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REFINEMENTS OF GENERALIZED HÖLDER'S INEQUALITY

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Abstract. In this paper, we present some refinements of a generalized Hölder's inequality, which is due to Vasić and Pečarić.

1. Introduction

If $a_k \geqslant 0, b_k \geqslant 0 \ (k = 1,2,\cdots,n)$, $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, then

$$\sum_{k=1}^{n} a_k b_k \leqslant \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}. \tag{1}$$

The sign of inequality is reversed for $p < 1$, $p \neq 0$. (For $p < 0$, we assume that a_k, $b_k > 0$.) Inequality (1) and its reversed version are called Hölder's inequalities and are important in analysis and their applications. The important inequalities have received considerable attention by many researchers, and have motivated a large number of research papers giving their different proofs providing various generalizations, improvements and analogues. For example, Aldaz [2] presents an improvement of Hölder's inequality for an arbitrary number of functions by using a refinement of the AM-GM inequality. Klaričić Bakula, Matković and Pečarić give several variants of Hölder's inequality in [6]. For more detail expositions, the interested reader may consult [1], [3–5], [7–9] and the references therein. Among various generalizations of (1), Vasić and Pečarić in [10] presented the following interesting theorems.

THEOREM A. *Let* $A_{ij} \geqslant 0 \ (i = 1,2,\cdots,n, \ j = 1,2,\cdots,m)$. *If* β_j *are positive numbers such that* $\sum_{j=1}^{m} \frac{1}{\beta_j} \geqslant 1$, *then*

$$\sum_{i=1}^{n} \prod_{j=1}^{m} A_{ij} \leqslant \prod_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij}^{\beta_j}\right)^{\frac{1}{\beta_j}}. \tag{2}$$

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THEOREM B. *Let* $A_{ij} > 0$ $(i = 1,2,\cdots,n,\ j = 1,2,\cdots,m)$.
(a) If $\beta_1 > 0,\ \beta_j < 0$ $(j = 2,3,\cdots,m)$, *and if* $\sum_{j=1}^{m} \frac{1}{\beta_j} \leqslant 1$, *then*

$$\sum_{i=1}^{n} \prod_{j=1}^{m} A_{ij} \geqslant \prod_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij}^{\beta_j} \right)^{\frac{1}{\beta_j}}. \tag{3}$$

(b) If $\beta_j < 0$ $(j = 1,2,\cdots,m)$,

$$\sum_{i=1}^{n} \prod_{j=1}^{m} A_{ij} \geqslant \prod_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij}^{\beta_j} \right)^{\frac{1}{\beta_j}}. \tag{4}$$

The main objective of this paper is to build some new refinements of inequalities (2), (3) and (4).

2. Refinements of generalized Hölder's inequality

We first introduce the following lemmas, which will be used in the sequel.

LEMMA 2.1. [10] *Let* $(\boldsymbol{a}) = (a_1, a_2, \cdots, a_n)$, $a_i > 0$, *and let* $t_r(\boldsymbol{a}) = \left(\sum_{i=1}^{n} a_i^r \right)^{\frac{1}{r}}$ $(r \neq 0)$. *Then for* $0 < r < s$, $r < s < 0$ *and* $s < 0 < r$

$$t_s(\boldsymbol{a}) < t_r(\boldsymbol{a}). \tag{5}$$

LEMMA 2.2. [3] *If* $x > -1$, $\alpha > 1$ *or* $\alpha < 0$, *then*

$$(1+x)^{\alpha} \geqslant 1 + \alpha x. \tag{6}$$

The sign of inequality is reversed for $0 < \alpha < 1$.

LEMMA 2.3. [4] *If* $x_i \geqslant 0$, $\lambda_i > 0$, $i = 1,2,\cdots,n$, $0 < p \leqslant 1$, *then*

$$\sum_{i=1}^{n} \lambda_i x_i^p \leqslant \left(\sum_{i=1}^{n} \lambda_i \right)^{1-p} \left(\sum_{i=1}^{n} \lambda_i x_i \right)^{p}. \tag{7}$$

The sign of inequality is reversed for $p \geqslant 1$ *or* $p < 0$.

Next, we present the refinement of the inequality (2).

