JORDAN TYPE INEQUALITIES USING MONOTONY OF FUNCTIONS

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Abstract. In this paper we will consider Jordan type inequalities involving hyperbolic trigonometric functions.

1. Introduction

In this section we give a brief overview of known results which pertain to the main results of this paper.

The following inequalities

$$\frac{2}{\pi} \leqslant \frac{\sin x}{x} \leqslant 1, \ \ (0 < |x| < \frac{\pi}{2}),$$

are due to Jordan ([7], p. 33). Its have attracted the attention of several mathematicians (See, e.g., [1–6], [8–14]).

Lazarević [5] gives us the following inequality:

$$\left(\frac{\sinh x}{x}\right)^q < \cosh x, \ \ (x \neq 0, \ q \geqslant 3). \tag{1}$$

The following inequalities

$$\sinh x < x + \frac{x^3}{5}, \ \ (0 < x < 1), \tag{2}$$

$$\frac{\sinh kx}{kx} \leqslant \frac{\sinh x}{x}, \ \ (x > 0), \tag{3}$$

$$\frac{1}{\cosh x} < 1 - \frac{x^2}{3}, \ \ (0 < x < 1), \tag{4}$$

and

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}, \ \ (0 < x < \frac{\pi}{2}), \tag{5}$$

have established by R. Klén, M. Visuri, and M. Vuorinen [4].

The inequality

$$\left(\frac{x}{\sinh x}\right)^\alpha < (1 - \eta) + \eta \left(\frac{1}{\cosh x}\right)^\alpha, \tag{6}$$

where $x > 0$, $\alpha > 0$, and $\eta \leqslant 1/3$ was studied recently by Zhu in [13].

In Section 2 we give upper and lower bounds for $(\sin x)/x$. Some Jordan's type inequalities with hyperbolic trigonometric functions are presented in Section 3.

2. Jordan's inequality

In this section we will find upper and lower bounds for $(\sin x)/x$ by using hyperbolic functions.

THEOREM 1. *For $x \in (0, \infty)$,*

$$\frac{\sin x}{x} < \sqrt{\cosh x}.$$

Proof. The inequality holds true if the function $f(x) = x\sqrt{\cosh x} - \sin x$ is positive on $(0, \infty)$. Since

$$f''(x) = \frac{1}{\sqrt{\cosh x}} \left[\sin\left(x\sqrt{\cosh x}\right)(2\cosh x + x\sinh x) + 3\sqrt{\cosh x}\sinh x + x\cosh^{\frac{3}{2}} x \right],$$

we have $f''(x) > 0$ for $x \in (0, \infty)$ and $f'(x)$ is increasing on $(0, \infty)$. Therefore

$$f'(x) = \frac{1}{2\sqrt{\cosh x}} \left(2\cosh x + x\sinh x - 2\cos x\sqrt{\cosh x} \right) > f'(0) = 0,$$

and the function $f(x)$ is increasing on $(0, \infty)$. Now $f(x) > f(0) = 0$ for $x \in (0, \infty)$. $\square$

COROLLARY 2. *For $x \in \mathbb{R}\backslash\{0\}$,*

$$\frac{\sin x}{x} < \cosh x.$$

Proof. Because $f(x) = \frac{\sinh x}{x} - \cosh x$ is even function we proof the inequality on $(0, \infty)$. Using Theorem 1 and the fact that $\sqrt{\cosh x} < \cosh x$ the assertion follows. In Figure 1 we present the graphics of functions $\frac{\sin x}{x}$ and $\cosh x$. $\square$
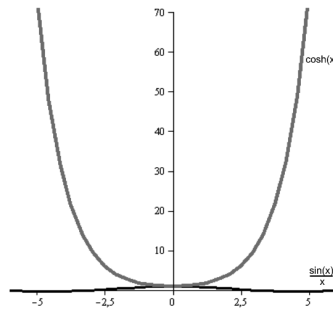

Figure 1.

In the above picture, with red is plotted the graph of the function $\frac{\sin x}{x}$ and with black is plotted the graph of the function $\cosh x$.

REMARK 3. Klén, Visuri, and Vuorinen have shown in [4] that the inequalities

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x} \tag{7}$$

are true for $x \in (0, \pi/2)$. By Corollary 2 and (7) we have

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \cosh x,$$

for $x \in (0, \pi/2)$.

