$ in $\{z : 1 < |z| \leqslant t\}$ (ignoring multiplicity), and

$$\overline{N}_0(R,f,a) = \int_{\frac{1}{R}}^{1} \frac{\overline{n}_1(t,f,a)}{t}dt + \int_1^R \frac{\overline{n}_2(t,f,a)}{t}dt.$$

For any positive integer k, we denote by $\overline{n}_1^k(t,f,a)$ the number of distinct zeros of order $\leqslant k$ of $f-a$ in $\{z : t < |z| \leqslant 1\}$ (ignoring multiplicity) and by $\overline{n}_2^k(t,f,a)$ the number of distinct zeros of order $\leqslant k$ of $f-a$ in $\{z : 1 < |z| \leqslant t\}$ (ignoring multiplicity). $\overline{N}_0^k(R,f,a)$ is defined as

$$\overline{N}_0^k(R,f,a) = \int_{\frac{1}{R}}^{1} \frac{\overline{n}_1^k(t,f,a)}{t}dt + \int_1^R \frac{\overline{n}_2^k(t,f,a)}{t}dt.$$

We denote also by $n_1^k(t,f,a)$ the number of zeros of $f-a$ in $\{z : t < |z| \leqslant 1\}$ and by $n_2^k(t,f,a)$ the number of zeros of $f-a$ in $\{z : 1 < |z| \leqslant t\}$, where a zero of order $< k$

is counted according to its multiplicity and a zero of order $\geqslant k$ is counted exactly k times. $N_0^k(R,f,a)$ is defined as

$$N_0^k(R,f,a) = \int_{\frac{1}{R}}^{1} \frac{n_1^k(t,f,a)}{t}dt + \int_{1}^{R} \frac{n_2^k(t,f,a)}{t}dt.$$

Under the above notations, we shall prove the following theorem.

THEOREM 2.2. (general form of Milloux inequality) *Suppose that $f(z)$ is an admissible meromorphic function in $A(R_0)$, where $1 < R_0 \leqslant +\infty$. Let $a^{[i]}$, $b^{[j]} \in (i = 1,2,\cdots,p;\ j = 1,2,\cdots,q)$ be distinct finite complex number such that $b^{[j]} \neq 0\ (j = 1,2,\cdots,q)$, and let m_i, $n_j\ (i = 1,2,\cdots,p;\ j = 1,2,\cdots,q)$ and l be any positive integers. Then*

$$\begin{aligned}
&\left\{pq - \left(\sum_{i=1}^{p} \tfrac{kq+1}{m_i+1} + \sum_{j=1}^{q} \tfrac{1}{n_j+1} + \tfrac{1}{l+1}\left(1 + k\sum_{j=1}^{q} \tfrac{1}{n_j+1}\right)\right)\right\} T_0(R,f) \\
&\leqslant \tfrac{l}{l+1}\left(1 + k\sum_{j=1}^{q} \tfrac{1}{n_j+1}\right)\overline{N}_0^l(R,f) + (kq+1)\sum_{i=1}^{p} \overline{N}_0^{m_i}\left(R, \tfrac{1}{f-a^{[i]}}\right) \\
&\quad + \sum_{j=1}^{q} \overline{N}_0^{n_j}\left(R, \tfrac{1}{f^{(k)}-b^{[j]}}\right) + S(R,f).
\end{aligned} \tag{2.1}$$

Letting $p = q = 1$ and l, m_i, n_j tend to infinity in (2.1), we can get Theorem 2.1. Here, we give a proof of Theorem 2.2.

Proof. From [8], we have

$$\begin{aligned}
T_0(R,f') &= T_0(R,f\tfrac{f'}{f}) \leqslant T_0(R,f) + T_0(R,\tfrac{f'}{f}) + O(1) \\
&= T_0(R,f) + m_0(R,\tfrac{f'}{f}) + N_0(R,\tfrac{f'}{f}) - 2m(1,\tfrac{f'}{f}) + O(1) \\
&\leqslant T_0(R,f) + \overline{N}_0(R,f) + S(R,f) \\
&\leqslant 2T_0(R,f) + S(R,f).
\end{aligned} \tag{2.2}$$