THEOREM 2.4. *Let* $A_{ij} \geqslant 0$, $(i = 1,2,\cdots,n,\ j = 1,2,\cdots,m)$, *let* $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_m > 0$, $\sum_{j=1}^{m} \frac{1}{\beta_j} \geqslant 1$, *and let* $\alpha(m) = \begin{cases} \frac{m}{2} & \text{if } m \text{ even} \\ \frac{m-1}{2} & \text{if } m \text{ odd} \end{cases}$. *Then*

$$\begin{aligned} \sum_{i=1}^{n} \prod_{j=1}^{m} A_{ij} &\leqslant \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij}^{\beta_j} \right)^{\frac{1}{\beta_j}} \right] \left\{ \prod_{j=1}^{\alpha(m)} \left[1 - \left(\frac{\sum_{i=1}^{n} \frac{1}{i} A_{i(2j-1)}^{\beta_{2j-1}}}{\sum_{i=1}^{n} A_{i(2j-1)}^{\beta_{2j-1}}} - \frac{\sum_{i=1}^{n} \frac{1}{i} A_{i(2j)}^{\beta_{2j}}}{\sum_{i=1}^{n} A_{i(2j)}^{\beta_{2j}}} \right)^2 \right]^{\frac{1}{2\beta_{2j}}} \right\} \\ &\leqslant \left[\prod_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij}^{\beta_j} \right)^{\frac{1}{\beta_j}} \right]. \end{aligned} \tag{8}$$

Proof. Preforming some simple computations, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\sum_{s=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(1-\frac{1}{i}+\frac{1}{s}\right)\\
&=\sum_{i=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\left(\prod_{t=1}^{m}A_{st}\right)-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{1}{i}\left(\prod_{j=1}^{m}A_{ij}\right)\left(\prod_{t=1}^{m}A_{st}\right)\\
&\quad+\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{1}{s}\left(\prod_{j=1}^{m}A_{ij}\right)\left(\prod_{t=1}^{m}A_{st}\right)\\
&=\left(\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij}\right)^{2}. \qquad (9)
\end{aligned}$$

Case (I). Let m is even. By using Lemma 2.3, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\sum_{s=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(1-\frac{1}{i}+\frac{1}{s}\right)^{\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&=\sum_{i=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\left(\prod_{t=1}^{m}A_{st}\right)\left(1-\frac{1}{i}+\frac{1}{s}\right)^{\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&\geqslant\left[\sum_{i=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\left(\prod_{t=1}^{m}A_{st}\right)\right]^{1-\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&\quad\times\left[\sum_{i=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\left(\prod_{t=1}^{m}A_{st}\right)\left(1-\frac{1}{i}+\frac{1}{s}\right)\right]^{\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&=\left[\sum_{i=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\left(\prod_{t=1}^{m}A_{st}\right)\right]^{1-\sum_{j=1}^{m}\frac{1}{\beta_j}}\left[\sum_{i=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\left(\prod_{t=1}^{m}A_{st}\right)\right]^{\sum_{j=1}^{m}\frac{1}{\lambda_j}}\\
&=\sum_{i=1}^{n}\sum_{s=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\left(\prod_{t=1}^{m}A_{st}\right)\\
&=\left[\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\right]^{2}. \qquad (10)
\end{aligned}$$

Moreover, according to $\sum_{j=1}^{m}\frac{1}{\beta_j}\geqslant 1$, we deduce from inequality (2)

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\sum_{s=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(1-\frac{1}{i}+\frac{1}{s}\right)^{\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&=\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\sum_{s=1}^{n}\prod_{t=1}^{m}A_{st}\left(1-\frac{1}{i}+\frac{1}{s}\right)^{\frac{1}{\beta_t}}\\
&\leqslant\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\left[\prod_{t=1}^{m}\left(\sum_{s=1}^{n}A_{st}^{\beta_t}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_t}}\right]
\end{aligned}$$

$$= \sum_{i=1}^{n}\left\{\prod_{j=1}^{\frac{m}{2}}\left[\left(A_{i(2j-1)}^{\beta_{2j-1}}\sum_{s=1}^{n}A_{s(2j-1)}^{\beta_{2j-1}}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_{2j-1}}-\frac{1}{\beta_{2j}}}\right.\right.$$
$$\times\left(A_{i(2j-1)}^{\beta_{2j-1}}\sum_{s=1}^{n}A_{s(2j)}^{\beta_{2j}}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_{2j}}}$$
$$\left.\left.\times\left(A_{i(2j)}^{\beta_{2j}}\sum_{s=1}^{n}A_{s(2j-1)}^{\beta_{(2j-1)}}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_{2j}}}\right]\right\}. \tag{11}$$

