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NEW CONVERSES OF THE JESSEN AND LAH–RIBARIČ INEQUALITIES II

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Abstract. New converses of the Jessen and Lah-Ribarič inequalities for continuous convex functions are studied. Applications are given for generalized and arithmetic means, Hölder's inequality, Hadamard's inequality, and the inequalities of Giaccardi and Petrović.

1. Introduction

The Jensen inequality for convex functions plays a very important role in the Theory of Inequalities due to the fact that it implies the whole series of the other classical inequalities such as the quasi-arithmetic mean and arithmetic mean inequalities, Hölder and Minkowski inequalities, Ky Fan's inequality etc.

In this paper we refer to a general form of the Jensen inequality for positive linear functionals. In order to present our results, we first introduce the appropriate settings.

Let E be a nonempty set and L be a linear class of real-valued functions $f\colon E \to \mathbb{R}$ having the properties:

L1: $f,g \in L \Rightarrow (af+bg) \in L$ for all $a,b \in \mathbb{R}$;

L2: $1 \in L$, i.e., if $f(t)=1$ for every $t \in E$, then $f \in L$.

We also consider positive linear functionals $A\colon L \to \mathbb{R}$. That is, we assume that:

A1: $A(af+bg) = aA(f)+bA(g)$ for $f,g \in L$ and $a,b \in \mathbb{R}$;

A2: $f \in L$, $f(t) \geqslant 0$ for every $t \in E \Rightarrow A(f) \geqslant 0$ (A is positive).

Jessen [7] gave the following generalization of Jensen's inequality for convex functions (see also [11, p. 47]):

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THEOREM 1.1. ([7]) *Let L satisfy properties $L1, L2$ on a nonempty set E, and assume that ϕ is a continous convex function on an interval $I \subset \mathbb{R}$. If A is a positive linear functional with $A(1) = 1$, then for all $f \in L$ such that $\phi(f) \in L$ we have $A(f) \in I$ and*

$$\phi(A(f)) \leqslant A(\phi(f)). \tag{1.1}$$

We also need to recall the following generalization of the Lah-Ribarič inequality for positive linear functionals which is proved in [1] by Beesack and Pečarić (see also [11, p. 98]):

THEOREM 1.2. ([1]) *Let ϕ be convex on $I = [m, M]$ $(-\infty < m < M < \infty)$. Let L satisfy conditions $L1, L2$ on E and let A be any positive linear functional on L with $A(1) = 1$. Then for every $f \in L$ such that $\phi(f) \in L$ (so that $m \leqslant f(t) \leqslant M$ for all $t \in E$), we have*

$$A(\phi(f)) \leqslant \frac{(M - A(f))\phi(m) + (A(f) - m)\phi(M)}{M - m}. \tag{1.2}$$

Let $(\Omega, \mathscr{A}, \mu)$ be a measure space consisting of a set Ω, a σ-algebra $\mathscr{A}$ of subsets of Ω and a countably additive and positive measure μ on $\mathscr{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w\colon \Omega \to \mathbb{R}$, with $w(x) \geqslant 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f\colon \Omega \to \mathbb{R}, f \text{ is } \mu - \text{measurable and } \textstyle\int_\Omega w(x)|f(x)|\mathrm{d}\mu(x) < \infty\}.$$

S.S.Dragomir [4] gave the following converse of Jensen's inequality:

THEOREM 1.3. ([4]) *Let $\phi\colon I \to \mathbb{R}$ be a continuous convex function on an interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I. Let $w > 0$ such that $\int w\mathrm{d}\mu = 1$. If $f\colon \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leqslant f(t) \leqslant M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that $f, \phi \circ f \in L_w(\Omega, \mu)$, then

$$\begin{aligned}
0 &\leqslant \int_\Omega w(t)\phi(f(t))\mathrm{d}\mu(t) - \phi(\bar{f}_{\Omega,w}) \\
&\leqslant \frac{(M - \bar{f}_{\Omega,w})(\bar{f}_{\Omega,w} - m)}{M - m} \sup_{t \in \langle m, M\rangle} \tilde{\Psi}_\phi(t; m, M) \\
&\leqslant (M - \bar{f}_{\Omega,w})(\bar{f}_{\Omega,w} - m)\frac{\phi'_-(M) - \phi'_+(m)}{M - m} \\
&\leqslant \frac{1}{4}(M - m)(\phi'_-(M) - \phi'_+(m)),
\end{aligned} \tag{1.3}$$

where $\bar{f}_{\Omega,w} := \int_\Omega w(t)f(t)\mathrm{d}\mu(t) \in [m, M]$ and $\tilde{\Psi}_\phi(\cdot; m, M)\colon \langle m, M\rangle \to \mathbb{R}$ is defined by

$$\tilde{\Psi}_\phi(t; m, M) = \frac{\phi(M) - \phi(t)}{M - t} - \frac{\phi(t) - \phi(m)}{t - m}.$$

We also have the inequalities

$$0 \leqslant \int_{\Omega} w(t)\phi(f(t))\mathrm{d}\mu(t) - \phi(\bar{f}_{\Omega,w}) \leqslant \frac{1}{4}(M-m)\tilde{\Psi}_{\phi}(\bar{f}_{\Omega,w};m,M)$$
$$\leqslant \frac{1}{4}(M-m)(\phi'_{-}(M) - \phi'_{+}(m)), \tag{1.4}$$

provided that $\bar{f}_{\Omega,w} \in \langle m,M\rangle$.

The main objective of this paper is to give improvements of the converses of Lah-Ribarič's and Jessen's inequalities for positive linear functionals obtained by the authors in [6]. Also, we shall give applications of these results to generalized means, power means, Hölder's inequality, Hadamard's inequality and to inequalities of Giaccardi and Petrović.

2. Results

The results in this section are converses of Jessen's and Lah-Ribarič's inequality for positive linear functionals.

THEOREM 2.1. *Let* ϕ *be a continuous convex function on an interval of real numbers* I *and* $m,M \in \mathbb{R}$, $m < M$ *with* $[m,M] \subset \mathring{I}$, *where* $\mathring{I}$ *is the interior of* I. *Let* L *satisfy conditions* $L1, L2$ *on* E *and let* A *be any positive linear functional on* L *with* $A(1) = 1$. *If* $f \in L$ *satisfies the bounds*

$$-\infty < m \leqslant f(t) \leqslant M < \infty \text{ for every } t \in E$$

and $\phi \circ f \in L$, *then*

$$\begin{aligned}0 &\leqslant A(\phi(f)) - \phi(A(f))\\ &\leqslant (M - A(f))(A(f) - m) \sup_{t \in \langle m,M\rangle} \Psi_{\phi}(t;m,M)\\ &\leqslant (M - A(f))(A(f) - m)\frac{\phi'_{-}(M) - \phi'_{+}(m)}{M-m}\\ &\leqslant \frac{1}{4}(M-m)(\phi'_{-}(M) - \phi'_{+}(m)).\end{aligned} \tag{2.1}$$

We also have the inequalities

$$0 \leqslant A(\phi(f)) - \phi(A(f)) \leqslant \frac{1}{4}(M-m)^2\Psi_{\phi}(A(f);m,M)$$
$$\leqslant \frac{1}{4}(M-m)(\phi'_{-}(M) - \phi'_{+}(m)), \tag{2.2}$$

where $\Psi_{\phi}(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ *is defined by*

$$\Psi_{\phi}(t;m,M) = \frac{1}{M-m}\left(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right) \tag{2.3}$$

and we assume that $\Psi_\phi(f;m,M) \in L$. *If* ϕ *is concave on* I, *then the inequality signs in (2.1) and (2.2) are reversed.*

Proof. First we assume that ϕ is convex. If $A(f) = m$ or $A(f) = M$, the inequalities are clear. Let us suppose that $A(f) \in \langle m,M\rangle$.

The first inequality in (2.1) and (2.2) follows directly from Theorem 1.1. By Theorem 1.2, we have

$$\begin{aligned}
A(\phi(f)) - \phi(A(f)) &\leqslant \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - \phi(A(f)) \\
&= \frac{(M-A(f))(A(f)-m)}{M-m}\left\{\frac{\phi(M)-\phi(A(f))}{M-A(f)} - \frac{\phi(A(f))-\phi(m)}{A(f)-m}\right\} \\
&= (M-A(f))(A(f)-m)\Psi_\phi(A(f);m,M) \\
&\leqslant (M-A(f))(A(f)-m)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M),
\end{aligned}$$

so we have proved the second inequality in (2.1).

$$\begin{aligned}
\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) &= \frac{1}{M-m}\sup_{t\in\langle m,M\rangle}\left\{\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right\} \\
&\leqslant \frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t} + \sup_{t\in\langle m,M\rangle}\frac{-(\phi(t)-\phi(m))}{t-m}\right) \\
&= \frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t} - \inf_{t\in\langle m,M\rangle}\frac{\phi(t)-\phi(m)}{t-m}\right) = \frac{\phi'_-(M)-\phi'_+(m)}{M-m},
\end{aligned}$$

which proves the third inequality in (2.1). To prove the last inequality in (2.1), we notice that for every $t \in [m,M]$, the inequality $\frac{(M-t)(t-m)}{M-m} \leqslant \frac{1}{4}(M-m)$ is valid. Since $A(f) \in [m,M]$, we can replace $t \leftrightarrow A(f)$ and the proof is completed.

The proof of the inequalities (2.2) is clear from the proof of the inequalities (2.1). If ϕ is concave, then $-\phi$ is convex, so we can apply (2.1) and (2.2) to function $-\phi$ and obtain reversed inequalities for ϕ. $\square$

REMARK 2.1. *Observe that the function* $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$, *defined by*

$$\Psi_\phi(t;m,M) = \frac{1}{M-m}\left(\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right),$$

is actually the second order divided difference of the function ϕ *at the points* m, t *and* M *for any* $t \in \langle m,M\rangle$.

