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INEQUALITIES AND ASYMPTOTIC EXPANSIONS OF THE WALLIS SEQUENCE AND THE SUM OF THE WALLIS RATIO

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Abstract. The asymptotic expansions for the Wallis sequence is studied in details, and explicit formulae for coefficients are given in their simplest form. This enables improvement of some basic inequalities connected with this sequence. Furthermore we establish identity, inequality and asymptotic expansion for the sum of the Wallis ratio.

1. Introduction

The famous Wallis sequence $\left(W_n\right)_{n\geqslant 1}$ is defined by

$$W_n = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \qquad (n \in \mathbb{N} := \{1,2,3,\ldots\}).$$

Wallis (1655) showed that $W_\infty = \pi/2$.

In [8], Hirschhorn proved that for $n \geqslant 1$,

$$\frac{\pi}{2}\left(1-\frac{1}{4n+\frac{7}{3}}\right) < W_n < \frac{\pi}{2}\left(1-\frac{1}{4n+\frac{8}{3}}\right), \tag{1.1}$$

which is stronger than Lampret's result [9]:

$$\frac{\pi}{2}\left(1-\frac{1.1}{4n}\right) < W_n < \frac{\pi}{2}\left(1-\frac{0.8}{4n}\right) \tag{1.2}$$

and Păltănea's result [10]:

$$\frac{\pi}{2}\sqrt{\frac{2n}{2n+1}} < W_n < \frac{\pi}{2}\sqrt{\frac{2n+1}{2n+2}} \qquad (n \geqslant 1). \tag{1.3}$$

The first aim of this paper is to establish more accurate bounds of these inequalities.

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It is known in the literature that

$$P_n := \frac{(2n-1)!!}{(2n)!!} = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)}$$

is called the Wallis ratio. Here, we employ the special double factorial notation as follows:

$$\begin{aligned} &(2n)!! = 2\cdot 4\cdot 6\cdots(2n) = 2^n n!,\\ &(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-1) = \pi^{-1/2}2^n\Gamma(n+\tfrac{1}{2}),\\ &0!! = 1, \qquad (-1)!! = 1 \end{aligned}$$

(see [1, p. 258]), Γ denotes the gamma function.

The second aim of this paper is to establish an identity, inequality and asymptotic expansion of the sum of the Wallis ratio $\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}$ (see Section 5).

In [8], the author pointed out that if the c_k are given by $x\sum_{k\geqslant 0} c_k x^{2k}/(2k)! = \tanh(x/4)$ then

$$\begin{aligned} W_n &\sim \frac{\pi}{2}\left(1+\frac{1}{2n}\right)^{-1}\prod_{k\geqslant 0}\exp\left(\frac{c_k}{n^{2k+1}}\right)\\ &\sim \frac{\pi}{2}\left(1-\frac{1}{4n}+\frac{5}{32n^2}-\frac{11}{128n^3}+\frac{83}{2048n^4}-\frac{143}{8192n^5}+\cdots\right) \quad (n\to\infty). \end{aligned} \tag{1.4}$$

We shall present how one can obtain explicit algorithm for the asymptotic expansion of W_n through the powers of the variable $n+\alpha$, where α is an independent parameter. Since this expansion is starting point for related inequalities, we shall start with this question.

2. Asymptotic expansion of the Wallis sequence

We will establish asymptotic expansion for W_n considering that

$$W_n = \frac{\pi}{2}\cdot\frac{1}{n+\frac{1}{2}}\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 = \frac{\pi}{2}\cdot\frac{\Gamma(n+1)^2}{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}. \tag{2.1}$$

The following asymptotic expansion of the multiple quotients of two gamma functions has been proved in [4]:

THEOREM A. *Let $t-s=v-u$ and let $r>0$. It holds*

$$\frac{\Gamma(x+t)\Gamma(x+u)}{\Gamma(x+s)\Gamma(x+v)} \sim \left(\sum_{m=0}^{\infty}\frac{P_m(t,s,u,v)}{x^m}\right)^{\frac{1}{r}}, \tag{2.2}$$

where polynomials (P_m) *are defined by*

$$\begin{aligned} &P_0(t,s,u,v)=1,\\ &P_m(t,s,u,v)=\frac{r}{m}\sum_{k=1}^{m}\frac{(-1)^{k+1}}{k+1}\left[B_{k+1}(t)-B_{k+1}(s)+B_{k+1}(u)-B_{k+1}(v)\right]P_{m-k}. \end{aligned} \tag{2.3}$$

$B_k(t)$ are Bernoulli polynomials defined by following generating function:

$$\frac{xe^{tx}}{e^x-1}=\sum_{k=0}^{\infty}B_k(t)\frac{x^k}{k!}. \tag{2.4}$$

Recall some properties of Bernoulli polynomials:

$$\begin{aligned} &B_{2k+1}=0, \qquad (k\geqslant 1),\\ &B_k(0)=(-1)^kB_k(1)=B_k,\\ &B_k(1-t)=(-1)^kB_k(t),\\ &B_k(t+1)-B_k(t)=kt^{k-1},\\ &(-1)^kB_k(-t)=B_k(t)+kt^{k-1},\\ &B_k(\tfrac{1}{2})=-(1-2^{1-k})B_k,\\ &B_k(\tfrac{1}{4})=(-1)^kB_k(\tfrac{3}{4})=-2^{-k}(1-2^{1-k})B_k-k4^{-k}E_{k-1}, \end{aligned} \tag{2.5}$$

where B_n is n-th Bernoulli number and E_n is n-th Euler number.

