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INEQUALITY CHAINS FOR WILKER, HUYGENS AND LAZAREVIĆ TYPE INEQUALITIES

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Abstract. We offer various refinements of inequalities related to the Wilker, Huygens, or Lazarević type inequalities for trigonometric and hyperbolic functions.

1. Introduction

It is known in the literature that

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2+\cos x}{3} \tag{1.1}$$

for $0 < |x| < \frac{\pi}{2}$, and

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{2+\cosh x}{3} \tag{1.2}$$

for $x \neq 0$. The left-hand side inequality (1.1) first appeared in [13, p. 238], while the right-hand side inequality (1.1) is due to Cusa and Huygens (see [20] for more details regarding this result). The first inequality in (1.2) was established by Lazarević [12] (see, e.g., [13, p. 238]), while the second inequality in (1.2) appeared in [17].

The first inequality in (1.1) can be re-written as

$$\left(\frac{\sin x}{x}\right)^2 \frac{\tan x}{x} > 1 \quad \left(\text{or} \quad \sqrt[3]{\left(\frac{\sin x}{x}\right)^2 \frac{\tan x}{x}} > 1\right) \quad \text{for all} \quad 0 < |x| < \frac{\pi}{2}. \tag{1.3}$$

Baricz and Sándor [4] have pointed out that inequality (1.3) implies two other inequalities

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \qquad 0 < |x| < \frac{\pi}{2} \tag{1.4}$$

and

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3, \qquad 0 < |x| < \frac{\pi}{2}, \tag{1.5}$$

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by using the arithmetic-geometric mean inequality. Inequality (1.4) was presented without proof by Wilker [22]. Wilker type inequality (1.4) has attracted much interest of many mathematicians and have motivated a large number of research papers involving different proofs and various generalizations and improvements (cf. [9, 10, 14, 15, 16, 17, 18, 21, 23, 24, 26, 27, 28, 29, 32, 33, 34] and the references cited therein). Inequality (1.5) is due to Huygens [11].

Wu and Srivastava [26, Lemma 3] established another inequality

$$\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}>2 \qquad \text{for all} \quad 0<|x|<\frac{\pi}{2}. \tag{1.6}$$

Neuman and Sándor [17, Theorem 2.3] proved that for $0<|x|<\pi/2$,

$$\frac{\sin x}{x}<\frac{2+\cos x}{3}<\frac{1}{2}\left(\frac{x}{\sin x}+\cos x\right). \tag{1.7}$$

By multiplying both sides of inequality (1.7) with $x/\sin x$, we obtain that for $0<|x|<\pi/2$,

$$\frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}\right]>\frac{2(x/\sin x)+x/\tan x}{3}>1. \tag{1.8}$$

The second inequality in (1.8) is equivalent to the second inequality in (1.1).

The first aim of this paper is to prove the following inequality chain.

THEOREM 1.1. *For* $0<|x|<\pi/2$,

$$\begin{aligned}&\frac{(\sin x/x)^2+\tan x/x}{2}>\left(\frac{\sin x}{x}\right)^2\left(\frac{\tan x}{x}\right)>\frac{2\,(\sin x/x)+\tan x/x}{3}\\ &>\left(\frac{\sin x}{x}\right)^{2/3}\left(\frac{\tan x}{x}\right)^{1/3}>\frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}\right]>\frac{2(x/\sin x)+x/\tan x}{3}>1.\end{aligned} \tag{1.9}$$

The first inequality in (1.2) can be re-written as

$$\left(\frac{\sinh x}{x}\right)^2\frac{\tanh x}{x}>1 \quad \left(\text{or} \quad \sqrt[3]{\left(\frac{\sinh x}{x}\right)^2\frac{\tanh x}{x}}>1\right) \quad \text{for} \quad x\neq 0. \tag{1.10}$$

Zhu [30] established hyperbolic versions of inequality (1.4):

$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}>2\,, \qquad x\neq 0. \tag{1.11}$$

Baricz and Sándor [4] have pointed out that (1.10) implies (1.11) and the following inequality

$$2\left(\frac{\sinh x}{x}\right)+\frac{\tanh x}{x}>3\,, \qquad x\neq 0, \tag{1.12}$$

by the arithmetic-geometric mean inequality. Neuman and Sándor [17, Theorem 2.4] proved a hyperbolic version of inequality (1.6):