In order to obtain a converse of the Lah-Ribarič inequality for convex functions, we need the following result:

LEMMA 2.1. *Let ϕ be a continuous convex function on an interval of real numbers I and $m,M \in \mathbb{R}$, $m < M$ with $[m,M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I. Then for any $t \in [m,M]$ the following inequalities are valid:*

$$\begin{aligned}\Delta_\phi(t;m,M) &= \frac{(t-m)\phi(M)+(M-t)\phi(m)}{M-m} - \phi(t) \\ &\leqslant (M-t)(t-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\leqslant \frac{(M-t)(t-m)}{M-m}(\phi'_-(M)-\phi'_+(m)) \\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).\end{aligned} \tag{2.4}$$

Also we have

$$\begin{aligned}\Delta_\phi(t;m,M) &\leqslant \frac{1}{4}(M-m)^2\Psi_\phi(t;m,M) \\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)),\end{aligned} \tag{2.5}$$

where $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ is defined by (2.3), and we assume that $\Psi_\phi(f;m,M) \in L$. If ϕ is concave, the inequality signs in (2.4) and (2.5) are reversed.

Proof. Let us suppose that ϕ is convex. If $t = m$ or $t = M$, the inequalities are clear. For any $t \in \langle m,M\rangle$ we have

$$\begin{aligned}&\Delta_\phi(t;m,M) = \frac{(t-m)\phi(M)+(M-t)\phi(m)}{M-m} - \phi(t) \\ &= \frac{(M-t)(t-m)}{M-m}\left[\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right] = (M-t)(t-m)\Psi_\phi(t;m,M) \\ &\leqslant (M-t)(t-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M),\end{aligned}$$

which is the first inequality in (2.4). The second inequality follows directly from

$$\begin{aligned}&\sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) = \frac{1}{M-m}\sup_{t\in\langle m,M\rangle}\left\{\frac{\phi(M)-\phi(t)}{M-t} - \frac{\phi(t)-\phi(m)}{t-m}\right\} \\ &\leqslant \frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t} + \sup_{t\in\langle m,M\rangle}\frac{-(\phi(t)-\phi(m))}{t-m}\right) \\ &= \frac{1}{M-m}\left(\sup_{t\in\langle m,M\rangle}\frac{\phi(M)-\phi(t)}{M-t} - \inf_{t\in\langle m,M\rangle}\frac{\phi(t)-\phi(m)}{t-m}\right) = \frac{\phi'_-(M)-\phi'_+(m)}{M-m}.\end{aligned}$$

To prove the last inequality in (2.4), we notice that for every $t \in [m,M]$, the inequality $\frac{(M-t)(t-m)}{M-m} \leqslant \frac{1}{4}(M-m)$ is valid. The proof of the inequalities (2.5) is clear from the proof of the inequalities (2.4). If ϕ is concave, then $-\phi$ is convex, so we can apply (2.4) and (2.5) to function $-\phi$ and obtain reversed inequalities for ϕ. $\square$

THEOREM 2.2. *Let us suppose that the assumptions from Theorem 2.1 hold. If $f \in L$ satisfies the bounds*

$$-\infty < m \leqslant f(t) \leqslant M < \infty \text{ for every } t \in E$$

and $\phi \circ f \in L$, then we have the following inequalities

(i)

$$\begin{aligned}0 &\leqslant \frac{(A(f)-m)\phi(M)+(M-A(f))\phi(m)}{M-m} - A(\phi(f))\\ &\leqslant A[(M-f)(f-m)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M)\\ &\leqslant \frac{A[(M-f)(f-m)]}{M-m}(\phi'_-(M)-\phi'_+(m))\\ &\leqslant \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M)-\phi'_+(m))\\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m))\end{aligned} \tag{2.6}$$

(ii)

$$\begin{aligned}0 &\leqslant \frac{(A(f)-m)\phi(M)+(M-A(f))\phi(m)}{M-m} - A(\phi(f))\\ &\leqslant A[(M-f)(f-m)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M)\\ &\leqslant (M-A(f))(A(f)-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M)\\ &\leqslant \frac{(M-A(f))(A(f)-m)}{M-m}(\phi'_-(M)-\phi'_+(m))\\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m))\end{aligned} \tag{2.7}$$

(iii)

$$\begin{aligned}0 &\leqslant \frac{(A(f)-m)\phi(M)+(M-A(f))\phi(m)}{M-m} - A(\phi(f))\\ &\leqslant \frac{1}{4}(M-m)^2 A(\Psi_\phi(f;m,M))\\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m))\end{aligned} \tag{2.8}$$

where $\Psi_\phi(\cdot;m,M)\colon \langle m,M\rangle \to \mathbb{R}$ is defined by (2.3), and we assume that $\Psi_\phi(f;m,M) \in L$. If ϕ is concave, the inequalities are reversed.

Proof. Let us assume that ϕ is convex. The first inequality in (2.6), (2.7) and (2.8) follows directly from Theorem 1.2.

Since f satisfies the bounds $m \leqslant f(t) \leqslant M$ for every $t \in [m,M]$, we can replace t with $f(t)$ in inequalities (2.4) and (2.5) from Lemma 2.1 and obtain

$$\begin{aligned}
&\frac{(f(t)-m)\phi(M)+(M-f(t))\phi(m)}{M-m}-\phi(f(t)) \\
&\leqslant (M-f(t))(f(t)-m)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) \\
&\leqslant \frac{(M-f(t))(f(t)-m)}{M-m}(\phi'_-(M)-\phi'_+(m)) \\
&\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m))
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&\frac{(f(t)-m)\phi(M)+(M-f(t))\phi(m)}{M-m}-\phi(f(t)) \\
&\leqslant \frac{1}{4}(M-m)^2\Psi_\phi(f;m,M) \\
&\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)).
\end{aligned} \tag{2.10}$$

Now we apply linear functional A, which is positive, to inequalities (2.9) and (2.10) and obtain inequalities (2.8) and first three inequalities in (2.6). To prove the fourth inequality in (2.6), we need to notice that the function $g(t)=(M-t)(t-m)$ is concave, so by the Jessen inequality we have $A(g(f)) \leqslant g(A(f))$. Since for every $t \in [m,M]$, the inequality $\frac{(M-t)(t-m)}{M-m} \leqslant \frac{1}{4}(M-m)$ is valid, we can replace $t \leftrightarrow A(f) \in [m,M]$ to obtain the last inequality in (2.6).

The first inequality in (2.7) is the first inequality in (2.6). Again, the function $g(t)=(M-t)(t-m)$ is concave, so from Jessen's inequality it follows that $A([M-f][f-m]) \leqslant (M-A(f))(A(f)-m)$, which proves the second inequality in (2.7). In the proof of Lemma 2.1 we have shown that the inequality $\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) \leqslant \frac{\phi'_-(M)-\phi'_+(m)}{M-m}$ is valid, so the third inequality in (2.7) directly follows. To prove the last inequality in (2.7), we notice that for every $t \in [m,M]$, the inequality $\frac{(M-t)(t-m)}{M-m} \leqslant \frac{1}{4}(M-m)$ is valid. Since $A(f) \in [m,M]$, we have $\frac{(M-A(f))(A(f)-m)}{M-m} \leqslant \frac{1}{4}(M-m)$ and thus the proof is complete.

If ϕ is concave, then $-\phi$ is convex, so we can apply (2.6), (2.7) and (2.8) to function $-\phi$ and obtain reversed inequalities for ϕ. $\square$

3. Applications

In this section we will give applications of the main results obtained in the previous section to generalized mean, quasi-arithmetic mean, Hölder's inequality and to inequalities of Giaccardi and Petrović. We will also compare those results with some related results known from the literature.

3.1. Generalized means

DEFINITION 3.1.1. *Let $I = \langle a,b\rangle$, $-\infty \leqslant a < b \leqslant \infty$, and let $\psi\colon I \to \mathbb{R}$ be continuous and strictly monotonic. Suppose that L and A satisfy the conditions $L1, L2$ and $A1, A2$ with $A(1) = 1$ on a non-empty set E, and that $\psi(f) \in L$ for some $f \in L$. Generalized mean with respect to the functional A and ψ for $f \in L$ is defined by*

$$M_\psi(f,A) = \psi^{-1}(A(\psi(f))). \tag{3.1.1}$$

The following result is a generalization to positive linear functionals of the general means inequality found in [11]:

THEOREM 3.1.1. ([11]) *Let $I = \langle a,b\rangle$, $-\infty \leqslant a < b \leqslant \infty$, and let $\psi, \chi\colon I \to \mathbb{R}$ be continuous and strictly monotonic. Suppose that L and A satisfy the conditions $L1, L2$ and $A1, A2$ with $A(1) = 1$ on a non-empty set E, and let $f \in L$ be such that $\psi(f), \chi(f) \in L$. Then the following inequality is valid*

$$M_\psi(f,A) \leqslant M_\chi(f,A), \tag{3.1.2}$$

provided either χ is increasing and $\phi = \chi \circ \psi^{-1}$ is convex, or χ is decreasing and $\phi = \chi \circ \psi^{-1}$ is concave.

THEOREM 3.1.2. ([11, p. 108, Theorem 4.3]) *Let L, A, ψ and χ be as in Theorem 3.1.1, but with $I = [m,M]$, $-\infty < m < M < \infty$. Then for every $f \in L$ such that $m \leqslant f(t) \leqslant M$, $t \in E$ we have*

$$(\psi(M) - \psi(m))A(\chi(f)) - (\chi(M) - \chi(m))A(\psi(f)) \leqslant \psi(M)\chi(m) - \chi(M)\psi(m), \tag{3.1.3}$$

provided that $\phi = \chi \circ \psi^{-1}$ is convex. The inequality in (3.1.3) is reversed if ϕ is concave.