Hence, by Theorem B and (2.2), we can get

$$S(R,f^{(k)}) = O(\log RT_0(R,f^{(k)})) = O(\log RT_0(R,f) = S(r,f), \tag{2.3}$$

and

$$m_0\left(R, \frac{f^k}{f-a^{[i]}}\right) = S(R,f), \tag{2.4}$$

holds for any positive any $a^{[i]}$. Put

$$F(z) = \sum_{i=1}^{p} \frac{1}{f(z)-a^{[i]}}.$$

Then as in [9] we have

$$
\begin{aligned}
m(R,F)+O(1) &\geqslant \sum_{i=1}^{p} m\left(R,\frac{1}{f(z)-a^{[i]}}\right),\\
m\left(\frac{1}{R},F\right)+O(1) &\geqslant \sum_{i=1}^{p} m\left(\frac{1}{R},\frac{1}{f(z)-a^{[i]}}\right).
\end{aligned}
\tag{2.5}
$$

In fact, (2.5) holds if $p=1$. If $p\geqslant 2$, put

$$\delta=\min_{i\neq j}|a^{[i]}-a^{[j]}|.$$

Let for the moment $i\in\{1,2,\cdots,q\}$ be fixed. Then we get in every point where

$$|f(z)-a^{[i]}|<\frac{\delta}{2q}\leqslant\frac{\delta}{4},$$

the inequality

$$|f(z)-a^{[j]}|\geqslant|a^{[i]}-a^{[j]}|-|f(z)-a^{[i]}|\geqslant\frac{3\delta}{4},$$

for $i\neq j$. Therefore the set of points on $\partial\mathbb{C}_r$ where $\mathbb{C}_r=\{z:|z|=r\}$ ($r=R$ or $r=\frac{1}{R}$). which is determined by $|f(z)-a^{[i]}|<\frac{\delta}{2q}$ is either empty or any two such sets for different i have empty intersection. In any case

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\log^+|F(re^{i\theta})|d\theta &\geqslant \frac{1}{2\pi}\sum_{i=1}^{q}\int_{|f(z)-a^{[i]}|<\frac{\delta}{2q},|z|=r}\log^+|F(re^{i\theta})|d\theta\\
&\geqslant \frac{1}{2\pi}\sum_{i=1}^{q}\int_{|f(z)-a^{[i]}|<\frac{\delta}{2q},|z|=r}\log^+\frac{1}{|f(re^{i\theta})-a^{[i]}|}d\theta.
\end{aligned}
$$

Because of

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{|f(z)-a^{[i]}|<\frac{\delta}{2q},|z|=r}\log^+\frac{1}{|f(re^{i\theta})-a^{[i]}|}d\theta\\
&=m\left(r,\frac{1}{f(z)-a^{[i]}}\right)-\frac{1}{2\pi}\int_{|f(z)-a^{[i]}|\geqslant\frac{\delta}{2q},|z|=r}\log^+\frac{1}{|f(re^{i\theta})-a^{[i]}|}d\theta\\
&\geqslant m\left(r,\frac{1}{f(z)-a^{[i]}}\right)-\log^+\frac{2q}{\delta},
\end{aligned}
$$

It follows

$$m(r,F)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|F(re^{i\theta})|d\theta\geqslant\sum_{i=1}^{q}m\left(r,\frac{1}{f(z)-a^{[i]}}\right)-q\log^+\frac{2q}{\delta}.$$

Hence (2.5) follows from the above inequality under the case of $r=R$ and $r=\frac{1}{R}$. Since

$$
\begin{aligned}
m(R,F) &\leqslant m\left(R,f^{(k)}F\right)+m\left(R,\frac{1}{f^{(k)}}\right)\\
&\leqslant \sum_{i=1}^{p} m\left(R,\frac{f^{(k)}}{f-a^{[i]}}\right)+m\left(R,\frac{1}{f^{(k)}}\right),
\end{aligned}
$$

and

$$m(\tfrac{1}{R},F) \leqslant m\left(\tfrac{1}{R}, f^{(k)}F\right) + m\left(\tfrac{1}{R}, \tfrac{1}{f^{(k)}}\right)$$
$$\leqslant \sum_{i=1}^{p} m\left(\tfrac{1}{R}, \tfrac{f^{(k)}}{f-a^{[i]}}\right) + m\left(\tfrac{1}{R}, \tfrac{1}{f^{(k)}}\right).$$

Therefore,

$$m_0(R,F) \leqslant \sum_{i=1}^{p} m_0\left(R, \tfrac{f^{(k)}}{f-a^{[i]}}\right) + m_0\left(R, \tfrac{1}{f^{(k)}}\right), \tag{2.6}$$

It follows from (2.3)–(2.6) and Theorem A that

$$\begin{aligned}\sum_{i=1}^{p} m_0\left(R, \tfrac{1}{f(z)-a^{[i]}}\right) &\leqslant m_0\left(R, \tfrac{1}{f^{(k)}}\right) + S(R,f)\\ &\leqslant T_0\left(R, f^{(k)}\right) - N_0\left(R, \tfrac{1}{f^{(k)}}\right) + S(R,f).\end{aligned} \tag{2.7}$$