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} > 2, \qquad x \neq 0. \tag{1.13}$$

In [17, p. 719] the authors proved that

$$\frac{\sinh x}{x} < \frac{2+\cosh x}{3} < \frac{1}{2}\left(\frac{x}{\sinh x} + \cosh x\right), \qquad x \neq 0. \tag{1.14}$$

By multiplying both sides of inequality (1.14) with $x/\sinh x$, we obtain that for $x \neq 0$,

$$\frac{1}{2}\left[\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x}\right] > \frac{2(x/\sinh x) + x/\tanh x}{3} > 1. \tag{1.15}$$

The second inequality in (1.15) is equivalent to the second inequality in (1.2).

In [17] not only hyperbolic versions of (1.4) and (1.5) are studied, but many other facts, e.g. Cusa-Huygens, Huygens, Wilker type inequalities and their connections to each others in the trigonometric and also the hyperbolic case. Wilker-type inequalities for hyperbolic functions were studied by Wu and Debnath [25]. Zhu [31] established some new inequalities of the Huygens type for trigonometric and hyperbolic functions. In [5, 8] inverse trigonometric and inverse hyperbolic versions of inequalities (1.4) and (1.5) were established. Very recently, Chen [6, 7] established Wilker and Huygens type inequalities for the lemniscate functions.

The second aim of this paper is to prove Theorem 1.2 below, which shows that the following inequality chain holds:

$$\begin{aligned}
&\frac{(\sinh x/x)^2 + \tanh x/x}{2} > \left(\frac{\sinh x}{x}\right)^2 \left(\frac{\tanh x}{x}\right) \\
&> \left(\frac{\sinh x}{x}\right)^{3/2} \left(\frac{\tanh x}{x}\right)^{3/4} > \frac{2\,(\sinh x/x) + \tanh x/x}{3} > \left(\frac{\sinh x}{x}\right) \left(\frac{\tanh x}{x}\right)^{1/2} \\
&> \frac{1}{2}\left[\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x}\right] > \frac{2(x/\sinh x) + x/\tanh x}{3} > 1
\end{aligned} \tag{1.16}$$

for $x \neq 0$.

THEOREM 1.2. (i) *For* $x \neq 0$,

$$\frac{(\sinh x/x)^2 + \tanh x/x}{2} > \left(\frac{\sinh x}{x}\right)^2 \frac{\tanh x}{x}. \tag{1.17}$$

(ii) *For* $x \neq 0$, $\alpha \geqslant 3/4$ *and* $\beta \leqslant 1/2$,

$$\left(\frac{\sinh x}{x}\right)^{2\alpha} \left(\frac{\tanh x}{x}\right)^{\alpha} > \frac{2\,(\sinh x/x) + \tanh x/x}{3} > \left(\frac{\sinh x}{x}\right)^{2\beta} \left(\frac{\tanh x}{x}\right)^{\beta} \tag{1.18}$$

with the best possible constants

$$\alpha = \frac{3}{4} \quad \text{and} \quad \beta = \frac{1}{2}.$$

(iii) *For* $x \neq 0$,

$$\left(\frac{\sinh x}{x}\right)\left(\frac{\tanh x}{x}\right)^{1/2} > \frac{1}{2}\left[\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x}\right] > \frac{2\,(x/\sinh x) + x/\tanh x}{3} > 1. \tag{1.19}$$

2. Lemmas

The following lemmas are needed in the sequel.

LEMMA 2.1. ([19]) *Let* $a_n \in \mathbb{R}$ *and* $b_n > 0$, $n = 0,1,2,\ldots$ *be real numbers with* $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ *being strictly increasing (respectively, decreasing). If the power series* $A(x) := \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) := \sum_{n=0}^{\infty} b_n x^n$ *are convergent for* $|x| < R$*, then the function* $A(x)/B(x)$ *is strictly increasing (respectively, decreasing) on* $(0,R)$.