THEOREM 3.1.3. *Let L, A, ψ, χ satisfy conditions of the Theorem 3.1.1. Let $I \supset [m,M]$, $-\infty < m < M < \infty$, and let us assume that the function $\phi = \chi \circ \psi^{-1}$ is convex. Then for every $f \in L$ such that $m \leqslant f(t) \leqslant M$ for $t \in [m,M]$ and $\psi(f), \chi(f) \in L$ we* *have*

$$\begin{aligned}
0 &\leqslant \chi(M_\chi(f,A)) - \chi(M_\psi(f,A)) \\
&\leqslant (M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi) \sup_{t\in\langle m,M\rangle} \Psi_{\chi\circ\psi^{-1}}(\psi(t);m_\psi,M_\psi) \\
&\leqslant (M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi) \frac{[\chi\circ\psi^{-1}]'_-(M_\psi) - [\chi\circ\psi^{-1}]'_+(m_\psi)}{M_\psi - m_\psi} \qquad (3.1.4) \\
&\leqslant \frac{1}{4}(M_\psi - m_\psi)([\chi\circ\psi^{-1}]'_-(M_\psi) - [\chi\circ\psi^{-1}]'_+(m_\psi))
\end{aligned}$$

We also have the inequalities

$$\begin{aligned}
0 &\leqslant \chi(M_\chi(f,A)) - \chi(M_\psi(f,A)) \leqslant \frac{1}{4}(M_\psi - m_\psi)^2 \Psi_{\chi\circ\psi^{-1}}(A(\psi(f));m_\psi,M_\psi) \\
&\leqslant \frac{1}{4}(M_\psi - m_\psi)([\chi\circ\psi^{-1}]'_-(M_\psi) - [\chi\circ\psi^{-1}]'_+(m_\psi)), \qquad (3.1.5)
\end{aligned}$$

where $[m_\psi, M_\psi] = \psi([m,M])$. *If* ϕ *is concave, then the inequalities are reversed.*

Proof. Function $\phi = \chi\circ\psi^{-1}$ is obviously continuous. Let us assume that ϕ is convex.

Since $m \leqslant f(t) \leqslant M$ for $t\in[m,M]$, we have $m_\psi \leqslant \psi(f(t)) \leqslant M_\psi$ for every $t\in[m,M]$ (if ψ is increasing, then $m_\psi = \psi(m)$ and $M_\psi = \psi(M)$; if ψ is decreasing, then $m_\psi = \psi(M)$ and $M_\psi = \psi(m)$). Conditions of Theorem 2.1 are satisfied, so we can obtain (3.1.4) and (3.1.5) by substituting $m \leftrightarrow m_\psi$, $M \leftrightarrow M_\psi$, $\phi \leftrightarrow \chi\circ\psi^{-1}$, $t \leftrightarrow \psi(t)$ and $f \leftrightarrow \psi\circ f$ in (2.1) and (2.2) respectively.

Now let us assume that $\phi = \chi\circ\psi^{-1}$ is concave. Then the function $-\phi = -\chi\circ\psi^{-1}$ is convex, so we can obtain reversed inequalities by replacing ϕ with $-\phi$. □

THEOREM 3.1.4. *Let us suppose that the assumptions from Theorem 3.1.3 hold. If* $f\in L$ *satisfies the bounds*

$$-\infty < m \leqslant f(t) \leqslant M < \infty \text{ for every } t\in E$$

and $\psi\circ f, \chi\circ f \in L$*, then we have the following inequalities*

(i)

$$\begin{aligned}
0 &\leqslant \frac{(A(\psi(f)) - \psi(m))\chi(M) + (\psi(M) - A(\psi(f)))\chi(m)}{\psi(M) - \psi(m)} - \chi(M_\chi(f,A)) \\
&\leqslant A[(M_\psi - \psi(f))(\psi(f) - m_\psi)] \sup_{t\in\langle m,M\rangle} \Psi_{\chi\circ\psi^{-1}}(\psi(t);m_\psi,M_\psi) \\
&\leqslant \frac{A[(M_\psi - \psi(f))(\psi(f) - m_\psi)]}{M - m}([\chi\circ\psi^{-1}]'_-(M_\psi) - [\chi\circ\psi^{-1}]'_+(m_\psi)) \\
&\leqslant \frac{(M_\psi - A(\psi(f)))(A(\psi(f)) - m_\psi)}{M_\psi - m_\psi}([\chi\circ\psi^{-1}]'_-(M_\psi) - [\chi\circ\psi^{-1}]'_+(m_\psi)) \\
&\leqslant \frac{1}{4}(M_\psi - m_\psi)([\chi\circ\psi^{-1}]'_-(M_\psi) - [\chi\circ\psi^{-1}]'_+(m_\psi)) \qquad (3.1.6)
\end{aligned}$$

(ii)

$$\begin{aligned}
0 &\leqslant \frac{(A(\psi(f))-\psi(m))\chi(M)+(\psi(M)-A(\psi(f)))\chi(m)}{\psi(M)-\psi(m)}-\chi(M_\chi(f,A)) \\
&\leqslant A[(M_\psi-\psi(f))(\psi(f)-m_\psi)] \sup_{t\in\langle m,M\rangle} \Psi_{\chi\circ\psi^{-1}}(\psi(t);m_\psi,M_\psi) \\
&\leqslant (M_\psi-A(\psi(f)))(A(\psi(f))-m_\psi) \sup_{t\in\langle m,M\rangle} \Psi_{\chi\circ\psi^{-1}}(\psi(t);m_\psi,M_\psi) \\
&\leqslant \frac{(M_\psi-A(\psi(f)))(A(\psi(f))-m_\psi)}{M_\psi-m_\psi}([\chi\circ\psi^{-1}]'_-(M_\psi)-[\chi\circ\psi^{-1}]'_+(m_\psi)) \\
&\leqslant \frac{1}{4}(M_\psi-m_\psi)([\chi\circ\psi^{-1}]'_-(M_\psi)-[\chi\circ\psi^{-1}]'_+(m_\psi))
\end{aligned} \tag{3.1.7}$$

(iii)

$$\begin{aligned}
0 &\leqslant \frac{(A(\psi(f))-\psi(m))\chi(M)+(\psi(M)-A(\psi(f)))\chi(m)}{\psi(M)-\psi(m)}-\chi(M_\chi(f,A)) \\
&\leqslant \frac{1}{4}(M_\psi-m_\psi)^2 A(\Psi_{\chi\circ\psi^{-1}}(\psi(f);m_\psi,M_\psi)) \qquad (3.1.8)\\
&\leqslant \frac{1}{4}(M_\psi-m_\psi)([\chi\circ\psi^{-1}]'_-(M_\psi)-[\chi\circ\psi^{-1}]'_+(m_\psi))
\end{aligned}$$

where $[m_\psi,M_\psi]=\psi([m,M])$. If ϕ is concave, then the inequalities are reversed.

Proof. Function $\phi=\chi\circ\psi^{-1}$ is obviously continuous. Let us assume that ϕ is convex.

Since $m\leqslant f(t)\leqslant M$ for $t\in[m,M]$, we have $m_\psi\leqslant\psi(f(t))\leqslant M_\psi$ for every $t\in[m,M]$ (if ψ is increasing, then $m_\psi=\psi(m)$ and $M_\psi=\psi(M)$; if ψ is decreasing, then $m_\psi=\psi(M)$ and $M_\psi=\psi(m)$). Conditions of Theorem 2.2 are satisfied, so we can obtain (3.1.6), (3.1.7) and (3.1.8) by substituting $m\leftrightarrow m_\psi$, $M\leftrightarrow M_\psi$, $\phi\leftrightarrow\chi\circ\psi^{-1}$, $t\leftrightarrow\psi(t)$ and $f\leftrightarrow\psi\circ f$ in (2.6), (2.7) and (2.8) respectively.

Now let us assume that $\phi=\chi\circ\psi^{-1}$ is concave. Then the function $-\phi=-\chi\circ\psi^{-1}$ is convex, so we can obtain reversed inequalities by replacing ϕ with $-\phi$. □

3.2. Power means

DEFINITION 3.2.1. *Suppose that L and A satisfy the conditions $L1,L2$ and $A1,A2$ with $A(1)=1$, on a non-empty set E. For $f\in L$, the power mean $M^{[r]}(f,A)$ is defined for $r\in\mathbb{R}$ with:*

$$M^{[r]}(f,A)=\begin{cases}(A(f^r))^{1/r} & : r\neq 0\\ \exp(A(\log f)) & : r=0\end{cases} \tag{3.2.1}$$

where $f(t)>0$ for $t\in E$, $f^r\in L$ for $r\in\mathbb{R}$ and $\log f\in L$.

From Theorem 3.1.1 ([11]) it follows as a special case:

THEOREM 3.2.1. ([11]) *Let* $-\infty < r \leqslant s < \infty$ *and let us assume that the assumptions from Definition 3.2.1 are valid. Then*

$$M^{[r]}(f,A) \leqslant M^{[s]}(f,A). \tag{3.2.2}$$

We can also obtain Goldman's inequality for positive functionals from (3.1.3) as a special case (see [2, p. 203]):

$$(M^r - m^r)(M^{[s]}(f,A))^s - (M^s - m^s)(M^{[r]}(f,A))^r \leqslant M^r m^s - M^s m^r \tag{3.2.3}$$

for $0 < r < s$ or $r < 0 < s$, and the inequality is reversed for $r < s < 0$.