THEOREM 4. *For* $x, k \in (0, \infty)$,

$$\frac{\sin x}{x} < \frac{\sinh kx}{kx}.$$

Proof. The inequality holds true if the function $f(x) = \sinh kx - k \sin x$ is positive on $(0, \infty)$. Since

$$f'(x) = k(\cosh kx - \cos x),$$

we have $f'(x) > 0$ for $x \in (0, \infty)$. Therefore that the function $f(x)$ is increasing on $(0, \infty)$. Now $f(x) > f(0) = 0$ for $x \in (0, \infty)$. $\square$

COROLLARY 5. *For* $x \in (0, \infty)$ *and* $k \in [1, \infty)$ *the following inequality*

$$\frac{\sin x}{x} < \frac{\sinh kx}{x}$$

holds.

REMARK 6. Similar inequalities to Theorem 4 have been considered by R. Klén, M. Visuri, and M. Vuorinen in [4, Theorem 4.3].

3. Hyperbolic Jordan's inequality

In this section we gave some hyperbolic Jordan's type inequalities with hyperbolic trigonometric functions.

THEOREM 7. *Let* $x \in (0, 1)$, $k \in (0, \infty)$ *and* $q \geqslant 3$, q *is natural number. Then the double inequality*

$$\sqrt[q]{\frac{3}{3 - k^2 x^2}} < \frac{\sinh x}{x} < 1 + \frac{x^2}{5}$$

holds.

Proof. From (1) result that

$$\sqrt[q]{\cosh x} < \frac{\sinh x}{x}. \tag{8}$$

By (3) and (8) we obtain

$$\sqrt[q]{\cosh kx} < \frac{\sinh kx}{kx} \leqslant \frac{\sinh x}{x} \tag{9}$$

The inequality (4) is equivalent with

$$\frac{3}{3 - x^2} < \cosh x,$$

therefore

$$\sqrt[q]{\frac{3}{3 - k^2 x^2}} < \sqrt[q]{\cosh kx} \tag{10}$$

By (2), (9) and (10) we obtain the conclusion. □

THEOREM 8. *Let* $x \in (0, 1)$, $\alpha > 0$ *and* $\eta \leqslant 1/3$, *then the following inequality*

$$\left(\frac{x}{\sinh x} \right)^\alpha < (1 - \eta) + \eta \left(1 - \frac{x^2}{3} \right)^\alpha$$

holds.

Proof. By (4) and (6) the assertion follows. □

When letting $\alpha = 1$ in Theorem 8, one can obtain the following result.

COROLLARY 9. *Let* $x \in (0, 1)$, *and* $\eta \leqslant 1/3$, *then the following inequality*

$$\frac{x}{\sinh x} < 1 - \eta \frac{x^2}{3}$$

holds.

THEOREM 10. *For* $x \in (0, 1)$ *the following inequalities*

$$1 - \frac{x^2}{2} \leqslant \frac{1}{\cosh x} \leqslant 1 - \frac{x^2}{3}$$

hold.

Proof. By the series expansion of $\frac{1}{\cosh x}$ we have

$$1 - \frac{x^2}{2} \leqslant \frac{1}{\cosh x} \leqslant 1 - \frac{x^2}{2} + 5 \cdot \frac{x^4}{24}. \tag{11}$$

By (4) and (11) we obtain the conclusion. □

THEOREM 11. *Let $x > 0$. Then the function*

$$f(t) = \frac{1}{\cosh^t \frac{x}{t}}$$

is decreasing on $(0, \infty)$.

Proof. Let us consider instead of $f(x)$ the function

$$f_1(y) = -\frac{x}{y} \log \cosh y$$

for $y \in (0, \infty)$. Note that $f(t) = \exp(f_1(\frac{x}{t}))$ and therefore the claim is equivalent to the function $f_1(y)$ being decreasing on $(0, \infty)$. We have

$$f_1'(y) = \frac{x}{y^2} (\log \cosh y - y \tanh y),$$

and

$$\frac{d(\log \cosh y - y \tanh y)}{dy} = -\frac{y}{\cosh^2 y} < 0.$$

and $\log \cosh y - y \tanh y \leqslant 0$. Therefore the function $f_1(y)$ is decreasing on $(0, \infty)$, and $f(t)$ is increasing on $(0, \infty)$. $\square$

COROLLARY 12. *Let $x \in (0, \infty)$. Then the following inequality*

$$\cosh \frac{x}{3} < \frac{2}{3} + \frac{1}{3} \cosh x$$

holds.