LEMMA 2.2. ([1, 2, 3]) *Let* $-\infty < a < b < \infty$*, and* $f,g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable in* (a,b)*. Suppose* $g' \neq 0$ *on* (a,b)*. If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b)*, then so are*

$$[f(x) - f(a)]/[g(x) - g(a)] \quad \text{and} \quad [f(x) - f(b)]/[g(x) - g(b)]\,.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Elementary calculations show that

$$\frac{(\sin x/x)^2 + \tan x/x}{2} - \left(\frac{\sin x}{x}\right)^2\left(\frac{\tan x}{x}\right) = \frac{\tan x}{x^3} U(x),$$

where

$$U(x) = x^2 + \frac{1}{2} x \sin(2x) - 1 + \cos(2x).$$

Differentiation yields

$$U'(x) = -\frac{3}{2}\sin(2x) + 2x + x\cos(2x),$$
$$U''(x) = 2\sin(2x)(\tan x - x) > 0 \quad \text{for} \quad 0 < x < \frac{\pi}{2}.$$

Hence, we have for $0 < x < \pi/2$,

$$U'(x) > U'(0) = 0 \Longrightarrow U(x) > U(0) = 0.$$

Therefore, the first inequality in (1.9) holds.

Elementary calculations show that

$$\left(\frac{\sin x}{x}\right)^2 \left(\frac{\tan x}{x}\right) - \frac{2(\sin x/x) + \tan x/x}{3} = \frac{\tan x}{x} Q(x),$$

where

$$Q(x) = \left(\frac{\sin x}{x}\right)^2 - \frac{1+2\cos x}{3}. \tag{3.20}$$

Differentiation yields

$$\frac{3x^3}{2\sin x} Q'(x) = W(x),$$

where

$$W(x) = x^3 - 3\sin x + 3x\cos x.$$

Since

$$W'(x) = 3x(x - \sin x) > 0 \quad \text{for} \quad 0 < x < \pi/2,$$

we have for $0 < x < \pi/2$,

$$W(x) > W(0) = 0 \Longrightarrow Q'(x) > 0 \Longrightarrow Q(x) > Q(0) = 0.$$

Therefore, the second inequality in (1.9) holds.

Using the arithmetic-geometric mean inequality, yields the third inequality in (1.9).

Consider the function $R(x)$ defined by

$$R(x) = \frac{\sqrt[3]{\left(\frac{\sin x}{x}\right)^2 \frac{\tan x}{x}}}{\frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}\right]} = \frac{2\sin^3 x}{x^2(x + \sin x\cos x)(\cos x)^{1/3}}, \quad 0 < x < \frac{\pi}{2}.$$

Differentiation yields

$$R'(x) = \frac{\sin^2 x S(x)}{3x^3\left(x^2 + x\sin(2x) + \frac{1}{4}\sin^2(2x)\right)(\cos x)^{4/3}}$$

where

$$\begin{aligned} S(x) &= \left(8x^2 + x\sin(2x)\right)\left(2\cos^2 x\right) - 5x\sin(2x) - 3\sin^2(2x) + 2x^2 \\ &= \left(8x^2 + x\sin(2x)\right)\left(1 + \cos(2x)\right) - 5x\sin(2x) - \frac{3\left(1 - \cos(4x)\right)}{2} + 2x^2 \\ &= -4x\sin(2x) + \frac{1}{2}x\sin(4x) + 8x^2\cos(2x) + \frac{3}{2}\cos(4x) - \frac{3}{2} + 10x^2. \end{aligned}$$

By using the power series expansions of $\sin x$ and $\cos x$, we have

$$2S(x) = \frac{128}{315}x^8 - \frac{256}{4725}x^{10} - \frac{128}{31185}x^{12} + \sum_{n=7}^{\infty}(-1)^{n-1}v_n(x), \tag{3.21}$$

where

$$v_n(x) = \frac{32n(n-1)\cdot 4^n + (n-6)\cdot 4^{2n}}{2\cdot(2n)!}x^{2n}.$$

Elementary calculations show that for $0 < x < \pi/2$ and $n \geqslant 7$,

$$\begin{aligned}\frac{v_{n+1}(x)}{v_n(x)} &= \frac{x^2\left(128n(n+1)\cdot 2^{2n} + (16n-80)\cdot 2^{4n}\right)}{2(2n+1)(n+1)\left(32n(n-1)\cdot 2^{2n} + (n-6)\cdot 2^{4n}\right)} \\ &< \frac{\left(\frac{\pi}{2}\right)^2\left(128n(n+1)\cdot 2^{2n} + (16n-80)\cdot 2^{4n}\right)}{2(2n+1)(n+1)\left(32n(n-1)\cdot 2^{2n} + (n-6)\cdot 2^{4n}\right)} \\ &< \frac{128n(n+1)\cdot 2^{2n} + (16n-80)\cdot 2^{4n}}{8(n+1)\left(32n(n-1)\cdot 2^{2n} + (n-6)\cdot 2^{4n}\right)}\end{aligned}$$