Similarly, for $r = 0$ and $s \in \mathbb{R}$ we obtain

$$(M^{[s]}(f,A))^s \log \frac{M}{m} - (M^s - m^s)\log(M^{[0]}(f,A)) \leqslant m^s \log M - M^s \log m. \tag{3.2.4}$$

Since power means are a special case of generalized means, from Theorem 3.1.3 and Theorem 3.1.4 it follows:

THEOREM 3.2.2. *Suppose that* L *and* A *satisfy the conditions* $L1, L2$ *and* $A1, A2$ *with* $A(1) = 1$*, on a non-empty set* E*. Let* $0 < m \leqslant f(t) \leqslant M < \infty$ *for* $t \in E$*,* f^r*,* f^s*,* $\log f \in L$ *for* $r, s \in \mathbb{R}$*,* $r < s$ *and let*

$$\phi(t) = \begin{cases} t^{s/r} & : r \neq 0,\ s \neq 0, \\ \frac{1}{r}\log t & : r \neq 0,\ s = 0, \\ e^{st} & : r = 0,\ s \neq 0. \end{cases} \tag{3.2.5}$$

If $0 < r < s$ *or* $r < 0 < s$ *then:*

$$\begin{aligned} 0 &\leqslant (M^{[s]}(f,A))^s - (M^{[r]}(f,A))^s \\ &\leqslant (M^r - A(f^r))(A(f^r) - m^r) \sup_{t \in \langle m,M \rangle} \Psi_\phi(t^r; m^r, M^r) \\ &\leqslant \frac{s}{r}(M^r - A(f^r))(A(f^r) - m^r)\frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\leqslant \frac{s}{4r}(M^r - m^r)(M^{s-r} - m^{s-r}) \end{aligned} \tag{3.2.6}$$

and we also have

$$\begin{aligned} 0 &\leqslant (M^{[s]}(f,A))^s - (M^{[r]}(f,A))^s \leqslant \frac{1}{4}(M^r - m^r)^2 \Psi_\phi(A(f^r); m^r, M^r) \\ &\leqslant \frac{s}{4r}(M^r - m^r)(M^{s-r} - m^{s-r}). \end{aligned} \tag{3.2.7}$$

If $r < s < 0$ *then:*

$$\begin{aligned} 0 &\geqslant (M^{[s]}(f,A))^s - (M^{[r]}(f,A))^s \\ &\geqslant (M^r - A(f^r))(A(f^r) - m^r) \sup_{t \in \langle m,M \rangle} \Psi_\phi(t^r; m^r, M^r) \\ &\geqslant \frac{s}{r}(M^r - A(f^r))(A(f^r) - m^r)\frac{M^{s-r} - m^{s-r}}{M^r - m^r} \\ &\geqslant \frac{s}{4r}(M^r - m^r)(M^{s-r} - m^{s-r}) \end{aligned} \tag{3.2.8}$$

and we also have

$$\begin{aligned}0 &\geqslant (M^{[s]}(f,A))^s - (M^{[r]}(f,A))^s \geqslant \frac{1}{4}(M^r - m^r)^2 \Psi_\phi(A(f^r);m^r,M^r) \\ &\geqslant \frac{s}{4r}(M^r - m^r)(M^{s-r} - m^{s-r}).\end{aligned} \tag{3.2.9}$$

If $s = 0$ *and* $r < 0$*, then:*

$$\begin{aligned}0 &\leqslant \log(M^{[0]}(f,A)) - \log(M^{[r]}(f,A)) \\ &\leqslant (M^r - A(f^r))(A(f^r) - m^r) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r;M^r,m^r) \\ &\leqslant -\frac{1}{r}\frac{(M^r - A(f^r))(A(f^r) - m^r)}{M^r m^r} \\ &\leqslant \frac{1}{4r}(m^r - M^r)(\frac{1}{m^r} - \frac{1}{M^r})\end{aligned} \tag{3.2.10}$$

and we also have

$$\begin{aligned}0 &\leqslant \log(M^{[0]}(f,A)) - \log(M^{[r]}(f,A)) \leqslant \frac{1}{4}(m^r - M^r)^2 \Psi_\phi(A(f^r);M^r,m^r) \\ &\leqslant \frac{1}{4r}(m^r - M^r)(\frac{1}{m^r} - \frac{1}{M^r}).\end{aligned} \tag{3.2.11}$$

If $r = 0$ *and* $s > 0$*, then:*

$$\begin{aligned}0 &\leqslant (M^{[s]}(f,A))^s - (M^{[0]}(f,A))^s \\ &\leqslant (\log M - A(\log f))(A(\log f) - \log m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(\log t;\log m,\log M) \\ &\leqslant s(\log M - A(\log f))(A(\log f) - \log m)\frac{M^s - m^s}{\log M - \log m} \\ &\leqslant s(M^s - m^s)\log\frac{M}{m}\end{aligned} \tag{3.2.12}$$

and we also have

$$\begin{aligned}0 &\leqslant (M^{[s]}(f,A))^s - (M^{[0]}(f,A))^s \leqslant \frac{1}{4}(\log M - \log m)^2 \Psi_\phi(A(\log f);\log m,\log M) \\ &\leqslant \frac{s}{4}(M^s - m^s)\log\frac{M}{m}.\end{aligned} \tag{3.2.13}$$

Proof. If we put $\chi(t) = t^s$ and $\psi(t) = t^r$, we have $\phi(t) = \chi(\psi^{-1}(t)) = t^{s/r}$, which is continuous, and convex for $0 < r < s$ and $r < 0 < s$. Function ψ is strictly increasing for $r > 0$, and the conditions of Theorem 3.1.3 are satisfied, so we can obtain (3.2.6) and (3.2.7) by replacing $m_\psi \leftrightarrow \psi(m) = m^r$, $M_\psi \leftrightarrow \psi(M) = M^r$, $\phi(t) = \chi\circ\psi^{-1}(t) = t^{s/r}$, $t \leftrightarrow \psi(t) = t^r$ and $\psi\circ f = f^r$ in (3.1.4) and (3.1.5). Function ψ is strictly decreasing for $r < 0$, so we can obtain (3.2.6) and (3.2.7) by replacing $M_\psi \leftrightarrow \psi(m) = m^r$, $m_\psi \leftrightarrow \psi(M) = M^r$, $\phi(t) = \chi\circ\psi^{-1}(t) = t^{s/r}$, $t \leftrightarrow \psi(t) = t^r$ and $\psi\circ f = f^r$ in (3.1.4) and (3.1.5).

In case $r<s<0$, function $\psi(t)=t^r$ is strictly decreasing and $\phi(t)=\chi(\psi^{-1}(t))=t^{s/r}$ is concave, so we obtain (3.2.8) and (3.2.9) by making substitutions $M_\psi \leftrightarrow \psi(m)=m^r$, $m_\psi \leftrightarrow \psi(M)=M^r$, $\phi(t)=-\chi\circ\psi^{-1}(t)=-t^{s/r}$, $t\leftrightarrow\psi(t)=t^r$ and $\psi\circ f=f^r$ in (3.1.4) and (3.1.5).

In case $r<0$ and $s=0$ we put $\chi(t)=\log t$ and $\psi(t)=t^r$. Then $\phi(t)=\chi(\psi^{-1}(t))=\dfrac{1}{r}\log t$ is continuous and convex, and ψ is strictly decreasing for $r<0$, so the conditions of Theorem 3.1.3 are satisfied and we can obtain (3.2.10) and (3.2.11) by making substitutions $M_\psi \leftrightarrow \psi(m)=m^r$, $m_\psi \leftrightarrow \psi(M)=M^r$, $\phi(t)=\chi\circ\psi^{-1}(t)=\dfrac{1}{r}\log t$, $t\leftrightarrow\psi(t)=t^r$ and $f\leftrightarrow\psi\circ f=f^r$ in (3.1.4) and (3.1.5).

In case $r=0$, $s>0$, we put $\chi(t)=t^s$ and $\psi(t)=\log t$. Then $\phi(t)=\chi(\psi^{-1}(t))=e^{st}$ is continuous and convex, and ψ is strictly increasing. The inequalities (3.2.12) and (3.2.13) are now obtained by replacing $m_\psi \leftrightarrow \psi(m)=\log m$, $M_\psi \leftrightarrow \psi(M)=\log M$, $\phi(t)=\chi\circ\psi^{-1}(t)=e^{st}$, $t\leftrightarrow\psi(t)=\log t$ and $f\leftrightarrow\psi\circ f=\log f$ in (3.1.4) and (3.1.5). $\square$

THEOREM 3.2.3. *Under the same hypothesis as in the previous theorem, if $0<r<s$ or $r<0<s$, then:*

(i)

$$\begin{aligned}
0 &\leqslant \frac{(A(f^r)-m^r)M^s+(M^r-A(f^r))m^s}{M^r-m^r}-(M^{[s]}(f,A))^s \\
&\leqslant A[(M^r-f^r)(f^r-m^r)]\sup_{t\in\langle m,M\rangle}\Psi_\phi(t^r;m^r,M^r) \\
&\leqslant \frac{s}{r}\frac{A[(M^r-f^r)(f^r-m^r)]}{M^r-m^r}(M^{s-r}-m^{s-r}) \\
&\leqslant \frac{s}{r}\frac{(M^r-A(f^r))(A(f^r)-m^r)}{M^r-m^r}(M^{s-r}-m^{s-r}) \\
&\leqslant \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r})
\end{aligned} \tag{3.2.14}$$

(ii)

$$\begin{aligned}
0 &\leqslant \frac{(A(f^r)-m^r)M^s+(M^r-A(f^r))m^s}{M^r-m^r}-(M^{[s]}(f,A))^s \\
&\leqslant A[(M^r-f^r)(f^r-m^r)]\sup_{t\in\langle m,M\rangle}\Psi_\phi(t^r;m^r,M^r) \\
&\leqslant (M^r-A(f^r))(A(f^r)-m^r)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t^r;m^r,M^r) \\
&\leqslant \frac{s}{r}\frac{(M^r-A(f^r))(A(f^r)-m^r)}{M^r-m^r}(M^{s-r}-m^{s-r}) \\
&\leqslant \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r})
\end{aligned} \tag{3.2.15}$$

(iii)

$$\begin{aligned}
0 &\leqslant \frac{(A(f^r)-m^r)M^s+(M^r-A(f^r))m^s}{M^r-m^r}-(M^{[s]}(f,A))^s \\
&\leqslant \frac{1}{4}(M^r-m^r)^2A(\Psi_\phi(f^r;m^r,M^r)) \\
&\leqslant \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r}).
\end{aligned} \tag{3.2.16}$$

If $r<s<0$, then:

(i)

$$\begin{aligned}
0 &\geqslant \frac{(A(f^r)-m^r)M^s+(M^r-A(f^r))m^s}{M^r-m^r}-(M^{[s]}(f,A))^s \\
&\geqslant A[(M^r-f^r)(f^r-m^r)]\sup_{t\in\langle m,M\rangle}\Psi_\phi(t^r;m^r,M^r) \\
&\geqslant \frac{s}{r}\frac{A[(M^r-f^r)(f^r-m^r)]}{M^r-m^r}(M^{s-r}-m^{s-r}) \\
&\geqslant \frac{s}{r}\frac{(M^r-A(f^r))(A(f^r)-m^r)}{M^r-m^r}(M^{s-r}-m^{s-r}) \\
&\geqslant \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r})
\end{aligned} \tag{3.2.17}$$