Hence, according to $\left(\frac{1}{\beta_1}-\frac{1}{\beta_2}\right)+\frac{1}{\beta_2}+\frac{1}{\beta_2}+\left(\frac{1}{\beta_3}-\frac{1}{\beta_4}\right)+\frac{1}{\beta_4}+\frac{1}{\beta_4}+\cdots+\left(\frac{1}{\beta_{m-1}}-\frac{1}{\beta_m}\right)+\frac{1}{\beta_m}+\frac{1}{\beta_m}\geqslant 1$, by using the inequality (2) on the right side of (11), we have

$$\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\sum_{s=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(1-\frac{1}{i}+\frac{1}{s}\right)^{\sum_{j=1}^{m}\frac{1}{\beta_j}}$$
$$\leqslant\prod_{j=1}^{\frac{m}{2}}\left[\left(\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}\sum_{s=1}^{n}A_{s(2j-1)}^{\beta_{2j-1}}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_{2j-1}}-\frac{1}{\beta_{2j}}}\right.$$
$$\times\left(\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}\sum_{s=1}^{n}A_{s(2j)}^{\beta_{2j}}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_{2j}}}$$
$$\left.\times\left(\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}\sum_{s=1}^{n}A_{s(2j-1)}^{\beta_{2j-1}}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_{2j}}}\right]$$
$$=\prod_{j=1}^{\frac{m}{2}}\left\{\left(\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}\right)^{\frac{2}{\beta_{2j-1}}-\frac{2}{\beta_{2j}}}\left[\left(\sum_{i=1}^{n}\sum_{s=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}A_{s(2j)}^{\beta_{2j}}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)\right.\right.$$
$$\left.\left.\times\left(\sum_{i=1}^{n}\sum_{s=1}^{n}A_{i(2j)}^{\beta_{2j}}A_{s(2j-1)}^{\beta_{2j-1}}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)\right]^{\frac{1}{\beta_{2j}}}\right\}$$
$$==\prod_{j=1}^{\frac{m}{2}}\left\{\left(\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}\right)^{\frac{2}{\beta_{2j-1}}-\frac{2}{\beta_{2j}}}\times\left[\left(\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}\sum_{s=1}^{n}A_{s(2j)}^{\beta_{2j}}\right.\right.\right.$$
$$\left.-\sum_{i=1}^{n}\frac{1}{i}A_{i(2j-1)}^{\beta_{2j-1}}\sum_{s=1}^{n}A_{s(2j)}^{\beta_{2j}}+\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}\sum_{s=1}^{n}\frac{1}{s}A_{s(2j)}^{\beta_{2j}}\right)$$
$$\times\left(\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}\sum_{s=1}^{n}A_{s(2j-1)}^{\beta_{2j-1}}-\sum_{i=1}^{n}\frac{1}{i}A_{i(2j)}^{\beta_{2j}}\sum_{s=1}^{n}A_{s(2j-1)}^{\beta_{2j-1}}\right.$$
$$\left.\left.\left.+\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}\sum_{s=1}^{n}\frac{1}{s}A_{s(2j-1)}^{\beta_{2j-1}}\right)\right]^{\frac{1}{\beta_{2j}}}\right\}$$
$$=\prod_{j=1}^{\frac{m}{2}}\left\{\left(\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}\right)^{\frac{2}{\beta_{2j-1}}-\frac{2}{\beta_{2j}}}\times\left[\left(\left(\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}\right)\left(\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}\right)\right)^{2}\right.\right.$$

$$-\left(\left(\sum_{i=1}^{n}\frac{1}{i}A_{i(2j-1)}^{\beta_{2j-1}}\right)\left(\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}\right)-\left(\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}\right)\left(\sum_{i=1}^{n}\frac{1}{i}A_{i(2j)}^{\beta_{2j}}\right)\right)^{2}\Bigg]^{\frac{1}{\beta_{2j}}}\Bigg\}$$

$$=\prod_{j=1}^{\frac{m}{2}}\Bigg\{\left(\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}\right)^{\frac{2}{\beta_{2j-1}}}\left(\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}\right)^{\frac{2}{\beta_{2j}}}$$