and

$$\begin{aligned}&8(n+1)\left(32n(n-1)\cdot 2^{2n} + (n-6)\cdot 2^{4n}\right) - \left(128n(n+1)\cdot 2^{2n} + (16n-80)\cdot 2^{4n}\right) \\ &= 128n(n+1)(2n-3)\cdot 2^{2n} + \left(8n(n-7)+32\right)\cdot 2^{4n} > 0.\end{aligned}$$

Hence, for every $x \in (0, \pi/2)$, the sequence $n \longmapsto v_n(x)$ is strictly decreasing for $n \geqslant 7$. Therefore, we obtain from (3.21) that

$$2S(x) > x^8\left(\frac{128}{315} - \frac{256}{4725}x^2 - \frac{128}{31185}x^4\right) > 0, \quad 0 < x < \frac{\pi}{2},$$

which implies $R'(x) > 0$ for $0 < x < \pi/2$. Hence, the function $R(x)$ is strictly increasing for $(0, \pi/2)$, and we have

$$R(x) > \lim_{x\to 0^+} R(x) = 1, \quad 0 < x < \frac{\pi}{2}.$$

Therefore, the fourth inequality in (1.9) holds.

By using the second inequality in (1.1), we find that $x \neq 0$,

$$\begin{aligned}
&\frac{(x/\sin x)^2 + x/\tan x}{2} - \frac{2(x/\sin x) + x/\tan x}{3} \\
&= \frac{x^2}{6\sin^2 x}\left[3 + \left(\frac{\sin x}{x}\right)\cos x - 4\left(\frac{\sin x}{x}\right)\right] \\
&> \frac{x^2}{6\sin^2 x}\left\{3 + \left(\frac{\sin x}{x}\right)\left[3\left(\frac{\sin x}{x}\right) - 2\right] - 4\left(\frac{\sin x}{x}\right)\right\} \\
&= \frac{x^2}{2\sin^2 x}\left(1 - \frac{\sin x}{x}\right)^2 > 0.
\end{aligned}$$

Hence, the fifth inequality in (1.9) holds.

The last inequality in (1.9) is equivalent to the second inequality in (1.1). This completes the proof of Theorem 1.1. □

REMARK 3.1. Neuman and Sándor [17, Theorem 2.3] proved the first inequality in (1.8), the proof is based on differential calculus. Here we present a new, algebraic proof of the first inequality in (1.8), without using derivatives. We discover the fact that the second inequality in (1.1) (i.e. the Cusa-Huygens inequality) implies the first inequality in (1.8).

REMARK 3.2. The first inequality in (1.1) can be separated. Indeed, we have

$$(\cos x)^{1/3} < \left(\frac{1+2\cos x}{3}\right)^{1/2} < \frac{\sin x}{x}, \quad 0 < |x| < \frac{\pi}{2}. \tag{3.22}$$

In fact, the first inequality in (3.22) may be written, after some elementary transformations, as $(1-\cos x)^2(1+8\cos x) > 0$ for $0 < |x| < \pi/2$, while the second inequality in (3.22) follows from the inequality $Q(x) > 0$ of (3.20).

The first inequality in (1.2) can be separated. Indeed, we have

$$(\cosh x)^{1/3} < \left(\frac{1+2\cosh x}{3}\right)^{1/2} < \frac{\sinh x}{x}, \quad x \neq 0. \tag{3.23}$$

In fact, the first inequality in (3.23) may be written, after some elementary transformations, as $(1-\cosh x)^2(1+8\cosh x) > 0$ for $x \neq 0$, while the second inequality in (3.23) follows from the following result:

$$\begin{aligned}
\left(\frac{\sinh x}{x}\right)^2 - \frac{1+2\cosh x}{3} &= \frac{-x^2 - 2x^2\cosh x - 3 + 3\cosh^2 x}{3x^2} \\
&= \sum_{n=3}^{\infty} \frac{3\cdot 2^{2n-1} - 4n(2n-1)}{3\cdot(2n)!} x^{2n-2} > 0, \qquad x \neq 0.
\end{aligned}$$