(ii)

$$\begin{aligned}
0 &\geqslant \frac{(A(f^r)-m^r)M^s+(M^r-A(f^r))m^s}{M^r-m^r}-(M^{[s]}(f,A))^s \\
&\geqslant A[(M^r-f^r)(f^r-m^r)]\sup_{t\in\langle m,M\rangle}\Psi_\phi(t^r;m^r,M^r) \\
&\geqslant (M^r-A(f^r))(A(f^r)-m^r)\sup_{t\in\langle m,M\rangle}\Psi_\phi(t^r;m^r,M^r) \\
&\geqslant \frac{s}{r}\frac{(M^r-A(f^r))(A(f^r)-m^r)}{M^r-m^r}(M^{s-r}-m^{s-r}) \\
&\geqslant \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r})
\end{aligned} \tag{3.2.18}$$

(iii)

$$\begin{aligned}
0 &\geqslant \frac{(A(f^r)-m^r)M^s+(M^r-A(f^r))m^s}{M^r-m^r}-(M^{[s]}(f,A))^s \\
&\geqslant \frac{1}{4}(M^r-m^r)^2A(\Psi_\phi(f^r;m^r,M^r)) \\
&\geqslant \frac{s}{4r}(M^r-m^r)(M^{s-r}-m^{s-r})
\end{aligned} \tag{3.2.19}$$

If $s=0$ and $r<0$, then:

(i)

$$
\begin{aligned}
0 &\leqslant \frac{(A(f^r)-m^r)\log M+(M^r-A(f^r))\log m}{M^r-m^r}-\log(M^{[0]}(f,A)) \\
&\leqslant A[(M^r-f^r)(f^r-m^r)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r;M^r,m^r) \\
&\leqslant \frac{1}{r}\frac{A[(M^r-f^r)(f^r-m^r)]}{m^r-M^r}\Big(\frac{1}{m^r}-\frac{1}{M^r}\Big) \\
&\leqslant \frac{1}{r}\frac{(M^r-A(f^r))(A(f^r)-m^r)}{m^r-M^r}\Big(\frac{1}{m^r}-\frac{1}{M^r}\Big) \\
&\leqslant \frac{1}{4r}(m^r-M^r)\Big(\frac{1}{m^r}-\frac{1}{M^r}\Big)
\end{aligned} \tag{3.2.20}
$$

(ii)

$$
\begin{aligned}
0 &\leqslant \frac{(A(f^r)-m^r)\log M+(M^r-A(f^r))\log m}{M^r-m^r}-\log(M^{[0]}(f,A)) \\
&\leqslant A[(M^r-f^r)(f^r-m^r)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r;M^r,m^r) \\
&\leqslant (M^r-A(f^r))(A(f^r)-m^r) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t^r;M^r,m^r) \\
&\leqslant \frac{1}{r}\frac{(M^r-A(f^r))(A(f^r)-m^r)}{m^r-M^r}\Big(\frac{1}{m^r}-\frac{1}{M^r}\Big) \\
&\leqslant \frac{1}{4r}(m^r-M^r)\Big(\frac{1}{m^r}-\frac{1}{M^r}\Big)
\end{aligned} \tag{3.2.21}
$$

(iii)

$$
\begin{aligned}
0 &\leqslant \frac{(A(f^r)-m^r)\log M+(M^r-A(f^r))\log m}{M^r-m^r}-\log(M^{[0]}(f,A)) \\
&\leqslant \frac{1}{4}(m^r-M^r)^2 A(\Psi_\phi(f^r;M^r,m^r)) \\
&\leqslant \frac{1}{4r}(m^r-M^r)\Big(\frac{1}{m^r}-\frac{1}{M^r}\Big)
\end{aligned} \tag{3.2.22}
$$

If $r=0$ and $s>0$, then:

(i)

$$
\begin{aligned}
0 &\leqslant \frac{(A(\log f)-\log m)M^s+(\log M-A(\log f))m^s}{\log M-\log m}-(M^{[s]}(f,A))^s \\
&\leqslant A[(\log M-\log f)(\log f-\log m)] \sup_{t\in\langle m,M\rangle} \Psi_\phi(\log t;\log m,\log M) \\
&\leqslant s\frac{A[(\log M-\log f)(\log f-\log m)]}{\log M-\log m}(M^s-m^s)
\end{aligned} \tag{3.2.23}
$$

$$
\begin{aligned}
&\leqslant s\frac{(\log M - A(\log f))(A(\log f)-\log m)}{\log M-\log m}(M^s-m^s)\\
&\leqslant \frac{s}{4}(M^s-m^s)\log\frac{M}{m}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
0 &\leqslant \frac{(A(\log f)-\log m)M^s+(\log M - A(\log f))m^s}{\log M-\log m}-(M^{[s]}(f,A))^s\\
&\leqslant A[(\log M-\log f)(\log f-\log m)]\sup_{t\in\langle m,M\rangle}\Psi_\phi(\log t;\log m,\log M)\\
&\leqslant (\log M - A(\log f))(A(\log f)-\log m)\sup_{t\in\langle m,M\rangle}\Psi_\phi(\log t;\log m,\log M) \qquad (3.2.24)\\
&\leqslant s\frac{(\log M - A(\log f))(A(\log f)-\log m)}{\log M-\log m}(M^s-m^s)\\
&\leqslant \frac{s}{4}(M^s-m^s)\log\frac{M}{m}
\end{aligned}
$$

(iii)

$$
\begin{aligned}
0 &\leqslant \frac{(A(\log f)-\log m)M^s+(\log M - A(\log f))m^s}{\log M-\log m}-(M^{[s]}(f,A))^s\\
&\leqslant \frac{1}{4}(\log M-\log m)^2 A(\Psi_\phi(\log f;\log m,\log M)) \qquad (3.2.25)\\
&\leqslant \frac{s}{4r}(\log M-\log m)(M^s-m^s).
\end{aligned}
$$

Proof. All the inequalities can be obtained directly from Theorem 3.1.4 by making the same supstitutions as in the proof of the previous theorem. □

REMARK 3.2.1. *It is easy to see that* $M^{[r]}(f,A)=(M^{[-r]}(f^{-1},A))^{-1}$ *holds for every* $f\in L$ *and* $r\in\mathbb{R}$. *Using that result, we can obtain analogue sequences of inequalities from Theorem 3.2.2 and Theorem 3.2.3 by replacing* $f\leftrightarrow f^{-1}$, $-r\leftrightarrow s$ *and* $-s\leftrightarrow r$.

3.3. The Hölder inequality

THEOREM 3.3.1. [11, p. 113] (Hölder′s inequality for positive functionals) *Let* L *satisfy conditions* $L1, L2$, *and* A *satisfy conditions* $A1, A2$ *on a non-empty set* E. *Let* $p>1$ *and* $q=p/(p-1)$. *If* $w,f,g\geqslant 0$ *on* E *and* $wf^p, wg^q, wfg\in L$, *then we have*

$$A(wfg)\leqslant A^{1/p}(wf^p)A^{1/q}(wg^q) \qquad (3.3.1)$$

In case $0<p<1$ *and* $A(wg^q)>0$ *(or* $p<0$ *and* $A(wf^p)>0$*) the inequality in (3.3.1) is reversed.*

THEOREM 3.3.2. [11, p. 114, Theorem 4.14] *Let L and A satisfy conditions $L1, L2$, and $A1, A2$ on a non-empty set E. Let $p > 1$ and $q = p/(p-1)$, and $w, f, g \geqslant 0$ on E with $wf^p, wg^q, wfg \in L$. If $0 < m \leqslant f(t)g^{-q/p}(t) \leqslant M$ for $t \in E$, then*

$$(M-m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \leqslant (M^p - m^p)A(wfg). \tag{3.3.2}$$

If $p < 0$, then (3.3.2) also holds provided either $A(wf^p) > 0$ or $A(wg^q) > 0$. If $0 < p < 1$, then the reversed inequality in (3.3.2) holds provided either $A(wf^p) > 0$ or $A(wg^q) > 0$.

We need analogues of Theorems 2.1 and 2.2 for the case when the condition $A(1) = 1$ is not satisfied:

THEOREM 2.1'. *Let ϕ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, where $\mathring{I}$ is the interior of I. Let L satisfy conditions $L1, L2$ on E, let A be any positive linear functional on L and let $w \geqslant 0$ on I such that $A(w) > 0$. If $f \in L$ satisfies the bounds*

$$-\infty < m \leqslant f(t) \leqslant M < \infty \text{ for every } t \in E$$

and $\phi \circ f \in L$, then

$$\begin{aligned} 0 &\leqslant \frac{A(w\phi(f))}{A(w)} - \phi\left(\frac{A(wf)}{A(w)}\right) \\ &\leqslant (M-\bar{f})(\bar{f}-m) \sup_{t \in \langle m,M \rangle} \Psi_\phi(t;m,M) \\ &\leqslant (M-\bar{f})(\bar{f}-m) \frac{\phi'_-(M) - \phi'_+(m)}{M-m} \\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)). \end{aligned} \tag{3.3.3}$$

We also have the inequalities

$$\begin{aligned} 0 &\leqslant \frac{A(w\phi(f))}{A(w)} - \phi\left(\frac{A(wf)}{A(w)}\right) \\ &\leqslant \frac{1}{4}(M-m)^2 \Psi_\phi(\bar{f};m,M) \\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M) - \phi'_+(m)), \end{aligned} \tag{3.3.4}$$

where $\bar{f} = \dfrac{A(wf)}{A(w)}$, $\Psi_\phi(\cdot;m,M)\colon \langle m,M \rangle \to \mathbb{R}$ is defined by (2.3), and we assume that $\Psi_\phi \in L$. If ϕ is concave, the inequalities are reversed.