Let $x=n+\alpha$, $t=1-\alpha$, $s=\frac{1}{2}-\alpha$, $u=1-\alpha$ and $v=\frac{3}{2}-\alpha$. Then from (2.1) and Theorem A follows

$$W_n\sim\frac{\pi}{2}\left(\sum_{m=0}^{\infty}P_m(\alpha)(n+\alpha)^{-m}\right)^{1/r}, \tag{2.6}$$

where

$$\begin{aligned} &P_0(\alpha)=1,\\ &P_m(\alpha)=\frac{r}{m}\sum_{k=1}^{m}\frac{(-1)^{k+1}}{k+1}\left[2B_{k+1}(1-\alpha)-B_{k+1}(\tfrac{1}{2}-\alpha)-B_{k+1}(\tfrac{3}{2}-\alpha)\right]P_{m-k}(\alpha). \end{aligned} \tag{2.7}$$

Applying (2.5) to $t=\frac{1}{2}-\alpha$ we get

$$P_m(\alpha)=\frac{2r}{m}\sum_{k=1}^{m}\frac{(-1)^{k+1}}{k+1}\Big[B_{k+1}(1-\alpha)-B_{k+1}(\tfrac{1}{2}-\alpha)-\tfrac{1}{2}(k+1)(\tfrac{1}{2}-\alpha)^k\Big]P_{m-k}(\alpha). \tag{2.8}$$

From the form of P_m we notice that the natural choice of α is 0 or $\frac{1}{2}$. We also want to choose other values of α such that coefficients of polynomials P_m are as simple as they can be, in the sense that we can express them in terms of Bernoulli and Euler numbers. According to properties of Bernoulli polynomials mentioned above, that would happen in the case $1-\alpha=1-(\frac{1}{2}-\alpha)$ or $1-\alpha=-(\frac{1}{2}-\alpha)$ wherefrom it follows $\alpha=\frac{1}{4}$ or $\alpha=\frac{3}{4}$.

We are now ready to state the following theorem.

THEOREM 2.1. *The following asymptotic expansion holds true:*

$$W_n \sim \frac{\pi}{2}\left(\sum_{m=0}^{\infty} P_m (n+\alpha)^{-m}\right)^{1/r}, \tag{2.9}$$

where $P_0=1$ *and*

1. *for* $\alpha=0$

$$P_m = \frac{2r}{m}\sum_{k=1}^{m}\frac{(-1)^{k+1}}{k+1}\left[(2-2^{-k})B_{k+1}-(k+1)2^{-k-1}\right]P_{m-k}; \tag{2.10}$$

2. *for* $\alpha=\frac{1}{4}$

$$P_m = \frac{r}{m}\sum_{k=1}^{m}(-1)^{k+1}4^{-k}(E_k-1)P_{m-k}; \tag{2.11}$$

3. *for* $\alpha=\frac{1}{2}$

$$P_m = \frac{r}{m}\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{2}{k}(4^{-k}-1)B_{2k}P_{m-2k+1}; \tag{2.12}$$

4. *for* $\alpha=\frac{3}{4}$

$$P_m = \frac{r}{m}\sum_{k=1}^{m}4^{-k}(E_k-1)P_{m-k}; \tag{2.13}$$

Proof. Let us denote

$$C_k(\alpha) = B_{k+1}(1-\alpha)-B_{k+1}(\tfrac{1}{2}-\alpha)-(k+1)(\tfrac{1}{2}-\alpha)^k. \tag{2.14}$$

We have

$$\begin{aligned} C_k(0) &= (-1)^{k+1}B_{k+1}+(1-2^{-k})B_{k+1}-(k+1)2^{-k-1} \\ &= ((-1)^{k+1}+1-2^{-k})B_{k+1}-(k+1)2^{-k-1} \\ &= (2-2^{-k})B_{k+1}-(k+1)2^{-k-1} \end{aligned}$$

which proves (2.10).

Furthermore,

$$C_k(\tfrac{1}{4}) = B_{k+1}(\tfrac{3}{4})-B_{k+1}(\tfrac{1}{4})-\frac{1}{2}(k+1)4^{-k}. \tag{2.15}$$

For odd k, using (2.5)

$$C_k(\tfrac{1}{4}) = -\tfrac{1}{2}(k+1)4^{-k}. \tag{2.16}$$

For even k we have

$$\begin{aligned} C_k(\tfrac{1}{4}) &= 2B_{k+1}(\tfrac{3}{4}) - \frac{1}{2}(k+1)4^{-k} \\ &= 2(k+1)4^{-k-1}E_k - \frac{1}{2}(k+1)4^{-k} \\ &= \frac{1}{2}(k+1)4^{-k}(E_k - 1), \end{aligned}$$

which coincides with (2.16) if k is odd. Hence, (2.11) is proved.

If $\alpha = \frac{1}{2}$ then we have

$$C_k(\tfrac{1}{2}) = B_{k+1}(\tfrac{1}{2}) - B_{k+1}(0) = -(2-2^{-k})B_{k+1}.$$

Since $B_{k+1} = 0$ for even k, easily follows (2.12).