Proof of Theorem 1.2. Elementary calculations show that

$$\frac{(\sinh x/x)^2+\tanh x/x}{2}-\left(\frac{\sinh x}{x}\right)^2\left(\frac{\tanh x}{x}\right)=\frac{\tanh x}{2x^3}V(x),$$

where

$$V(x)=x^2+\frac{1}{2}x\sinh(2x)+1-\cosh(2x).$$

Differentiation yields

$$V'(x)=-\frac{3}{2}\sinh(2x)+2x+x\cosh(2x),$$
$$V''(x)=2\sinh(2x)(x-\tan x)>0 \quad \text{for} \quad x>0.$$

Hence, we have for $x>0$,

$$V'(x)>V'(0)=0\Longrightarrow V(x)>V(0)=0.$$

Theorem, inequality (1.17) holds.

In order to prove (1.18), we consider the function $f(x)$ defined for $x>0$ by

$$f(x)=\frac{\ln\left(\frac{2(\sinh x/x)+\tanh x/x}{3}\right)}{\ln\left(\left(\frac{\sinh x}{x}\right)^2\frac{\tanh x}{x}\right)}=\frac{f_1(x)}{f_2(x)},$$

where

$$f_1(x)=\ln\left(\frac{2(\sinh x/x)+\tanh x/x}{3}\right) \quad \text{and} \quad f_2(x)=\ln\left(\left(\frac{\sinh x}{x}\right)^2\frac{\tanh x}{x}\right).$$

Elementary calculations show that

$$\frac{f_1'(x)}{f_2'(x)}=\frac{-2\sinh x\cosh^2 x-\sinh x\cosh x+2x\cosh^3 x+x}{(2\cosh x+1)(-3\sinh x\cosh x+2x\cosh^2 x+x)}=\frac{f_3(x)}{f_4(x)},$$

where

$$\begin{aligned}f_3(x)&=-2\sinh x\cosh^2 x-\sinh x\cosh x+2x\cosh^3 x+x\\&=-\frac{1}{2}\sinh(3x)-\frac{1}{2}\sinh x-\frac{1}{2}\sinh(2x)+\frac{1}{2}x\cosh(3x)+\frac{3}{2}x\cosh x+x\\&=\sum_{n=2}^{\infty}\frac{(n-1)\cdot 9^n-4^n+3n+1}{(2n+1)!}x^{2n+1}=\sum_{n=2}^{\infty}a_nx^{2n+1}\end{aligned}$$

and

$$\begin{aligned}f_4(x)&=(2\cosh x+1)(-3\sinh x\cosh x+2x\cosh^2 x+x)\\&=-\frac{3}{2}\sinh(3x)-\frac{3}{2}\sinh x+x\cosh(3x)+5x\cosh x-\frac{3}{2}\sinh(2x)+x\cosh(2x)+2x\\&=\sum_{n=2}^{\infty}\frac{(2n-\frac{7}{2})\cdot 9^n+2(n-1)\cdot 4^n+10n+\frac{7}{2}}{(2n+1)!}x^{2n+1}=\sum_{n=2}^{\infty}b_nx^{2n+1}.\end{aligned}$$

We claim that the function $\frac{f_1'(x)}{f_2'(x)}$ is strictly decreasing on $(0,\infty)$. By Lemma 2.1, it suffices to show that

$$\frac{a_n}{b_n} > \frac{a_{n+1}}{b_{n+1}}, \qquad n \geqslant 2. \tag{3.24}$$

Direct calculations show that

$$\begin{aligned}\frac{a_n}{b_n} - \frac{a_{n+1}}{b_{n+1}} &= \frac{(n-1)\cdot 9^n - 4^n + 3n + 1}{(2n-\frac{7}{2})\cdot 9^n + 2(n-1)\cdot 4^n + 10n + \frac{7}{2}} \\ &\quad - \frac{n\cdot 9^{n+1} - 4^{n+1} + 3(n+1) + 1}{\left(2(n+1)-\frac{7}{2}\right)\cdot 9^{n+1} + 2n\cdot 4^{n+1} + 10(n+1) + \frac{7}{2}} \\ &= \frac{2c_n}{\left((36n-27)9^n + 16n\cdot 4^n + 20n + 27\right)\left((4n-7)9^n + (n-1)\cdot 2^{2n+2} + 20n + 7\right)},\end{aligned}$$

where

$$\begin{aligned}c_n &= 27\cdot 81^n - \left((20n^2+1)\cdot 36^n + (96n + 64n^2 + 26)9^n + 16\cdot 16^n\right) \\ &\quad + (36n^2 + 72n + 17)\cdot 4^n - 1.\end{aligned}$$