Proof. We define functional $B(f) = \dfrac{A(wf)}{A(w)}$. $B(1) = \dfrac{A(w)}{A(w)} = 1$, so B satisfies the conditions of Theorem 2.1. If ϕ is convex, inequalities in (3.3.3) and (3.3.4) are now obtained from the inequalities in (2.1) and (2.2) by replacing A with B. If ϕ is concave, the reversed inequalities follow from replacing ϕ with $-\phi$. $\square$

THEOREM 2.2'. *With the same assumptions as in Theorem 2.1', the following inequalities are valid.*

(i)

$$\begin{aligned} 0 &\leqslant \frac{(\bar{f}-m)\phi(M)+(M-\bar{f})\phi(m)}{M-m}-\frac{A(w\phi(f))}{A(w)} \\ &\leqslant \frac{A(w[(M-f)(f-m)])}{A(w)} \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\leqslant \frac{A(w[(M-f)(f-m)])}{(M-m)A(w)}(\phi'_-(M)-\phi'_+(m)) \\ &\leqslant \frac{(M-\bar{f})(\bar{f}-m)}{M-m}(\phi'_-(M)-\phi'_+(m)) \\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) \end{aligned} \tag{3.3.5}$$

(ii)

$$\begin{aligned} 0 &\leqslant \frac{(\bar{f}-m)\phi(M)+(M-\bar{f})\phi(m)}{M-m}-\frac{A(w\phi(f))}{A(w)} \\ &\leqslant \frac{A(w[(M-f)(f-m)])}{A(w)} \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\leqslant (M-\bar{f})(\bar{f}-m) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \\ &\leqslant \frac{(M-\bar{f})(\bar{f}-m)}{M-m}(\phi'_-(M)-\phi'_+(m)) \\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) \end{aligned} \tag{3.3.6}$$

(iii)

$$\begin{aligned} 0 &\leqslant \frac{(\bar{f}-m)\phi(M)+(M-\bar{f})\phi(m)}{M-m}-\frac{A(w\phi(f))}{A(w)} \\ &\leqslant \frac{1}{4}(M-m)^2\frac{A(w\Psi_\phi(f;m,M))}{A(w)} \\ &\leqslant \frac{1}{4}(M-m)(\phi'_-(M)-\phi'_+(m)) \end{aligned} \tag{3.3.7}$$

If ϕ is concave, the inequalities are reversed.

Proof. Same as in the proof of the Theorem 2.1', we define functional $B(f)=\frac{A(wf)}{A(w)}$. $B(1)=\frac{A(w)}{A(w)}=1$, so B satisfies the conditions of Theorem 2.2. Inequalities in (3.3.5), (3.3.6) and (3.3.7) are now obtained from the inequalities in (2.6), (2.7) and

(2.8) respectively by replacing A with B. If ϕ is concave, the reversed inequalities follow from replacing ϕ with $-\phi$. $\square$

The following results are converses of Hölder's inequality:

THEOREM 3.3.3. *Let L satisfy conditions $L1, L2$, and A satisfy conditions $A1, A2$ on a non-empty set E. Let $p > 1$ and $q = p/(p-1)$. If $w, f, g \geqslant 0$ on E and $wf^p, wg^q, wfg \in L$, $A(wg^q) > 0$, then we have*

$$\begin{aligned}
0 &\leqslant A(wf^p)A^{p/q}(wg^q) - A^p(wfg) \\
&\leqslant (MA(wg^q) - A(wfg))(A(wfg) - mA(wg^q)) \sup_{t \in \langle m, M \rangle} \Psi_\phi(t; m, M) A^{p-2}(wg^q) \\
&\leqslant (MA(wg^q) - A(wfg))(A(wfg) - mA(wg^q)) p \frac{M^{p-1} - m^{p-1}}{M - m} A^{p-2}(wg^q) \\
&\leqslant \frac{p}{4}(M - m)(M^{p-1} - m^{p-1}) A^p(wg^q).
\end{aligned} \tag{3.3.8}$$

We also have the inequalities

$$\begin{aligned}
0 &\leqslant A(wf^p)A^{p/q}(wg^q) - A^p(wfg) \\
&\leqslant \frac{1}{4}(M - m)^2 \Psi_\phi\Big(\frac{A(wfg)}{A(wg^q)}; m, M\Big) A^p(wg^q) \\
&\leqslant \frac{p}{4}(M - m)(M^{p-1} - m^{p-1}) A^p(wg^q),
\end{aligned} \tag{3.3.9}$$

where $m \leqslant f(t)g^{-q/p}(t) \leqslant M$ for $t \in E$ and $\phi(t) = t^p$. If $A(wfg) > 0$, then the inequalities also hold for $p < 0$. In case $0 < p < 1$ the inequalities are reversed.

Proof. Function $\phi(t) = t^p$ is continuous, and convex for $p > 1$ and $p < 0$, so we can obtain the inequalities (3.3.8) and (3.3.9) from (3.3.3) and (3.3.4) by replacing $w \leftrightarrow wg^q$ and $f \leftrightarrow fg^{-q/p}$.

For $0 < p < 1$, $\phi(t) = t^p$ is concave, so we obtain the reversed inequalities in the same way as above. $\square$

THEOREM 3.3.4. *With the assumptions in Theorem 3.3.3, if $p > 1$ or $p < 0$ the following inequalities are valid*

(i)

$$\begin{aligned}
0 &\leqslant \frac{(A(wfg) - mA(wg^q))M^p + (MA(wg^q) - A(wfg))m^p}{M - m} - A(wf^p) \\
&\leqslant A(wg^q[(M - fg^{-q/p})(fg^{-q/p} - m)]) \sup_{t \in \langle m, M \rangle} \Psi_\phi(t; m, M) \\
&\leqslant \frac{A(wg^q[(M - fg^{-q/p})(fg^{-q/p} - m)])}{M - m} p(M^{p-1} - m^{p-1})
\end{aligned} \tag{3.3.10}$$

$$\leqslant \frac{(MA(wg^q)-A(wfg))(A(wfg)-mA(wg^q))}{(M-m)A(wg^q)}p(M^{p-1}-m^{p-1})$$
$$\leqslant \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A(wg^q)$$

(ii)

$$\begin{aligned}
0 &\leqslant \frac{(A(wfg)-mA(wg^q))M^p+(MA(wg^q)-A(wfg))m^p}{M-m}-A(wf^p)\\
&\leqslant A(wg^q[(M-fg^{-q/p})(fg^{-q/p}-m)])\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)\\
&\leqslant \frac{(MA(wg^q)-A(wfg))(A(wfg)-mA(wg^q))}{A(wg^q)}\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)\\
&\leqslant \frac{(MA(wg^q)-A(wfg))(A(wfg)-mA(wg^q))}{(M-m)A(wg^q)}p(M^{p-1}-m^{p-1})\\
&\leqslant \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A(wg^q) \qquad (3.3.11)
\end{aligned}$$

(iii)

$$\begin{aligned}
0 &\leqslant \frac{(A(wfg)-mA(wg^q))M^p+(MA(wg^q)-A(wfg))m^p}{M-m}-A(wf^p)\\
&\leqslant \frac{1}{4}(M-m)^2A(wg^q\Psi_\phi(fg^{-q/p};m,M)) \qquad (3.3.12)\\
&\leqslant \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A(wg^q)
\end{aligned}$$

where $m\leqslant f(t)g^{-q/p}(t)\leqslant M$ for $t\in E$ and $\phi(t)=t^p$. If $0<p<1$, the inequalities are reversed.

Proof. Function $\phi(t)=t^p$ is continuous, and convex for $p>1$ and $p<0$, so we can obtain the inequalities (3.3.10), (3.3.11) and (3.3.12) from (3.3.5), (3.3.6) and (3.3.7) respectively by replacing $w\leftrightarrow wg^q$ and $f\leftrightarrow fg^{-q/p}$.

For $0<p<1$, $\phi(t)=t^p$ is concave, so we obtain the reversed inequalities in the same way as above. □

THEOREM 3.3.5. *Let L satisfy conditions $L1,L2$, and A satisfy conditions $A1,A2$ on a non-empty set E. Let $0<p<1$ and $q=p/(p-1)$. If $f,g\geqslant 0$ on E and $f^p,g^q,fg\in L$, $A(g^q)>0$, then we have*

$$\begin{aligned}
0 &\leqslant A(fg)-A^{1/p}(f^p)A^{1/q}(g^q)\\
&\leqslant \frac{(MA(g^q)-A(f^p))(A(f^p)-mA(g^q))}{A(g^q)}\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M)\\
&\leqslant (MA(g^q)-A(f^p))(A(f^p)-mA(g^q))\frac{M^{-1/q}-m^{-1/q}}{p(M-m)A(g^q)} \qquad (3.3.13)
\end{aligned}$$

$$\leqslant \frac{1}{4p}(M-m)(M^{-1/q}-m^{-1/q})A(g^q)$$

We also have the inequalities

$$\begin{aligned} 0 &\leqslant A(fg)-A^{1/p}(f^p)A^{1/q}(g^q) \\ &\leqslant \frac{1}{4}(M-m)^2\Psi_\phi(\frac{A(f^p)}{A(g^q)};m,M)A(g^q) \\ &\leqslant \frac{1}{4p}(M-m)(M^{-1/q}-m^{-1/q})A(g^q), \end{aligned} \tag{3.3.14}$$

where $m \leqslant f^p(t)g^{-q}(t) \leqslant M$ *for* $t \in E$ *and* $\phi(t)=t^{1/p}$. *If* $A(f^p)>0$, *the inequalities hold for* $p<0$. *In case* $p>1$ *the inequalities are reversed.*

Proof. Function $\phi(t)=t^{1/p}$ is continuous, and for $p<1$ convex, so we can obtain the inequalities (3.3.13) and (3.3.14) from (3.3.3) and (3.3.4) by replacing $w \leftrightarrow \frac{g^q}{A(g^q)}$ and $f \leftrightarrow \frac{f^p}{g^q}$.