$$\times\left[1-\left(\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j-1)}^{\beta_{2j-1}}}{\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}}-\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j)}^{\beta_{2j}}}{\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}}\right)^{2}\right]^{\frac{1}{\beta_{2j}}}\Bigg\}$$

$$=\left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{\frac{2}{\beta_j}}\right]\left\{\prod_{j=1}^{\frac{m}{2}}\left[1-\left(\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j-1)}^{\beta_{2j-1}}}{\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}}-\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j)}^{\beta_{2j}}}{\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}}\right)^{2}\right]^{\frac{1}{\beta_{2j}}}\right\} \tag{12}$$

Combining inequalities (10) and (12) we can get

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij}\leqslant\left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{\frac{1}{\beta_j}}\right]$$
$$\times\left\{\prod_{j=1}^{\frac{m}{2}}\left[1-\left(\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j-1)}^{\beta_{2j-1}}}{\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}}-\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j)}^{\beta_{2j}}}{\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}}\right)^{2}\right]^{\frac{1}{2\beta_{2j}}}\right\}. \tag{13}$$

Since

$$0<\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j-1)}^{\beta_{2j-1}}}{\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}},\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j)}^{\beta_{2j}}}{\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}}\leqslant 1, \tag{14}$$

we have

$$\left|\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j-1)}^{\beta_{2j-1}}}{\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}}-\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j)}^{\beta_{2j}}}{\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}}\right|<1. \tag{15}$$

Consequently, from (13) and (15), we obtain the desired inequality (9) for m is even.

Case (II). Let m is odd, and let $\sum_{j=1}^{m}\frac{1}{\beta_j}\geqslant 1$. By the same method as in the above case (I), we have

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij}\leqslant\left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{\frac{1}{\beta_j}}\right]$$
$$\times\left\{\prod_{j=1}^{\frac{m-1}{2}}\left[1-\left(\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j-1)}^{\beta_{2j-1}}}{\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}}-\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j)}^{\beta_{2j}}}{\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}}\right)^{2}\right]^{\frac{1}{2\beta_{2j}}}\right\}. \tag{16}$$

The proof of Theorem 2.4 is completed. □

If we set $0<\frac{1}{2\beta_{2j}}<1$, then from Theorem 2.4 and Lemma 2.2 we obtain the following refinement of the generalized Hölder's inequality (2).

COROLLARY 2.5. *Let* $A_{ij} \geqslant 0$ $(i=1,2,\cdots,n,\ j=1,2,\cdots,m)$, *let* $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_m > 0$, $\sum_{j=1}^{m}\frac{1}{\beta_j} \geqslant 1$, *and let* $\alpha(m)=\begin{cases}\frac{m}{2} & \text{if } m \text{ even}\\ \frac{m-1}{2} & \text{if } m \text{ odd}\end{cases}$. *Then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} \leqslant \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{\frac{1}{\beta_j}}\right] \times\left\{\prod_{j=1}^{\alpha(m)}\left[1-\frac{1}{2\beta_{2j}}\left(\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j-1)}^{\beta_{2j-1}}}{\sum_{i=1}^{n}A_{i(2j-1)}^{\beta_{2j-1}}}-\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i(2j)}^{\beta_{2j}}}{\sum_{i=1}^{n}A_{i(2j)}^{\beta_{2j}}}\right)^2\right]\right\}. \tag{17}$$

Finally, we will give the refinements of inequalities (3) and (4).

THEOREM 2.6. *Let* $A_{ij} > 0(i=1,2,\cdots,n,\ j=1,2,\cdots,m)$.
(a) If $\beta_1 > 0,\ \beta_j < 0(j=2,3,\cdots,m)$, *and if* $0 < \sum_{j=1}^{m}\frac{1}{\beta_j} \leqslant 1$, *then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} \geqslant \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{\frac{1}{\beta_j}}\right] \times\left\{\prod_{j=2}^{m}\left[1-\frac{1}{2\beta_j}\left(\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i1}^{\beta_1}}{\sum_{i=1}^{n}A_{i1}^{\beta_1}}-\frac{\sum_{i=1}^{n}\frac{1}{i}A_{ij}^{\beta_j}}{\sum_{i=1}^{n}A_{ij}^{\beta_j}}\right)^2\right]\right\}. \tag{18}$$