Finally, using (2.5)

$$\begin{aligned} C_k(\tfrac{3}{4}) &= B_{k+1}(\tfrac{1}{4}) - B_{k+1}(-\tfrac{1}{4}) - (-1)^k \tfrac{1}{2}(k+1)4^{-k} \\ &= (-1)^{k+1}B_{k+1}(\tfrac{3}{4}) - (-1)^{k+1}B_{k+1}(\tfrac{1}{4}) - (-1)^{k+1}\tfrac{1}{2}(k+1)4^{-k} \\ &= (-1)^{k+1}C_k(\tfrac{1}{4}). \end{aligned}$$

The proof of the theorem is completed. □

Calculating the first few values of $P_m(\alpha)$ for observed shifts α from Theorem 2.1 we get, using $r = 1$:

$$W_n \sim \frac{\pi}{2}\left[1 - \frac{1}{4n} + \frac{5}{32n^2} - \frac{11}{128n^3} + \frac{83}{2048n^4} - \frac{143}{8192n^5} + \cdots\right], \tag{2.17}$$

$$\begin{aligned} W_n \sim \frac{\pi}{2}\Bigg[1 &- \frac{1}{4\left(n+\frac{1}{4}\right)} + \frac{3}{32\left(n+\frac{1}{4}\right)^2} - \frac{3}{128\left(n+\frac{1}{4}\right)^3} \\ &+ \frac{3}{2048\left(n+\frac{1}{4}\right)^4} - \frac{3}{8192\left(n+\frac{1}{4}\right)^5} + \cdots\Bigg], \end{aligned} \tag{2.18}$$

$$\begin{aligned} W_n \sim \frac{\pi}{2}\Bigg[1 &- \frac{1}{4\left(n+\frac{1}{2}\right)} + \frac{1}{32\left(n+\frac{1}{2}\right)^2} + \frac{1}{128\left(n+\frac{1}{2}\right)^3} \\ &- \frac{5}{2048\left(n+\frac{1}{2}\right)^4} - \frac{23}{8192\left(n+\frac{1}{2}\right)^5} + \cdots\Bigg], \end{aligned} \tag{2.19}$$

$$W_n \sim \frac{\pi}{2}\left[1 - \frac{1}{4\left(n+\frac{3}{4}\right)} - \frac{1}{32\left(n+\frac{3}{4}\right)^2} + \frac{1}{128\left(n+\frac{3}{4}\right)^3} + \frac{11}{2048\left(n+\frac{3}{4}\right)^4} - \frac{11}{8192\left(n+\frac{3}{4}\right)^5} + \cdots\right], \tag{2.20}$$

Notice that substitution $N = 4n+1$ gives us simpler form of Wallis expansion. For example, from (2.18):

$$W_n \sim \frac{\pi}{2}\left[1 - \frac{1}{N} + \frac{3}{2N^2} - \frac{3}{2N^3} + \frac{3}{8N^4} - \frac{3}{8N^5} + \frac{159}{16N^6} - \frac{159}{16N^7} + \cdots\right]. \tag{2.21}$$

Also, another choice of r gives expansions which can be useful in determining lower and upper bounds in various forms. For example, taking $r = 2$ and using (2.8) we obtain:

$$W_n \sim \frac{\pi}{2}\sqrt{\sum_{m=0}^{\infty} P_m(\alpha)(n+\alpha)^{-m}} \tag{2.22}$$

where

$$\begin{aligned}
P_0(\alpha) &= 1,\\
P_1(\alpha) &= \tfrac{1}{2}\alpha - \tfrac{3}{8},\\
P_2(\alpha) &= -\tfrac{1}{2}\alpha^2 + \tfrac{3}{4}\alpha - \tfrac{1}{4},\\
P_3(\alpha) &= -\tfrac{1}{2}\alpha^3 + \tfrac{9}{8}\alpha - \tfrac{3}{4}\alpha + \tfrac{19}{128},\\
P_4(\alpha) &= -\tfrac{1}{2}\alpha^4 + \tfrac{3}{2}\alpha^3 - \tfrac{3}{2}\alpha^2 + \tfrac{19}{32}\alpha - \tfrac{21}{256}\\
P_5(\alpha) &= -\tfrac{1}{2}\alpha^5 + \tfrac{15}{8}\alpha^4 - \tfrac{5}{2}\alpha^3 + \tfrac{95}{64}\alpha^2 - \tfrac{105}{256}\alpha + \tfrac{49}{1024}
\end{aligned} \tag{2.23}$$

For $\alpha = 0$ the standard expansion can be written:

$$W_n \sim \frac{\pi}{2}\sqrt{1 - \frac{1}{2n} + \frac{3}{8n^2} - \frac{1}{4n^3} + \frac{19}{128n^4} - \frac{21}{256} + \dots}$$

However, the inspection of the coefficients in (2.23) shows that $P_2(\alpha) = 0$ for $\alpha = \frac{1}{2}$ or $\alpha = 1$. This leads to the following expansions which are related with inequality (1.3):

$$W_n \sim \frac{\pi}{2}\sqrt{1 - \frac{1}{2(n+\frac{1}{2})} + \frac{1}{8(n+\frac{1}{4})^2} - \frac{1}{128(n+\frac{1}{2})^4} - \frac{1}{256(n+\frac{1}{2})^5} + \dots} \tag{2.24}$$

$$\sim \frac{\pi}{2}\sqrt{1 - \frac{1}{2(n+1)} - \frac{1}{8(n+1)^2} + \frac{3}{128(n+1)^4} + \frac{3}{256(n+1)^5} + \dots}. \tag{2.25}$$

The lower bound in (1.3) is given by the first term in (2.24).