For $p>1$, the function $\phi(t)=t^{1/p}$ is concave, so we obtain the reversed inequalities in the same way as above. □

THEOREM 3.3.6. *With the assumptions in Theorem 3.3.5, if* $p<1$ *the following inequalities are valid*

(i)

$$\begin{aligned} 0 &\leqslant \frac{(A(f^p)-mA(g^q))M^{1/p}+(MA(g^q)-A(f^p))m^{1/p}}{M-m}-A(fg) \\ &\leqslant A(g^{-q}[(Mg^q-f^p)(f^p-mg^q)])\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) \\ &\leqslant \frac{A(g^{-q}[(Mg^q-f^p)(f^p-mg^q)])}{p(M-m)}(M^{-1/q}-m^{-1/q}) \\ &\leqslant \frac{(MA(g^q)-A(f^p))(A(f^p)-mA(g^q))}{p(M-m)A(g^q)}(M^{-1/q}-m^{-1/q}) \\ &\leqslant \frac{1}{4p}(M-m)(M^{-1/q}-m^{-1/q})A(g^q) \end{aligned} \tag{3.3.15}$$

(ii)

$$\begin{aligned} 0 &\leqslant \frac{(A(f^p)-mA(g^q))M^{1/p}+(MA(g^q)-A(f^p))m^{1/p}}{M-m}-A(fg) \\ &\leqslant A(g^{-q}[(Mg^q-f^p)(f^p-mg^q)])\sup_{t\in\langle m,M\rangle}\Psi_\phi(t;m,M) \end{aligned}$$

$$\leqslant \frac{(MA(g^q)-A(f^p))(A(f^p)-mA(g^q))}{A(g^q)} \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M) \tag{3.3.16}$$
$$\leqslant \frac{(MA(g^q)-A(f^p))(A(f^p)-mA(g^q))}{p(M-m)A(g^q)}(M^{-1/q}-m^{-1/q})$$
$$\leqslant \frac{1}{4p}(M-m)(M^{-1/q}-m^{-1/q})A(g^q)$$

(iii)

$$0 \leqslant \frac{(A(f^p)-mA(g^q))M^{1/p}+(MA(g^q)-A(f^p))m^{1/p}}{M-m} - A(fg)$$
$$\leqslant \frac{1}{4}(M-m)^2 A(g^q\Psi_\phi(\frac{f^p}{g^q};m,M)) \tag{3.3.17}$$
$$\leqslant \frac{1}{4p}(M-m)(M^{-1/q}-m^{-1/q})A(g^q)$$

where $m \leqslant f^p(t)g^{-q}(t) \leqslant M$ for $t \in E$ and $\phi(t)=t^{1/p}$. If $p>1$, the inequalities in are reversed.

Proof. Function $\phi(t)=t^{1/p}$ is continuous, and convex for $p<1$, so we can obtain the inequalities (3.3.15), (3.3.16) and (3.3.17) from (3.3.5), (3.3.6) and (3.3.7) by replacing $w \leftrightarrow \dfrac{g^q}{A(g^q)}$ and $f \leftrightarrow \dfrac{f^p}{g^p}$.

For $p>1$, $\phi(t)=t^{1/p}$ is concave, so we obtain the reversed inequalities by applying the inequalities (3.3.15), (3.3.16) and (3.3.17) to $-\phi$. □

THEOREM 3.3.7. *Let L satisfy conditions $L1,L2$, and A satisfy conditions $A1,A2$ on a non-empty set E. Let $p>1$ or $p<0$ and $q=p/(p-1)$. If $f,g \geqslant 0$ on E and $g^q, fg \in L$, $A(g^q)>0$, then we have*

$$0 \leqslant A(f^p)A^{p/q}(g^q)-A^p(fg)$$
$$\leqslant (MA(g^q)-A(fg))(A(fg)-mA(g^q)) \sup_{t\in\langle m,M\rangle} \Psi_\phi(t;m,M)A^{p-2}(g^q)$$
$$\leqslant (MA(g^q)-A(fg))(A(fg)-mA(g^q))p\frac{M^{p-1}-m^{p-1}}{M-m}A^{p-2}(g^q)$$
$$\leqslant \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A^p(g^q). \tag{3.3.18}$$

We also have the inequalities

$$0 \leqslant A(f^p)A^{p/q}(g^q)-A^p(fg)$$
$$\leqslant \frac{1}{4}(M-m)^2\Psi_\phi(\frac{A(fg)}{A(g^q)};m,M)A^p(g^q) \tag{3.3.19}$$
$$\leqslant \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})A^p(g^q),$$

where $m \leqslant f(t)g^{1-q}(t) \leqslant M$ for $t \in E$ and $\phi(t) = t^p$. In case $0 < p < 1$ the inequalities are reversed.

Proof. Function $\phi(t) = t^p$ is continuous, and convex for $p > 1$ and $p < 0$, so we can obtain the inequalities (3.3.18) and (3.3.19) from (3.3.3) and (3.3.4) by replacing $w \leftrightarrow g^q$ and $f \leftrightarrow fg^{1-q}$.

For $0 < p < 1$, the function $\phi(t) = t^p$ is concave, so we obtain the reversed inequalities in the same way as above. □

THEOREM 3.3.8. *With the assumptions in Theorem 3.3.7, if $p > 1$ or $p < 0$ the following inequalities are valid*

(i)

$$\begin{aligned}
0 &\leqslant \frac{(A(fg) - mA(g^q))M^p + (MA(g^q) - A(fg))m^p}{M-m} - A(f^p) \\
&\leqslant A(g^{-q}[(Mg^q - fg)(fg - mg^q)]) \sup_{t \in \langle m,M \rangle} \Psi_\phi(t;m,M) \\
&\leqslant \frac{A(g^{-q}[(Mg^q - fg)(fg - mg^q)])}{(M-m)} p(M^{p-1} - m^{p-1}) \qquad (3.3.20)\\
&\leqslant \frac{(MA(g^q) - A(fg))(A(fg) - mA(g^q))}{(M-m)A(g^q)} p(M^{p-1} - m^{p-1}) \\
&\leqslant \frac{p}{4}(M-m)(M^{p-1} - m^{p-1})A(g^q)
\end{aligned}$$

(ii)

$$\begin{aligned}
0 &\leqslant \frac{(A(fg) - mA(g^q))M^p + (MA(g^q) - A(fg))m^p}{M-m} - A(f^p) \\
&\leqslant A(g^{-q}[(Mg^q - fg)(fg - mg^q)]) \sup_{t \in \langle m,M \rangle} \Psi_\phi(t;m,M) \\
&\leqslant \frac{(MA(g^q) - A(fg))(A(fg) - mA(g^q))}{A(g^q)} \sup_{t \in \langle m,M \rangle} \Psi_\phi(t;m,M) \\
&\leqslant \frac{(MA(g^q) - A(fg))(A(fg) - mA(g^q))}{(M-m)A(g^q)} p(M^{p-1} - m^{p-1}) \qquad (3.3.21)\\
&\leqslant \frac{p}{4}(M-m)(M^{p-1} - m^{p-1})A(g^q)
\end{aligned}$$

(iii)

$$\begin{aligned}
0 &\leqslant \frac{(A(fg) - mA(g^q))M^p + (MA(g^q) - A(fg))m^p}{M-m} - A(f^p) \\
&\leqslant \frac{1}{4}(M-m)^2 A(g^q \Psi_\phi(fg^{1-q};m,M)) \qquad (3.3.22)\\
&\leqslant \frac{p}{4}(M-m)(M^{p-1} - m^{p-1})A(g^q)
\end{aligned}$$

where $m \leqslant f(t)g^{1-q}(t) \leqslant M$ *for* $t \in E$ *and* $\phi(t) = t^p$. *If* $0 < p < 1$, *the inequalities are reversed.*

Proof. Function $\phi(t) = t^p$ is continuous, and convex for $p > 1$ and $p < 0$, so we can obtain the inequalities (3.3.20), (3.3.21) and (3.3.22) from (3.3.5), (3.3.6) and (3.3.7) by replacing $w \leftrightarrow g^q$ and $f \leftrightarrow fg^{1-q}$.

For $0 < p < 1$, $\phi(t) = t^p$ is concave, so we obtain the reversed inequalities by applying (3.3.20), (3.3.21) and (3.3.22) to $-\phi$. □

3.4. Hadamard's inequality

THEOREM 3.4.1. ([9]) (Hermite – Hadamard's inequality) *Let* $-\infty < a < b < \infty$ *and* $f\colon [a,b] \to \mathbb{R}$. *If f is convex, then*

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \leqslant \frac{f(a)+f(b)}{2} \tag{3.4.1}$$

If f is concave, the inequalities in (3.4.1) are reversed.

THEOREM 3.4.2. *Let* $a < b$ *and let us assume that f is a continuous convex function on an open interval of real numbers* $I \supset [a,b]$. *Then*

$$\begin{aligned} 0 &\leqslant \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\Big(\frac{a+b}{2}\Big) \\ &\leqslant \frac{1}{4}(b-a)^2 \sup_{t\in\langle a,b\rangle} \Psi_f(t;a,b) \\ &\leqslant \frac{1}{4}(b-a)(f'_-(b) - f'_+(a)) \end{aligned} \tag{3.4.2}$$

We also have the inequalities

$$\begin{aligned} 0 &\leqslant \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\Big(\frac{a+b}{2}\Big) \\ &\leqslant \frac{1}{4}(b-a)^2 \Psi_f(\frac{a+b}{2};a,b) \\ &\leqslant \frac{1}{4}(b-a)(f'_-(b) - f'_+(a)) \end{aligned} \tag{3.4.3}$$

If f is concave, the inequalities are reversed.

Proof. Inequalities (3.4.2) and (3.4.3) are obtained from (2.1) and (2.2) by replacing $A(f) = \dfrac{1}{b-a}\int_a^b f(t)\mathrm{d}t$, $f(t) = t$ and $\phi \leftrightarrow f$.

If f is concave, the reversed inequalities follow from the convexity of $-f$. □

THEOREM 3.4.3. *Let $a<b$ and let us assume that f is a continuous convex function on an open interval of real numbers $I \supset [a,b]$. Then*

$$
\begin{aligned}
0 &\leqslant \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\
&\leqslant \frac{1}{6}(b-a)^2 \sup_{t\in\langle a,b\rangle} \Psi_f(t;a,b) \\
&\leqslant \frac{1}{6}(b-a)(f'_-(b)-f'_+(a))
\end{aligned}
\tag{3.4.4}
$$

If f is concave, the inequalities in (3.4.4) are reversed.