(b) If $\beta_1 > 0,\ \beta_j < 0(j=2,3,\cdots,m)$, *and if* $\sum_{j=1}^{m}\frac{1}{\beta_j} < 0$, *then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} \geqslant \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{\frac{1}{\beta_j}}\right] \times\left\{\prod_{j=2}^{m}\left[1-\frac{1}{2\beta_j}\left(\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i1}^{\beta_1^*}}{\sum_{i=1}^{n}A_{i1}^{\beta_1^*}}-\frac{\sum_{i=1}^{n}\frac{1}{i}A_{ij}^{\beta_j}}{\sum_{i=1}^{n}A_{ij}^{\beta_j}}\right)^2\right]\right\}. \tag{19}$$

(c) If $\beta_j < 0(j=1,2,\cdots,m)$, *then*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij} \geqslant \left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{\frac{1}{\beta_j}}\right] \times\left\{\prod_{j=2}^{m}\left[1-\frac{1}{2\beta_j}\left(\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i1}^{\beta_1^*}}{\sum_{i=1}^{n}A_{i1}^{\beta_1^*}}-\frac{\sum_{i=1}^{n}\frac{1}{i}A_{ij}^{\beta_j}}{\sum_{i=1}^{n}A_{ij}^{\beta_j}}\right)^2\right]\right\}, \tag{20}$$

where $\frac{1}{\beta_1^*}=1-\sum_{j=2}^{m}\frac{1}{\beta_j}$.

Proof. (a). On the one hand, by the inequality (7), we have

$$\sum_{s=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\left(1-\frac{1}{i}+\frac{1}{s}\right)^{\sum_{j=1}^{m}\frac{1}{\beta_j}}$$
$$=\sum_{s=1}^{n}\sum_{i=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(\prod_{j=1}^{m}A_{ij}\right)\left(1-\frac{1}{i}+\frac{1}{s}\right)^{\sum_{j=1}^{m}\frac{1}{\beta_j}}$$

$$
\begin{aligned}
&\leqslant \left[\sum_{s=1}^{n}\sum_{i=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(\prod_{j=1}^{m}A_{ij}\right)\right]^{1-\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&\quad\times\left[\sum_{s=1}^{n}\sum_{i=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(\prod_{j=1}^{m}A_{ij}\right)\left(1-\frac{1}{i}+\frac{1}{s}\right)\right]^{\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&=\left[\sum_{s=1}^{n}\sum_{i=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(\prod_{j=1}^{m}A_{ij}\right)\right]^{1-\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&\quad\times\left[\sum_{s=1}^{n}\sum_{i=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(\prod_{j=1}^{m}A_{ij}\right)-\sum_{s=1}^{n}\sum_{i=1}^{n}\frac{1}{i}\left(\prod_{t=1}^{m}A_{st}\right)\left(\prod_{j=1}^{m}A_{ij}\right)\right.\\
&\qquad\left.+\sum_{s=1}^{n}\sum_{i=1}^{n}\frac{1}{s}\left(\prod_{t=1}^{m}A_{st}\right)\left(\prod_{j=1}^{m}A_{ij}\right)\right]^{\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&=\left[\sum_{s=1}^{n}\sum_{i=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(\prod_{j=1}^{m}A_{ij}\right)\right]^{1-\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&\quad\times\left[\sum_{s=1}^{n}\sum_{i=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(\prod_{j=1}^{m}A_{ij}\right)\right]^{\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&=\sum_{s=1}^{n}\sum_{i=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left(\prod_{j=1}^{m}A_{ij}\right)=\left[\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\right]^{2}. \qquad (21)
\end{aligned}
$$

On the other hand, by the inequality (3), we obtain

$$
\begin{aligned}
&\sum_{s=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\left(1-\frac{1}{i}+\frac{1}{s}\right)^{\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&=\sum_{s=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij}\left(1-\frac{1}{i}+\frac{1}{s}\right)^{\frac{1}{\beta_j}}\\
&\geqslant\sum_{s=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_j}}\right]\\
&=\sum_{s=1}^{n}\left\{\left(A_{s1}^{\beta_1}\sum_{i=1}^{n}A_{i1}^{\beta_1}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_1}-\sum_{j=2}^{m}\frac{1}{\beta_j}}\right.\\
&\qquad\times\left[\prod_{j=2}^{m}\left(A_{s1}^{\beta_1}\sum_{i=1}^{n}A_{ij}^{\beta_j}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_j}}\right]\\
&\qquad\left.\times\left[\prod_{j=2}^{m}\left(A_{sj}^{\beta_j}\sum_{i=1}^{n}A_{i1}^{\beta_1}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_j}}\right]\right\}. \qquad (22)
\end{aligned}
$$