Further, we can see that for $\alpha = \frac{3}{4}$ the second term $P_1(\frac{3}{4})$ is equal to zero. This will imply the new, better lower bound in (1.3) which will be proved in the Section 4. The corresponding asymptotic expansion is

$$W_n \sim \frac{\pi}{2}\sqrt{1 - \frac{1}{2(n+\frac{3}{4})} + \frac{1}{32(n+\frac{3}{4})^3} + \frac{1}{128(n+\frac{3}{4})^4} - \frac{3}{512(n+\frac{3}{4})^5} + \dots}. \tag{2.26}$$

3. Asymptotic expansions of the second type

Wallis product can be written as:

$$W_n = \frac{\pi}{2} \cdot \frac{1}{n+\frac{1}{2}} \left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \right]^2. \tag{3.1}$$

This form is convenient for applying results proved in [2]:

THEOREM B. *It holds*

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{\frac{1}{t-s}} \sim \sum_{m=0}^{\infty} Q_m(t,s) x^{-m+1} \tag{3.2}$$

where Q_m are polynomials of order m defined by

$$\begin{aligned} &Q_0(t,s) = 1, \\ &Q_m(t,s) = \frac{1}{m} \sum_{k=1}^{m} (-1)^{k+1} \frac{B_{k+1}(t) - B_{k+1}(s)}{(k+1)(t-s)} Q_{m-k}(t,s). \end{aligned} \tag{3.3}$$

Now we have

$$\begin{aligned} \frac{1}{x+u} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{\frac{1}{t-s}} &\sim \frac{1}{x} \cdot \frac{1}{1+\frac{u}{x}} \left(\sum_{m=0}^{\infty} Q_m(t,s) x^{-m+1} \right) \\ &= \left(\sum_{m=0}^{\infty} (-1)^m \frac{u^m}{x^m} \right) \left(\sum_{m=0}^{\infty} Q_m(t,s) \frac{1}{x^m} \right) = \sum_{m=0}^{\infty} d_m x^{-m} \end{aligned}$$

where

$$d_m = \sum_{k=0}^{m} (-1)^{m-k} u^{m-k} Q_k(t,s).$$

Substitution $x = n+\alpha$, $t = 1-\alpha$ and $s = u = \frac{1}{2} - \alpha$ gives us asymptotic expansion

$$W_n \sim \sum_{m=0}^{\infty} d_m (n+\alpha)^{-m} \tag{3.4}$$

where

$$d_m = \sum_{k=0}^{m} (-1)^{m-k} (\tfrac{1}{2} - \alpha)^{m-k} Q_k(1-\alpha, \tfrac{1}{2} - \alpha). \tag{3.5}$$

In the case $\alpha = 0$ we have

$$\begin{aligned} Q_m(1,\tfrac{1}{2}) &= \frac{1}{m} \sum_{k=1}^{m} (-1)^{k+1} \frac{B_{k+1}(1) - B_{k+1}(\frac{1}{2})}{\frac{1}{2}(k+1)} Q_{m-k}(1,\tfrac{1}{2}) \\ &= \frac{1}{m} \sum_{k=1}^{m} (-1)^{k+1} \frac{2\left[(-1)^k + 1 - 2^{-k}\right] B_{k+1}}{k+1} Q_{m-k}(1,\tfrac{1}{2}) \\ &= \frac{1}{m} \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} \frac{2(1-2^{-2k})}{k} B_{2k} Q_{m-2k+1}(1,\tfrac{1}{2}). \end{aligned} \tag{3.6}$$

If $\alpha = \frac{1}{4}$, then

$$\begin{aligned} Q_m(\tfrac{3}{4},\tfrac{1}{4}) &= \frac{1}{m}\sum_{k=1}^{m}(-1)^{k+1}\frac{B_{k+1}(\frac{3}{4})-B_{k+1}(\frac{1}{4})}{\frac{1}{2}(k+1)}Q_{m-k}(\tfrac{3}{4},\tfrac{1}{4}) \\ &= -\frac{1}{m}\sum_{k=1}^{\lfloor\frac{m}{2}\rfloor}\frac{2B_{2k+1}}{\frac{1}{2}(2k+1)}Q_{m-2k}(\tfrac{3}{4},\tfrac{1}{4}) \\ &= -\frac{1}{m}\sum_{k=1}^{\lfloor\frac{m}{2}\rfloor}4^{-2k}E_{2k}Q_{m-2k}(\tfrac{3}{4},\tfrac{1}{4}). \end{aligned} \tag{3.7}$$

For $\alpha = \frac{1}{2}$

$$\begin{aligned} Q_m(\tfrac{1}{2},0) &= \frac{1}{m}\sum_{k=1}^{m}(-1)^{k+1}\frac{B_{k+1}(\frac{1}{2})-B_{k+1}(0)}{\frac{1}{2}(k+1)}Q_{m-k}(\tfrac{1}{2},0) \\ &= \frac{1}{m}\sum_{k=1}^{m}\frac{-(2-2^{-k})}{\frac{1}{2}(k+1)}B_{k+1}Q_{m-k}(\tfrac{1}{2},0) \\ &= \frac{1}{m}\sum_{k=1}^{\lfloor\frac{m+1}{2}\rfloor}\frac{2}{k}(4^{-k}-1)B_{2k}Q_{m-2k+1}(\tfrac{1}{2},0). \end{aligned} \tag{3.8}$$

We notice that $Q_m(\frac{1}{2},0)$ satisfy the same recursive formula (2.12) as polynomials P_m from Theorem 2.1.