Thus

$$pT_0(R,f) \leqslant \sum_{i=1}^{p} N_0\left(R, \tfrac{1}{f(z)-a^{[i]}}\right) + T_0\left(R, f^{(k)}\right) - N_0\left(R, \tfrac{1}{f^{(k)}}\right) + S(R,f). \tag{2.8}$$

Now it follows from Theorems A, B and (2.3) that

$$\begin{aligned}qT_0(R,f^{(k)}) &\leqslant \sum_{j=1}^{q} N_0\left(R, \tfrac{1}{f^{(k)}-b^{[j]}}\right) + N_0\left(R, \tfrac{1}{f^{(k)}}\right) + N_0(R, f^{(k)})\\ &\quad -(N_0\left(R, \tfrac{1}{f^{(k+1)}}\right) + 2N_0(R, f^{(k)}) - N_0(R, f^{(k+1)})) + S(R, f^{(k)})\\ &= \sum_{j=1}^{q} N_0\left(R, \tfrac{1}{f^{(k)}-b^{[j]}}\right) + N_0\left(R, \tfrac{1}{f^{(k)}}\right) - N_0(R, f^{(k)})\\ &\quad +N_0(R, f^{(k+1)}) - N_0\left(R, \tfrac{1}{f^{(k+1)}}\right) + S(R,f)\\ &\leqslant \sum_{j=1}^{q} N_0\left(R, \tfrac{1}{f^{(k)}-b^{[j]}}\right) + N_0\left(R, \tfrac{1}{f^{(k)}}\right) + \overline{N}_0(R,f)\\ &\quad -N_0\left(R, \tfrac{1}{f^{(k+1)}}\right) + S(R,f)\end{aligned} \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$\begin{aligned}pqT_0(R,f) &\leqslant \overline{N}_0(R,f) + (q-1)\left\{\sum_{i=1}^{p} N_0\left(R, \tfrac{1}{f(z)-a^{[i]}}\right) - N_0\left(R, \tfrac{1}{f^{(k)}}\right)\right\}\\ &\quad + \left\{\sum_{i=1}^{p} N_0\left(R, \tfrac{1}{f(z)-a^{[i]}}\right) + \sum_{j=1}^{q} N_0\left(R, \tfrac{1}{f^{(k)}-b^{[j]}}\right) - N_0\left(R, \tfrac{1}{f^{(k+1)}}\right)\right\}\\ &\quad +S(R,f).\end{aligned} \tag{2.10}$$

By [4], we have

$$\begin{aligned}&\sum_{i=1}^{p} N_0\left(R, \tfrac{1}{f(z)-a^{[i]}}\right) + \sum_{j=1}^{q} N_0\left(R, \tfrac{1}{f^{(k)}-b^{[j]}}\right) - N_0\left(R, \tfrac{1}{f^{(k+1)}}\right)\\ &\leqslant \sum_{i=1}^{p} N_0^{k+1}(R, \tfrac{1}{f-a^{[i]}}) + \sum_{j=1}^{q} \overline{N}_0(R, \tfrac{1}{f^{(k)}-b^{[j]}}),\end{aligned} \tag{2.11}$$

and

$$\sum_{i=1}^{p} N_0\left(R, \frac{1}{f(z)-a^{[i]}}\right) - N_0\left(R, \frac{1}{f^{(k)}}\right) \leqslant \sum_{i=1}^{p} N_0^k(R, \frac{1}{f-a^{[i]}}). \tag{2.12}$$

Substituting (2.11) and (2.12) to (2.10), we obtain

$$\begin{aligned} pqT_0(R,f) \leqslant \overline{N}_0(R,f) + (q-1)\sum_{i=1}^{p} N_0^k(R, \frac{1}{f-a^{[i]}}) + \sum_{i=1}^{p} N_0^{k+1}(R, \frac{1}{f-a^{[i]}}) \\ + \sum_{j=1}^{q} \overline{N}_0(R, \frac{1}{f^{(k)}-b^{[j]}}) + S(R,f). \end{aligned} \tag{2.13}$$