Consequently, according to $\left(\frac{1}{\beta_1}-\sum_{j=2}^{m}\frac{1}{\beta_j}\right)+\frac{1}{\beta_2}+\frac{1}{\beta_3}+\cdots+\frac{1}{\beta_m}+\frac{1}{\beta_2}+\frac{1}{\beta_3}+\cdots+\frac{1}{\beta_m}\leqslant 1$, by using the inequality (3) on the right side of (22), we observe that

$$
\begin{aligned}
&\sum_{s=1}^{n}\left(\prod_{t=1}^{m}A_{st}\right)\sum_{i=1}^{n}\left(\prod_{j=1}^{m}A_{ij}\right)\left(1-\frac{1}{i}+\frac{1}{s}\right)^{\sum_{j=1}^{m}\frac{1}{\beta_j}}\\
&\geqslant\left(\sum_{s=1}^{n}\sum_{i=1}^{n}A_{s1}^{\beta_1}A_{i1}^{\beta_1}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_1}-\sum_{j=2}^{m}\frac{1}{\beta_j}}\\
&\quad\times\left[\prod_{j=2}^{m}\left(\sum_{s=1}^{n}\sum_{i=1}^{n}A_{s1}^{\beta_1}A_{ij}^{\beta_j}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_j}}\right]\\
&\quad\times\left[\prod_{j=2}^{m}\left(\sum_{s=1}^{n}\sum_{i=1}^{n}A_{sj}^{\beta_j}A_{i1}^{\beta_1}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)^{\frac{1}{\beta_j}}\right]\\
&=\left(\sum_{i=1}^{n}A_{i1}^{\beta_1}\right)^{\frac{2}{\beta_1}-\sum_{j=2}^{m}\frac{2}{\beta_j}}\times\left\{\prod_{j=2}^{m}\left[\left(\sum_{s=1}^{n}\sum_{i=1}^{n}A_{s1}^{\beta_1}A_{ij}^{\beta_j}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)\right.\right.\\
&\quad\left.\left.\times\left(\sum_{s=1}^{n}\sum_{i=1}^{n}A_{sj}^{\beta_j}A_{i1}^{\beta_1}\left(1-\frac{1}{i}+\frac{1}{s}\right)\right)\right]^{\frac{1}{\beta_j}}\right\}\\
&=\left(\sum_{i=1}^{n}A_{i1}^{\beta_1}\right)^{\frac{2}{\beta_1}-\sum_{j=2}^{m}\frac{2}{\beta_j}}\times\left\{\prod_{j=2}^{m}\left[\left(\sum_{s=1}^{n}A_{s1}^{\beta_1}\sum_{i=1}^{n}A_{ij}^{\beta_j}\right.\right.\right.\\
&\quad\left.-\sum_{s=1}^{n}A_{s1}^{\beta_1}\sum_{i=1}^{n}\frac{1}{i}A_{ij}^{\beta_j}+\sum_{s=1}^{n}\frac{1}{s}A_{s1}^{\beta_1}\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)\\
&\quad\left.\left.\times\left(\sum_{s=1}^{n}A_{sj}^{\beta_j}\sum_{i=1}^{n}A_{i1}^{\beta_1}-\sum_{s=1}^{n}A_{sj}^{\beta_j}\sum_{i=1}^{n}\frac{1}{i}A_{i1}^{\beta_1}+\sum_{s=1}^{n}\frac{1}{s}A_{sj}^{\beta_j}\sum_{i=1}^{n}A_{i1}^{\beta_1}\right)\right]^{\frac{1}{\beta_j}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&=\left(\sum_{i=1}^{n}A_{i1}^{\beta_1}\right)^{\frac{2}{\beta_1}-\sum_{j=2}^{m}\frac{2}{\beta_j}}\times\left\{\prod_{j=2}^{m}\left[\left(\left(\sum_{i=1}^{n}A_{i1}^{\beta_1}\right)\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)\right)^2\right.\right.\\
&\qquad\left.\left.-\left(\left(\sum_{i=1}^{n}A_{i1}^{\beta_1}\right)\left(\sum_{i=1}^{n}\frac{1}{i}A_{ij}^{\beta_j}\right)-\left(\sum_{i=1}^{n}\frac{1}{i}A_{i1}^{\beta_1}\right)\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)\right)^2\right]^{\frac{1}{\beta_j}}\right\}\\
&=\left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{\frac{2}{\beta_j}}\right]\left\{\prod_{j=2}^{m}\left[1-\left(\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i1}^{\beta_1}}{\sum_{i=1}^{n}A_{i1}^{\beta_1}}-\frac{\sum_{i=1}^{n}\frac{1}{i}A_{ij}^{\beta_j}}{\sum_{i=1}^{n}A_{ij}^{\beta_j}}\right)^2\right]^{\frac{1}{\beta_j}}\right\}. \qquad (23)
\end{aligned}
$$