In the last case, $\alpha = \frac{3}{4}$,

$$\begin{aligned} Q_m(\tfrac{1}{4},-\tfrac{1}{4}) &= \frac{1}{m}\sum_{k=1}^{m}(-1)^{k+1}\frac{B_{k+1}(\frac{1}{4})-B_{k+1}(-\frac{1}{4})}{\frac{1}{2}(k+1)}Q_{m-k}(\tfrac{1}{4},-\tfrac{1}{4}) \\ &= \frac{1}{m}\sum_{k=1}^{m}(-1)^{k+1}\frac{-(k+1)4^{-k}(E_k-1)}{\frac{1}{2}(k+1)}Q_{m-k}(\tfrac{1}{4},-\tfrac{1}{4}) \\ &= \frac{1}{m}\sum_{k=1}^{m}2(-1)^{k+1}4^{-k}(E_k-1)Q_{m-k}(\tfrac{1}{4},-\tfrac{1}{4}). \end{aligned} \tag{3.9}$$

4. Inequalities for the Wallis sequence

The beginning of asymptotic expansion (2.6) for $r=1$ read as

$$W_n \sim \frac{\pi}{2}\left(1-\frac{\frac{1}{4}}{n+\alpha}+\frac{\frac{5}{32}-\frac{\alpha}{4}}{(n+\alpha)^2}+\frac{-\frac{11}{128}+\frac{5\alpha}{16}-\frac{\alpha^2}{4}}{(n+\alpha)^3}+\dots\right).$$

Therefore, for $\alpha = \frac{5}{8}$ the second coefficient is equal to zero and it holds

$$W_n \sim \frac{\pi}{2}\left(1-\frac{1}{4n+\frac{5}{2}}\right)+O\left(\frac{1}{n^3}\right).$$

The first few terms in this expansion are

$$W_n \sim \frac{\pi}{2}\left(1-\frac{\frac{1}{4}}{n+\frac{5}{8}}+\frac{\frac{3}{256}}{(n+\frac{5}{8})^3}+\frac{\frac{3}{2048}}{(n+\frac{5}{8})^4}-\frac{\frac{51}{16384}}{(n+\frac{5}{8})^5}-\frac{\frac{75}{65536}}{(n+\frac{5}{8})^6}+\frac{\frac{2253}{1048576}}{(n+\frac{5}{8})^7}+\cdots\right).$$

This fact motivated us to observe Theorem 4.1, which gives more accurate bounds for the Wallis sequence W_n.

THEOREM 4.1. *For all integers $n \geqslant 1$,*

$$\frac{\pi}{2}\left(1-\frac{\frac{1}{4}}{n+\frac{5}{8}}+\frac{\frac{3}{256}}{(n+\frac{5}{8})^3}+\frac{\frac{3}{2048}}{(n+\frac{5}{8})^4}-\frac{\frac{51}{16384}}{(n+\frac{5}{8})^5}-\frac{\frac{75}{65536}}{(n+\frac{5}{8})^6}\right) < W_n < \frac{\pi}{2}\left(1-\frac{\frac{1}{4}}{n+\frac{5}{8}}+\frac{\frac{3}{256}}{(n+\frac{5}{8})^3}+\frac{\frac{3}{2048}}{(n+\frac{5}{8})^4}\right). \tag{4.1}$$

In order to prove Theorem 4.1, we need the following result.

LEMMA A. [see [12]] *For $n \geqslant 0$, the following Brouncker's continued fraction formula holds true:*

$$\left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2 = \cfrac{4}{1+4n+\cfrac{1^2}{2+8n+\cfrac{3^2}{2+8n+\cfrac{5^2}{2+8n+\ddots}}}}. \tag{4.2}$$

By (4.2), we have the following inequality [7, p. 742]:

$$\frac{16(19+92n+96n^2+128n^3)}{105+704n+1920n^2+2048n^3+2048n^4} = \cfrac{4}{1+4n+\cfrac{1^2}{2+8n+\cfrac{3^2}{2+8n+\cfrac{5^2}{2+8n}}}}$$

$$< \left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2 < \cfrac{4}{1+4n+\cfrac{1^2}{2+8n+\cfrac{3^2}{2+8n+\cfrac{5^2}{2+8n+\cfrac{7^2}{2+8n}}}}}$$

$$= \frac{4(789+2912n+6848n^2+4096n^3+4096n^4)}{945+6756n+18880n^2+32000n^3+20480n^4+16384n^5} \qquad (n \in \mathbb{N}). \tag{4.3}$$

Proof of Theorem 4.1. The Wallis sequence can be expressed as

$$W_n = \frac{\pi}{2n+1}\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2.$$

By using the second inequality in (4.3), we have for $n \geqslant 1$,

$$\begin{aligned}
&\frac{2}{\pi}W_n - \left(1 - \frac{\frac{1}{4}}{n+\frac{5}{8}} + \frac{\frac{3}{256}}{(n+\frac{5}{8})^3} + \frac{\frac{3}{2048}}{(n+\frac{5}{8})^4} - \frac{\frac{51}{16384}}{(n+\frac{5}{8})^5} - \frac{\frac{75}{65536}}{(n+\frac{5}{8})^6}\right)\\
&= \frac{2}{2n+1}\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2\\
&\quad - \left(1 - \frac{\frac{1}{4}}{n+\frac{5}{8}} + \frac{\frac{3}{256}}{(n+\frac{5}{8})^3} + \frac{\frac{3}{2048}}{(n+\frac{5}{8})^4} - \frac{\frac{51}{16384}}{(n+\frac{5}{8})^5} - \frac{\frac{75}{65536}}{(n+\frac{5}{8})^6}\right)\\
&> \frac{2}{2n+1}\left(\frac{4(789+2912n+6848n^2+4096n^3+4096n^4)}{945+6756n+18880n^2+32000n^3+20480n^4+16384n^5}\right)^{-1}\\
&\quad - \left(1 - \frac{\frac{1}{4}}{n+\frac{5}{8}} + \frac{\frac{3}{256}}{(n+\frac{5}{8})^3} + \frac{\frac{3}{2048}}{(n+\frac{5}{8})^4} - \frac{\frac{51}{16384}}{(n+\frac{5}{8})^5} - \frac{\frac{75}{65536}}{(n+\frac{5}{8})^6}\right)\\
&= \frac{\begin{array}{c}3\Big(8933109+29070400(n-1)+34036160(n-1)^2\\ +17031680(n-1)^3+3076096(n-1)^4\Big)\end{array}}{2(789+2912n+6848n^2+4096n^3+4096n^4)(2n+1)(8n+5)^6} > 0.
\end{aligned}$$