We now show that

$$c_n > 0 \quad \text{for} \quad n \geqslant 2. \tag{3.25}$$

Direct computation would yield

$$c_2 = 34560, \quad c_3 = 5225472, \quad c_4 = 612873216,$$
$$c_5 = 63709797888, \quad c_6 = 6054331765248.$$

In order to show that $c_n > 0$ for $n \geqslant 7$, it suffices to show that for $n \geqslant 7$,

$$27\cdot 81^n > \left((20n^2+1) + (96n + 64n^2 + 26) + 16\right)36^n = \left(84n^2 + 96n + 43\right)36^n,$$

i.e.,

$$\left(\frac{4}{3}\right)^{2n} > \frac{28}{9}n^2 + \frac{32}{9}n + \frac{43}{27}. \tag{3.26}$$

The proof of the inequality (3.26) is not difficult, and is left with the readers. Hence, (3.25) and (3.24) hold. This proves the claim.

By Lemma 2.2, the function

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x) - f_1(0)}{f_2(t) - f_2(0)}$$

is strictly decreasing on $(0,\infty)$, and we have

$$\frac{1}{2} < \lim_{x\to\infty} f(x) < f(x) = \frac{\ln\left(\frac{2(\sinh x/x)+\tanh x/x}{3}\right)}{\ln\left(\left(\frac{\sinh x}{x}\right)^2 \frac{\tanh x}{x}\right)} < \lim_{x\to 0} f(x) = \frac{3}{4}, \quad x>0.$$

Hence, inequality (1.18) holds for $x\neq 0$ with the best possible constants $\alpha=\frac{3}{4}$ and $\beta=\frac{1}{2}$.

Elementary calculations show that

$$\left(\frac{\sinh x}{x}\right)^2\left(\frac{\tanh x}{x}\right)-\left(\frac{(x/\sinh x)^2+x/\tanh x}{2}\right)^2=\frac{G(x)}{4x^3\cosh^2 x\sinh^4 x},$$

where

$$\begin{aligned}
G(x) &= 2x^4\sinh x\cosh x-4\sinh x\cosh x+12\sinh x\cosh^3 x-12\sinh x\cosh^5 x\\
&\quad -2x^4\sinh x\cosh^3 x+4\sinh x\cosh^7 x-x^7\cosh^2 x+3x\cosh^4 x-x\cosh^6 x\\
&\quad +x-3x\cosh^2 x\\
&=-\frac{3}{16}\sinh(6x)+\frac{5}{16}x-\frac{1}{2}x^7-\frac{1}{2}x^7\cosh(2x)-\frac{1}{4}x^4\sinh(4x)+\frac{3}{16}x\cosh(4x)\\
&\quad -\frac{15}{32}x\cosh(2x)-\frac{1}{32}x\cosh(6x)+\frac{7}{16}\sinh(4x)-\frac{7}{16}\sinh(2x)\\
&\quad +\frac{1}{2}x^4\sinh(2x)+\frac{1}{32}\sinh(8x)\\
&=\sum_{n=4}^{\infty}\frac{\omega_n x^{2n+1}}{(2n+1)!},
\end{aligned}$$

with

$$\begin{aligned}
\omega_n &= \frac{1}{4}\cdot 64^n-\left(\frac{37}{32}+\frac{1}{16}n\right)\cdot 36^n-\left(-\frac{31}{16}-\frac{23}{64}n-\frac{1}{64}n^2-\frac{1}{16}n^3+\frac{1}{16}n^4\right)\cdot 16^n\\
&\quad -\left(\frac{43}{32}-\frac{19}{16}n+\frac{81}{16}n^2+\frac{65}{16}n^3-\frac{37}{2}n^4+\frac{35}{2}n^5-7n^6+n^7\right)\cdot 4^n.
\end{aligned}$$