Combining inequalities (21) and (23) we get

$$
\begin{aligned}
\sum_{i=1}^{n}\prod_{j=1}^{m}A_{ij}\geqslant&\left[\prod_{j=1}^{m}\left(\sum_{i=1}^{n}A_{ij}^{\beta_j}\right)^{\frac{1}{\beta_j}}\right]\\
&\times\left\{\prod_{j=2}^{m}\left[1-\left(\frac{\sum_{i=1}^{n}\frac{1}{i}A_{i1}^{\beta_1}}{\sum_{i=1}^{n}A_{i1}^{\beta_1}}-\frac{\sum_{i=1}^{n}\frac{1}{i}A_{ij}^{\beta_j}}{\sum_{i=1}^{n}A_{ij}^{\beta_j}}\right)^2\right]^{\frac{1}{2\beta_j}}\right\}. \qquad (24)
\end{aligned}
$$

Furthermore, noting that

$$0 < \frac{\sum_{i=1}^{n} \frac{1}{i} A_{i1}^{\beta_1}}{\sum_{i=1}^{n} A_{i1}^{\beta_1}}, \frac{\sum_{i=1}^{n} \frac{1}{i} A_{ij}^{\beta_j}}{\sum_{i=1}^{n} A_{ij}^{\beta_j}} \leqslant 1, \tag{25}$$

we have

$$\left| \frac{\sum_{i=1}^{n} \frac{1}{i} A_{i1}^{\beta_1}}{\sum_{i=1}^{n} A_{i1}^{\beta_1}} - \frac{\sum_{i=1}^{n} \frac{1}{i} A_{ij}^{\beta_j}}{\sum_{i=1}^{n} A_{ij}^{\beta_j}} \right| < 1. \tag{26}$$

Consequently, from Lemma 2.2 and the inequalities (24) and (26), we have the desired inequality (18).

(b). Since

$$\frac{1}{\beta_1^*} + \sum_{j=2}^{m} \frac{1}{\beta_j} = 1,$$

where $\beta_1^* > 0, \beta_j < 0 (j = 2, 3, \cdots, m)$, from Case (a), we have

$$\sum_{i=1}^{n} \prod_{j=1}^{m} A_{ij} \geqslant \left(\sum_{i=1}^{n} A_{i1}^{\beta_1^*} \right)^{\frac{1}{\beta_1^*}} \left[\prod_{j=2}^{m} \left(\sum_{i=1}^{n} A_{ij}^{\beta_j} \right)^{\frac{1}{\beta_j}} \right]$$
$$\times \left\{ \prod_{j=2}^{m} \left[1 - \frac{1}{2\beta_j} \left(\frac{\sum_{i=1}^{n} \frac{1}{i} A_{i1}^{\beta_1^*}}{\sum_{i=1}^{n} A_{i1}^{\beta_1^*}} - \frac{\sum_{i=1}^{n} \frac{1}{i} A_{ij}^{\beta_j}}{\sum_{i=1}^{n} A_{ij}^{\beta_j}} \right)^2 \right] \right\}. \tag{27}$$

By using Lemma 2.1 with $\beta_1 > \beta_1^* > 0$, we get that inequality (19) is valid.

(c). The proof of Case (c) is similar to the one of Case (b). So the details are omitted. □

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