Using the first inequality in (4.3), we have for $n \geqslant 1$,

$$\begin{aligned}
&\frac{2}{\pi}W_n - \left(1 - \frac{\frac{1}{4}}{n+\frac{5}{8}} + \frac{\frac{3}{256}}{(n+\frac{5}{8})^3} + \frac{\frac{3}{2048}}{(n+\frac{5}{8})^4} - \frac{\frac{51}{16384}}{(n+\frac{5}{8})^5}\right)\\
&= \frac{2}{2n+1}\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 - \left(1 - \frac{\frac{1}{4}}{n+\frac{5}{8}} + \frac{\frac{3}{256}}{(n+\frac{5}{8})^3} + \frac{\frac{3}{2048}}{(n+\frac{5}{8})^4}\right)\\
&< \frac{2}{2n+1}\,\frac{105+704n+1920n^2+2048n^3+2048n^4}{16(19+92n+96n^2+128n^3)}\\
&\quad - \left(1 - \frac{\frac{1}{4}}{n+\frac{5}{8}} + \frac{\frac{3}{256}}{(n+\frac{5}{8})^3} + \frac{\frac{3}{2048}}{(n+\frac{5}{8})^4}\right)\\
&= -\frac{3\Big(16405+43504(n-1)+34752(n-1)^2+8704(n-1)^3\Big)}{8(19+92n+96n^2+128n^3)(2n+1)(8n+5)^4} < 0.
\end{aligned}$$

The proof of Theorem 4.1 is complete. □

From the inequality (4.1), the better lower bound than (1.1) can be obtained.

COROLLARY 4.2. *For all integers $n \geqslant 1$ it holds*

$$W_n > \frac{\pi}{2}\left(1 - \frac{1}{4n + \frac{5}{2}}\right). \tag{4.4}$$

The constant $\frac{5}{2}$ is the best possible.

Proof. It is sufficient to prove that it holds

$$A = \tfrac{3}{256}(n+\tfrac{5}{8})^3 + \tfrac{3}{2048}(n+\tfrac{5}{8})^2 - \tfrac{51}{16384}(n+\tfrac{5}{8}) - \tfrac{75}{65536} \geqslant 0$$

for all integers $n \geqslant 1$. But this is evident since

$$A = \tfrac{3}{256}n^3 + \tfrac{3}{128}n^2 + \tfrac{51}{4096}n + \tfrac{45}{131072}.$$

Let us prove that $\frac{5}{2}$ cannot be replaced by any $a > \frac{5}{2}$. From (4.1) one conclude that

$$W_n - \frac{\pi}{2}\left(1 - \frac{1}{4n + \frac{5}{2}}\right) = O(n^{-3}). \tag{4.5}$$

Therefore, for any constant $a > \frac{5}{2}$, we have

$$\begin{aligned} W_n - \frac{\pi}{2}\left(1 - \frac{1}{4n+a}\right) &= W_n - \frac{\pi}{2}\left(1 - \frac{1}{4n + \frac{5}{2} + (a - \frac{5}{2})}\right) \\ &= W_n - \frac{\pi}{2}\left(1 - \frac{1}{4n+\frac{5}{2}} \cdot \left[1 - \frac{a-\frac{5}{2}}{4n+\frac{5}{2}} + O(n^{-2})\right]\right) \\ &= W_n - \frac{\pi}{2}\left(1 - \frac{1}{4n+\frac{5}{2}}\right) - \frac{\pi}{2} \cdot \frac{a-\frac{5}{2}}{(4n+\frac{5}{2})^2} + O(n^{-3}) \end{aligned}$$

and this is in contradiction with (4.5). □

REMARK 4.3. In the inequality (1.1) the left side is asymptotic, inequality is always strict and one gets equality when $n \to \infty$. In the right side however, equality can be obtained for $n = 1$, if one replace the constant $\frac{8}{3}$ with better one, say b, such that it holds:

$$W_1 = \frac{\pi}{2}\left(1 - \frac{1}{4+b}\right) \tag{4.6}$$

wherefrom it follows

$$b = \frac{32 - 9\pi}{3\pi - 8} \approx 2.6149.$$

The constant $\frac{8}{3}$ is just a nice rational approximation greater than this value of b. In searching for the better constant, interesting continued fraction appears:

$$b = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2.08\cdots}}}}} < \frac{34}{13} = 2.6153\cdots.$$

The constant $\frac{8}{3}$ is the third approximant in this sequence.

The refinement of the inequality (1.3) will be proved using another technique and results which can be also interesting for itself.

THEOREM 4.4. *Let function f be defined as*

$$f(x) = \frac{\Gamma(x+1)^2}{\Gamma(x+1-s)\Gamma(x+1+s)}\sqrt{\frac{x+s^2+\frac{1}{2}}{x-s^2+\frac{1}{2}}}, \qquad (s>0). \tag{4.7}$$

Then the function $\log(f(x))$ is completely monotonic on $(s-1,\infty)$.