Since

$$\begin{aligned} N_0^{k+1}(R, \frac{1}{f-a^{[i]}}) &\leqslant (k+1)\overline{N}_0(R, \frac{1}{f-a^{[i]}}) \\ &\leqslant \frac{k+1}{m_i+1}\left\{m_i\overline{N}_0^{m_i}(R, \frac{1}{f-a^{[i]}}) + N_0(R, \frac{1}{f-a^{[i]}})\right\} \\ &\leqslant \frac{k+1}{m_i+1}\left\{m_i\overline{N}_0^{m_i}(R, \frac{1}{f-a^{[i]}}) + T_0(R,f)\right\} + O(1), \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} N_0^k(R, \frac{1}{f-a^{[i]}}) &\leqslant k\overline{N}_0(R, \frac{1}{f-a^{[i]}}) \\ &\leqslant \frac{k}{m_i+1}\left\{m_i\overline{N}_0^{m_i}(R, \frac{1}{f-a^{[i]}}) + N_0(R, \frac{1}{f-a^{[i]}})\right\} \\ &\leqslant \frac{k}{m_i+1}\left\{m_i\overline{N}_0^{m_i}(R, \frac{1}{f-a^{[i]}}) + T_0(R,f)\right\} + O(1), \end{aligned} \tag{2.15}$$

Similarly, we can get

$$\overline{N}_0(R, \frac{1}{f^{(k)}-b^{[j]}}) \leqslant \frac{1}{n_j+1}\left\{n_j\overline{N}_0^{n_j}(R, \frac{1}{f^{(k)}-b^{[j]}}) + T_0(R, f^{(k)})\right\} + O(1), \tag{2.16}$$

and

$$\overline{N}_0(R,f) \leqslant \frac{1}{l+1}\left\{l\overline{N}_0^{l}(R,f) + T_0(R,f)\right\}. \tag{2.17}$$

By (2.4), we have

$$\begin{aligned} T_0(R, f^{(k)}) &= m_0(r, f^{(k)}) + m_0(R, f^{(k)}) - m(1, f^{(k)}) \\ &\leqslant m_0(R,f) + m_0(R, \frac{f^{(k)}}{f}) + N_0(R,f) + k\overline{N}_0(R,f) + O(1) \\ &= T_0(R,f) + k\overline{N}_0(R,f) + S(R,f). \end{aligned} \tag{2.18}$$

Substituting (2.14)–(2.18) to (2.13), we obtain

$$\begin{aligned}
pqT_0(R,f) \leqslant{}& \overline{N}_0(R,f) + (q-1)\sum_{i=1}^{p}\frac{k}{m_i+1}\left\{m_i\overline{N}_0^{m_i}(R,\tfrac{1}{f-a^{[i]}}) + T_0(R,f)\right\}\\
&+\sum_{i=1}^{p}\frac{k+1}{m_i+1}\left\{m_i\overline{N}_0^{m_i}(R,\tfrac{1}{f-a^{[i]}}) + T_0(R,f)\right\}\\
&+\sum_{j=1}^{q}\frac{1}{n_j+1}\left\{n_j\overline{N}_0^{n_j}(R,\tfrac{1}{f^{(k)}-b^{[j]}}) + T_0(R,f^{(k)})\right\} + S(R,f)\\
\leqslant{}& \left(1+\sum_{j=1}^{q}\frac{k}{n_j+1}\right)\overline{N}_0(R,f) + (kq+1)\sum_{i=1}^{p}\frac{m_i}{m_i+1}\overline{N}_0^{m_i}(R,\tfrac{1}{f-a^{[i]}})\\
&+\sum_{i=1}^{p}\frac{kq+1}{m_i+1}T_0(R,f) + \sum_{j=1}^{q}\frac{n_j}{n_j+1}\overline{N}_0^{n_j}(R,\tfrac{1}{f^{(k)}-b^{[j]}})\\
&+\sum_{j=1}^{q}\frac{1}{n_j+1}T_0(R,f) + S(R,f)\\
\leqslant{}& \left(1+\sum_{j=1}^{q}\frac{k}{n_j+1}\right)\frac{l}{l+1}\overline{N}_0^{l}(R,f) + (kq+1)\sum_{i=1}^{p}\overline{N}_0^{m_i}(R,\tfrac{1}{f-a^{[i]}})\\
&+\left\{\sum_{i=1}^{p}\frac{kq+1}{m_i+1} + \sum_{j=1}^{q}\frac{1}{n_j+1} + \frac{1}{l+1}\left(1+k\sum_{j=1}^{q}\frac{1}{n_j+1}\right)\right\}T_0(R,f)\\
&+\sum_{j=1}^{q}\frac{1}{n_j+1}T_0(R,f) + \sum_{j=1}^{q}\overline{N}_0^{n_j}(R,\tfrac{1}{f^{(k)}-b^{[j]}}) + S(R,f).
\end{aligned} \tag{2.19}$$

Hence, (2.1) follows from (2.19). □

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