Proof. Inequalities (3.4.4) are obtained from (2.6) by replacing

$$
A(f)=\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t, \quad f(t)\leftrightarrow t \quad \text{and} \quad \phi \leftrightarrow f.
$$

If f is concave, the reversed inequalities follow from the convexity of $-f$. □

REMARK 3.4.1. *Let $a<b$ and let us assume that f is continuous convex function on an open interval of real numbers $I \supset [a,b]$. By combining the above results, we obtain*

$$
\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{1}{6}(b-a)^2 \sup_{t\in\langle a,b\rangle} \Psi_f(t;a,b) \leqslant \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \\
&\leqslant f\Big(\frac{a+b}{2}\Big) + \frac{1}{4}(b-a)^2 \sup_{t\in\langle a,b\rangle} \Psi_f(t;a,b).
\end{aligned}
\tag{3.4.5}
$$

If f is concave, the inequalities in (3.4.5) are reversed.

3.5. The inequalities of Giaccardi and Petrović

THEOREM 3.5.1. (Giaccardi, [14]) *Let $\boldsymbol{p}$ be an n-tuple of nonnegative real numbers and $\boldsymbol{x}$ an n-tuple of real numbers such that*

$$
(x_i-x_0)\Big(\sum_{j=1}^n p_jx_j - x_i\Big)\geqslant 0, \quad i=1,\dots,n; \quad \sum_{i=1}^n p_ix_i \neq x_0; \quad x_0,\ \sum_{i=1}^n p_ix_i \in [a,b]. \tag{3.5.1}
$$

If $f\colon [a,b]\to\mathbb{R}$ is a convex function, then

$$
\sum_{i=1}^n p_i f(x_i) \leqslant A f\Big(\sum_{i=1}^n p_ix_i\Big) + B\Big(\sum_{i=1}^n p_i - 1\Big) f(x_0) \tag{3.5.2}
$$

where

$$
A=\frac{\sum_{i=1}^n p_i(x_i-x_0)}{\sum_{i=1}^n p_ix_i - x_0}, \quad B=\frac{\sum_{i=1}^n p_ix_i}{\sum_{i=1}^n p_ix_i - x_0}. \tag{3.5.3}
$$

Our next results is a convesre of Giaccardi's inequality obtained directly from Theorem 2.2:

THEOREM 3.5.2. *Let **p** be an n-tuple of nonnegative real numbers and let **x** be an n-tuple of real numbers such that (3.5.1) holds. Let I be an open interval of real numbers. If $f: I \supset [a,b] \to \mathbb{R}$ is a continuous convex function, then*

(i)

$$\begin{aligned} 0 &\leqslant Af\Big(\sum_{i=1}^n p_i x_i\Big) + B\Big(\sum_{i=1}^n p_i - 1\Big) f(x_0) - \sum_{i=1}^n p_i f(x_i) \\ &\leqslant \sum_{j=1}^n p_j \Big(\sum_{i=1}^n p_i x_i - x_j\Big)\Big(x_j - x_0\Big) \sup_{t\in\langle m,M\rangle} \Psi_f\Big(t; x_0, \sum_{i=1}^n p_i x_i\Big) \\ &\leqslant \frac{\sum_{j=1}^n p_j (\sum_{i=1}^n p_i x_i - x_j)(x_j - x_0)}{M-m} (f'_-(M) - f'_+(m)) \\ &\leqslant \Big(M - \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i}\Big)\Big(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i} - m\Big) \frac{f'_-(M) - f'_+(m)}{M-m} \sum_{i=1}^n p_i \\ &\leqslant \frac{1}{4}(M-m)(f'_-(M) - f'_+(m)) \end{aligned} \tag{3.5.4}$$

(ii)

$$\begin{aligned} 0 &\leqslant Af\Big(\sum_{i=1}^n p_i x_i\Big) + B\Big(\sum_{i=1}^n p_i - 1\Big) f(x_0) - \sum_{i=1}^n p_i f(x_i) \\ &\leqslant \sum_{j=1}^n p_j \Big(\sum_{i=1}^n p_i x_i - x_j\Big)\Big(x_j - x_0\Big) \sup_{t\in\langle m,M\rangle} \Psi_f\Big(t; x_0, \sum_{i=1}^n p_i x_i\Big) \\ &\leqslant \Big(M - \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i}\Big)\Big(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i} - m\Big) \sup_{t\in\langle m,M\rangle} \Psi_f\Big(t; x_0, \sum_{i=1}^n p_i x_i\Big) \sum_{i=1}^n p_i \\ &\leqslant \Big(M - \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i}\Big)\Big(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i} - m\Big) \frac{f'_-(M) - f'_+(m)}{M-m} \sum_{i=1}^n p_i \\ &\leqslant \frac{1}{4}(M-m)(f'_-(M) - f'_+(m)) \sum_{i=1}^n p_i \end{aligned} \tag{3.5.5}$$

(iii)

$$\begin{aligned} 0 &\leqslant Af\Big(\sum_{i=1}^n p_i x_i\Big) + B\Big(\sum_{i=1}^n p_i - 1\Big) f(x_0) - \sum_{i=1}^n p_i f(x_i) \\ &\leqslant \frac{1}{4}(M-m)^2 \sum_{i=1}^n p_i \Psi_f\Big(x_i; x_0, \sum_{i=1}^n p_i x_i\Big) \\ &\leqslant \frac{1}{4}(M-m)(f'_-(M) - f'_+(m)) \sum_{i=1}^n p_i \end{aligned} \tag{3.5.6}$$

where $m=\min\{x_0,\sum_{i=1}^n p_ix_i\}$, $M=\max\{x_0,\sum_{i=1}^n p_ix_i\}$, *and* A, B *are defined in* (3.5.3). *If* f *is concave, the inequalities are reversed.*

Proof. Let f be a convex function. The inequalities (3.5.4), (3.5.5) and (3.5.6) are obtained from (2.6), (2.7) and (2.8) by substituting $A(\mathbf{x})=\frac{\sum_{i=1}^n p_ix_i}{\sum_{i=1}^n p_i}$ and $\phi\leftrightarrow f$.

If f is concave, then the reversed inequalities follow by substituting $f\leftrightarrow -f$ which is convex. □

The well-known Petrović's inequality [13] for a convex function $f\colon[0,a]\to\mathbb{R}$ is given by

$$\sum_{i=1}^n f(x_i)\leqslant f\Big(\sum_{i=1}^n x_i\Big)+(n-1)f(0) \tag{3.5.7}$$

where $x_i, i=1,\dots,n$ are nonnegative numbers such that $x_1,\dots,x_n,\sum_{i=1}^n x_i\in[0,a]$.

The following result follows directly by applying Theorem 2.2 to Petrović's inequality, but can also be obtained as a special case of Theorem 3.5.2 for $p_1=\dots=p_n=1$ and $x_0=0$.

THEOREM 3.5.3. *Let* f *be a continuous convex function on an open interval of real numbers* $I\supset[0,a]$ *If* $x_1,\dots,x_n\in[0,a]$ *are real numbers such that* $\sum_{i=1}^n x_i\in\langle 0,a]$, *then*

(i)

$$\begin{aligned}0&\leqslant f\Big(\sum_{i=1}^n x_i\Big)+(n-1)f(0)-\sum_{i=1}^n f(x_i)\\&\leqslant\sum_{j=1}^n x_j\Big(\sum_{i=1}^n x_i-x_j\Big)\sup_{t\in\langle 0,\sum_{i=1}^n x_i\rangle}\Psi_f\Big(t;0,\sum_{i=1}^n x_i\Big)\\&\leqslant\frac{\sum_{j=1}^n x_j(\sum_{i=1}^n x_i-x_j)}{\sum_{i=1}^n x_i}\Big(f'_-\Big(\sum_{i=1}^n x_i\Big)-f'_+(0)\Big)\\&\leqslant\frac{n-1}{n}\Big(\sum_{i=1}^n x_i\Big)\Big(f'_-\Big(\sum_{i=1}^n x_i\Big)-f'_+(0)\Big)\\&\leqslant\frac{n}{4}\Big(\sum_{i=1}^n x_i\Big)\Big(f'_-\Big(\sum_{i=1}^n x_i\Big)-f'_+(0)\Big)\end{aligned}\tag{3.5.8}$$

(ii)

$$\begin{aligned}0&\leqslant f\Big(\sum_{i=1}^n x_i\Big)+(n-1)f(0)-\sum_{i=1}^n f(x_i)\\&\leqslant\sum_{j=1}^n x_j\Big(\sum_{i=1}^n x_i-x_j\Big)\sup_{t\in\langle 0,\sum_{i=1}^n x_i\rangle}\Psi_f\Big(t;0,\sum_{i=1}^n x_i\Big)\\&\leqslant\frac{n-1}{n}\Big(\sum_{i=1}^n x_i\Big)^2\sup_{t\in\langle 0,\sum_{i=1}^n x_i\rangle}\Psi_f\Big(t;0,\sum_{i=1}^n x_i\Big)\end{aligned}\tag{3.5.9}$$

$$\leqslant \frac{n-1}{n}\Big(\sum_{i=1}^{n} x_i\Big)\Big(f'_-\Big(\sum_{i=1}^{n} x_i\Big) - f'_+(0)\Big)$$
$$\leqslant \frac{n}{4}\Big(\sum_{i=1}^{n} x_i\Big)\Big(f'_-\Big(\sum_{i=1}^{n} x_i\Big) - f'_+(0)\Big)$$

(iii)

$$\begin{aligned} 0 &\leqslant f\Big(\sum_{i=1}^{n} x_i\Big) + (n-1)f(0) - \sum_{i=1}^{n} f(x_i) \\ &\leqslant \frac{1}{4}\Big(\sum_{i=1}^{n} x_i\Big)^2 \sum_{i=1}^{n} \Psi_f\Big(x_i; 0, \sum_{i=1}^{n} x_i\Big) \qquad (3.5.10) \\ &\leqslant \frac{n}{4}\Big(\sum_{i=1}^{n} x_i\Big)\Big(f'_-\Big(\sum_{i=1}^{n} x_i\Big) - f'_+(0)\Big) \end{aligned}$$

If f is concave, the inequalities are reversed.

Proof. Let f be a convex function. The inequalities (3.5.8), (3.5.9) and (3.5.10) are obtained from (2.6), (2.7) and (2.8) by substituting $A(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$, $m = 0$, $M = \sum_{i=1}^{n} x_i$ and $\phi \leftrightarrow f$.

If f is concave, then the reversed inequalities follow by substituting $f \leftrightarrow -f$ which is convex. $\square$

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