It follows that $f(x) \geqslant 1$, and for $s=\frac{1}{2}$ we have the following:

COROLLARY 4.5. *For all integers $n \geqslant 1$ it holds*

$$W_n > \frac{\pi}{2}\sqrt{\frac{4n+1}{4n+3}}. \tag{4.8}$$

The constant $\frac{3}{2}$ in the expression

$$W_n > \frac{\pi}{2}\sqrt{1-\frac{1}{2n+\frac{3}{2}}}$$

is the best possible.

Proof of Theorem 4.4. Using the following integral representations [1]:

$$\log\Gamma(x) = \int_0^\infty \left[(x-1)e^{-t} - \frac{1-e^{(1-x)t}}{e^t-1}\right]\frac{dt}{t},$$

$$\log x = \int_0^\infty [e^{-t}-e^{-xt}]\frac{dt}{t}$$

it is easy to obtain

$$\log(f(x)) = \int_0^\infty h(t)\frac{e^{-xt}}{t(e^t-1)}\,dt.$$

Here, function $h(t)$ is defined by

$$h(t) = 2-2\cosh(st)+\cosh(s^2+\tfrac{1}{2})t-\cosh(s^2-\tfrac{1}{2})t = \sum_{k=1}^\infty \frac{c_{2k}}{(2k)!}t^{2k}$$

where

$$c_{2k} = (s^2+\tfrac{1}{2})^{2k}-(s^2-\tfrac{1}{2})^{2k}-2s^{2k}.$$

It is obvious that $c_2=0$. For $k>1$ one has

$$\begin{aligned} c_{2k} &> (s^2+\tfrac{1}{2})^{2k}-(s^2-\tfrac{1}{2})^{2k}-2^k s^{2k} \\ &= (s^4+s^2+1)^k-(s^4-s^2+1)^k-(2s^2)^k > 0. \end{aligned}$$

Hence, it holds $h(t)>0$ for all $t>0$ and the claim follows. □

REMARK 4.6. In the inequality (1.3) one can obtain best upper bound by taking the value for $n = 1$. For example, from

$$W_n \leqslant \frac{\pi}{2}\sqrt{1-\frac{1}{4n+b}}$$

for $n = 1$ one obtain

$$b = \frac{128-9\pi^2}{9\pi^2-64} = 1.5779\cdots.$$

Continued fraction approximants of b, which are greater than b are: 2, $\frac{8}{5} = 1.6$, $\frac{30}{19} = 1.5789\cdots$, etc. Taking $\frac{8}{5}$ instead of b, one obtain upper bound

$$W_n < \frac{\pi}{2}\sqrt{\frac{10n+3}{10n+8}}.$$

5. Sum of the Wallis ratio

In this section, we consider the sum of the Wallis ratio $\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}$. Theorem 5.1 below establishes a more general result. As a consequence, we obtain an identities as follows:

$$\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!} = \frac{(2n+1)!!}{(2n)!!} \qquad (n \geqslant 0). \tag{5.1}$$

THEOREM 5.1. *Let $n \geqslant 1$ be an integer, and $a, b > 0$ be two real numbers. Then*

$$\sum_{k=1}^{n}\frac{\Gamma(k+a)}{\Gamma(k+b)} = \frac{1}{a-b+1}\left[\frac{\Gamma(n+1+a)}{\Gamma(n+b)} - \frac{\Gamma(1+a)}{\Gamma(b)}\right]. \tag{5.2}$$

Proof. We prove the representation formula (5.2) by using the principle of mathematical induction. In our proof of the representation formula (5.2), we make use of the recurrence formula:

$$\Gamma(z+1) = z\Gamma(z).$$

For $n = 1$, we find that

$$\sum_{k=1}^{1}\frac{\Gamma(k+a)}{\Gamma(k+b)} = \frac{\Gamma(1+a)}{\Gamma(1+b)} = \frac{1}{a-b+1}\left[\frac{(1+a)\Gamma(1+a)}{\Gamma(1+b)} - \frac{b\Gamma(1+a)}{\Gamma(1+b)}\right].$$

which shows that the formula (5.2) holds true for $n = 1$. We assume now that the formula (5.2) holds true for a fixed positive integer n. Then, for $n \mapsto n+1$ in (5.2), we have

$$\begin{aligned}\sum_{k=1}^{n+1}\frac{\Gamma(k+a)}{\Gamma(k+b)}&=\sum_{k=1}^{n}\frac{\Gamma(k+a)}{\Gamma(k+b)}+\frac{\Gamma(n+1+a)}{\Gamma(n+1+b)}\\&=\frac{1}{a-b+1}\left[\frac{\Gamma(n+1+a)}{\Gamma(n+b)}-\frac{\Gamma(1+a)}{\Gamma(b)}\right]+\frac{\Gamma(n+1+a)}{\Gamma(n+1+b)}\\&=\frac{1}{a-b+1}\left[\frac{(n+b)\Gamma(n+1+a)}{\Gamma(n+1+b)}-\frac{\Gamma(1+a)}{\Gamma(b)}\right.\\&\qquad\left.+\frac{(a-b+1)\Gamma(n+1+a)}{\Gamma(n+1+b)}\right]\\&=\frac{1}{a-b+1}\left[\frac{(n+1+a)\Gamma(n+1+a)}{\Gamma(n+1+b)}-\frac{\Gamma(1+a)}{\Gamma(b)}\right]\\&=\frac{1}{a-b+1}\left[\frac{\Gamma(n+2+a)}{\Gamma(n+1+b)}-\frac{\Gamma(1+a)}{\Gamma(b)}\right].\end{aligned}$$

The proof of the formula (5.2) is thus completed by means of the principle of mathematical induction on n. □

In particular, by taking $a=\frac{1}{2}$ and $b=1$ in (5.2) we obtain the following identity:

$$\sqrt{\pi}+\sum_{k=1}^{n}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}=\frac{(2n+1)\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\qquad(n\geqslant 1),\tag{5.3}$$

i.e.,

$$\sum_{k=0}^{n}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}=\frac{(2n+1)\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\qquad(n\geqslant 0),\tag{5.4}$$

which can be written as (5.1).