We claim that $\omega_n>0$ for $n\geqslant 4$. By direct computation we obtain that $\omega_n>0$ for $n=4,5,\ldots,49$. We prove now that $\omega_n>0$ for $n\geqslant 50$, it suffices to show that

$$\begin{aligned}
\frac{1}{4}\cdot 64^n &> \left(\frac{37}{32}+\frac{1}{16}n\right)\cdot 36^n+\left(-\frac{31}{16}-\frac{23}{64}n-\frac{1}{64}n^2-\frac{1}{16}n^3+\frac{1}{16}n^4\right)\cdot 36^n\\
&\quad +\left(\frac{43}{32}+\frac{19}{16}n+\frac{81}{16}n^2+\frac{65}{16}n^3-\frac{37}{2}n^4+\frac{35}{2}n^5-7n^6+n^7\right)\cdot 36^n\\
&=\left(\frac{9}{16}-\frac{95}{64}n+\frac{323}{64}n^2+4n^3-\frac{295}{16}n^4+\frac{35}{2}n^5-7n^6+n^7\right)\cdot 36^n, \quad n\geqslant 50,
\end{aligned}$$

i.e.,

$$\left(\frac{4}{3}\right)^{2n} > \frac{9}{4} - \frac{95}{16}n + \frac{323}{16}n^2 + 16n^3 - \frac{295}{4}n^4 + 70n^5 - 28n^6 + 4n^7, \quad n \geqslant 50. \quad (3.27)$$

We now prove the inequality (3.27) by using the principle of mathematical induction. For $n = 50$, elementary calculations show that

$$\left[\left(\frac{4}{3}\right)^{2n} - \left(\frac{9}{4} - \frac{95}{16}n + \frac{323}{16}n^2 + 16n^3 - \frac{295}{4}n^4 + 70n^5 - 28n^6 + 4n^7\right)\right]_{n=50}$$
$$= \frac{1686588593904015004538077919379151952850149684940649775363615}{41230201658560906482916890381249701816168601760008} > 0,$$

which shows that the inequality (3.27) holds true for $n = 50$.

We assume now that the inequality (3.27) holds true for a fixed positive integer $n \geqslant 50$, we try to obtain it for $n+1$. By inductive assumption, we have

$$\begin{aligned}
&\left(\frac{4}{3}\right)^{2(n+1)} - \left(\frac{9}{4} - \frac{95}{16}(n+1) + \frac{323}{16}(n+1)^2 + 16(n+1)^3 \right.\\
&\qquad\qquad \left. - \frac{295}{4}(n+1)^4 + 70(n+1)^5 - 28(n+1)^6 + 4(n+1)^7\right)\\
&> \left(\frac{4}{3}\right)^2 \left(\frac{9}{4} - \frac{95}{16}n + \frac{323}{16}n^2 + 16n^3 - \frac{295}{4}n^4 + 70n^5 - 28n^6 + 4n^7\right)\\
&\quad - \left(\frac{9}{4} - \frac{95}{16}(n+1) + \frac{323}{16}(n+1)^2 + 16(n+1)^3 \right.\\
&\qquad\qquad \left. - \frac{295}{4}(n+1)^4 + 70(n+1)^5 - 28(n+1)^6 + 4(n+1)^7\right)\\
&= -\frac{3}{4} - \frac{1151}{144}n + \frac{6653}{144}n^2 + \frac{247}{9}n^3 - \frac{4585}{36}n^4 + \frac{1246}{9}n^5 - \frac{448}{9}n^6 + \frac{28}{9}n^7\\
&= \frac{122057942787421}{72} + \frac{361738603041 49}{144}(n-50) + \frac{2292645499453}{144}(n-50)^2\\
&\quad + \frac{5035920997}{9}(n-50)^3 + \frac{424041415}{36}(n-50)^4 + \frac{1336846}{9}(n-50)^5\\
&\quad + \frac{9352}{9}(n-50)^6 + \frac{28}{9}(n-50)^7 > 0.
\end{aligned}$$

The proof of the inequality (3.27) is thus completed by means of the principle of mathematical induction on n.

Hence, $\omega_n > 0$ for $n \geqslant 4$, and then $G(x) > 0$ for $x > 0$. Thus, the first inequality in (1.19) holds.

Following the same method used in the proof of the fourth inequality in (1.9), we can prove the second inequality in (1.19) by using the second inequality in (1.2).

The last inequality in (1.19) is equivalent to the second inequality in (1.2). The proof of Theorem 1.2 is complete. □

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