Theorem 5.2 provides sharp bounds for the sum of the Wallis ratio $\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}$.

THEOREM 5.2. *For all integers $n\geqslant 0$, then*

$$2\sqrt{n+\frac{3}{4}}<\sum_{k=0}^{n}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\leqslant 2\sqrt{n+\frac{\pi}{4}}\,.\tag{5.5}$$

The constants $\frac{3}{4}$ and $\frac{\pi}{4}$ are the best possible.

Proof. By (5.4), inequality (5.5) can be rewritten as

$$\frac{3}{4}<\left(\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right)^2-n\leqslant\frac{\pi}{4}\qquad(n\geqslant 0),$$

and this inequality follows from [5], Theorem 2. □

Inequality (5.5) can be written as

$$\frac{2}{\sqrt{\pi}}\sqrt{n+\frac{3}{4}} < \sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!} \leqslant \frac{2}{\sqrt{\pi}}\sqrt{n+\frac{\pi}{4}}\,. \tag{5.6}$$

From inequality (5.6) we obtain the limit formula for constant π:

$$\frac{2}{\sqrt{\pi}} = \lim_{n\to\infty} n^{-1/2}\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}. \tag{5.7}$$

Sofo [11] summarized some known representations for the constant π and established new representations.

In order to present an asymptotic expansion for the sum of the Wallis ratio $\sum_{k=0}^{n}\frac{(2k-1)!!}{(2k)!!}$, we will need the following theorem from [3].

THEOREM C. *It holds*

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim x^{t-s}\left(\sum_{m=0}^{\infty} R_m(t,s)x^{-m}\right)^{1/r} \tag{5.8}$$

where polynomials R_m are defined by:

$$\begin{aligned} &R_0(t,s) = 1,\\ &R_m(t,s) = \frac{r}{m}\sum_{k=1}^{m}(-1)^{k+1}\frac{B_{k+1}(t)-B_{k+1}(s)}{k+1}R_{m-k}(t,s). \end{aligned} \tag{5.9}$$

THEOREM 5.3. *The following asymptotic expansion holds true:*

$$\sum_{k=0}^{n}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} \sim \sum_{m=0}^{\infty}\frac{2r_m(\alpha)}{(n+\alpha)^{m+\frac{1}{2}}} \qquad (n\to\infty), \tag{5.10}$$

where

$$\begin{aligned} &r_0(\alpha) = 0,\\ &r_m(\alpha) = \frac{1}{m}\sum_{k=1}^{m}(-1)^{k+1}\frac{B_{k+1}(\frac{3}{2}-\alpha)-B_{k+1}(1-\alpha)}{k+1}r_{m-k}(\alpha). \end{aligned} \tag{5.11}$$

Proof. We apply Theorem 5.1 for $a=\frac{1}{2}$ and $b=1$:

$$\begin{aligned} \sum_{k=0}^{n}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} &= \frac{\Gamma(\frac{1}{2})}{\Gamma(1)} + \sum_{k=1}^{n}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} = \sqrt{\pi}+2\left[\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} - \frac{\Gamma(\frac{3}{2})}{\Gamma(1)}\right]\\ &= 2\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)} \end{aligned} \tag{5.12}$$

We use Theorem C for $x = n+\alpha$, $t = \frac{3}{2}-\alpha$, $s = 1-\alpha$ and $r = 1$. In that case $t-s = \frac{1}{2}$ and we obtain

$$\sum_{k=0}^{n} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} \sim 2(n+\alpha)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{R_m(\frac{3}{2}-\alpha, 1-\alpha)}{(n+\alpha)^m}. \tag{5.13}$$

Letting $r_m(\alpha) = R_m(\frac{3}{2}-\alpha, 1-\alpha)$ the proof of theorem is complete. □

REMARK 5.4. The same methods of this paper can be applied to find asymptotic expansions of the following sums, see [6], 0.1670.2 and 0.157.4:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(k+b)}{\Gamma(k+a)} = \frac{\Gamma(b)}{\Gamma(a-b)} \cdot \frac{\Gamma(n+a-b)}{\Gamma(n+a)},$$

or

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = \frac{(2n-1)!}{[(n-1)!]^2} = \frac{4^n-1}{\sqrt{\pi}} \cdot \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}$$

REMARK 5.5. Wallis product can be written in the form

$$\prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k)^2}\right) = \frac{2}{\pi}.$$

The similar result is also well known, [6], 0.262.3:

$$\prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k+1)^2}\right) = \frac{\pi}{4}.$$

Denote

$$V_n = \prod_{k=1}^{n-1} \left(1 - \frac{1}{(2k+1)^2}\right).$$

Then, it holds

$$V_n = \frac{\pi}{4} \cdot \frac{\Gamma(n)\Gamma(n+1)}{\Gamma(n+\frac{1}{2})^2}$$

and methods of Theorem 2.1 can be applied to this series.

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