Proof. By Theorem 11 we obtain that

$$\frac{1}{\cosh^2 \frac{x}{2}} > \frac{1}{\cosh^3 \frac{x}{3}}. \tag{12}$$

By using the identities

$$\cosh^2 \frac{x}{2} = \frac{1 + \cosh x}{2},$$

and

$$\cosh^3 \frac{x}{3} = \frac{3}{4} \cosh \frac{x}{3} + \frac{1}{4} \cosh x$$

in relation (12) we obtain the conclusion. $\square$

THEOREM 13. *Let $x \in \mathbb{R} \backslash \{0\}$. Then the following inequality*

$$\frac{\sinh x}{x} > \frac{1}{\cosh \frac{x}{3}}$$

holds.

Proof. Because

$$f(x) = \frac{\sinh x}{x} - \frac{1}{\cosh \frac{x}{3}}$$

is even function we proof the inequality on $(0, \infty)$. The inequality holds true if the function

$$g(x) = \sinh x \cosh \frac{x}{3} - x$$

is positive on $(0, \infty)$. Since

$$g'(x) = \frac{1}{3} \cosh \frac{2}{3} x + \frac{2}{3} \cosh \frac{4}{3} x - 1,$$

we have $g'(x) > 0$ for $x \in (0, \infty)$. Therefore that the function $g(x)$ is increasing on $(0, \infty)$. Now $g(x) > g(0) = 0$ for $x \in (0, \infty)$.
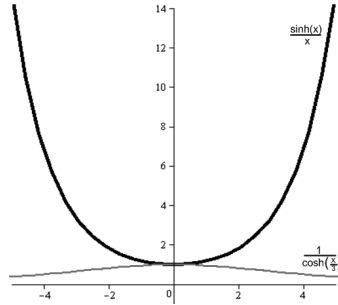

Figure 2.

In Figure 2 we present the graphics of functions $\frac{\sinh x}{x}$ and $\frac{1}{\cosh \frac{x}{3}}$.   □

In the above picture, with red is plotted the graph of the function $\frac{\sinh x}{x}$ and with black is plotted the graph of the function $\frac{1}{\cosh \frac{x}{3}}$.

COROLLARY 14. *Let $x \in (0, \infty)$. Then the following inequality*

$$\frac{x}{\sinh x} < \frac{2}{3} + \frac{1}{3} \cosh x$$

holds.

Proof. By Corollary 12 and Theorem 13 we obtain the conclusion.   □

REFERENCES

[1] R. P. AGARWAL, Y.-H. KIM, S. K. SEN, *A new refined Jordan's inequality and its application*, Mathematical Inequalities & Applications, vol. 12, no. 2, pp. 255–264, 2009.

[2] L. DEBNATH, C.-J. ZHAO, *New strengthened Jordan's inequality and its applications*, Applied Mathematics Letters, vol. 16, no. 4, pp. 557–560, 2003.

[3] W. D. JIANG, H. YUN, *Sharpening of Jordan's inequality and its applications*, Journal of Inequalities in Pure and Applied Mathematics, vol. 7, no. 3, article 102, pp. 1–4, 2006.

[4] R. KLÉN, M. VISURI, M. VUORINEN, *On Jordan Type Inequalities for Hyperbolic Functions*, Journal of Inequalities and Applications, Volume 2010, Article ID 362548.

[5] I. LAZAREVIĆ, *Neke nejednakosti sa hiperbolickim funkcijama*, Univerzitet u Beogradu, Publikacije Elektrotehničkog Fakulteta, Serija Matematika i Fizika, vol. 170, pp. 41–48, 1966.

[6] J.-L. LI, Y.-L. LI, *On the strengthened Jordan's inequality*, Journal of Inequalities and Applications, vol. 2007, Article ID 74328, 8 pages, 2007.

[7] D. S. MITRINOVIĆ, P. M. VASIĆ, *Analytic Inequalities*, Die Grundlehren der mathematischen Wissenschaften, vol. 16, Springer, New York, NY, USA, p 31, 1970.

[8] A. Z. ÖZBAN, *A new refined form of Jordan's inequality and its applications*, Applied Mathematics Letters, vol. 19, no. 2, pp. 155–160, 2006.

[9] F. QI, D.-W. NIU, B.-N. GUO, *Rafinements, generalizations, and applications of Jordan's inequality and related problems*, Journal of Inequalities and Applications, vol. 2009, Article ID 271923, 52 pages, 2009.

[10] J. SÁNDOR, *On the concavity of sin x/x*, Octogon Mathematical Magazine, vol. 13, no. 1, pp. 406–407, 2005.

[11] S.-H. WU, H. M. SRIVASTAVA, *A further refinement of a Jordan type inequality and its application*, Applied Mathematics and Computation, vol. 197, no. 2, pp. 914–923, 2008.

[12] X. ZHANG, G. WANG, Y. CHU, *Extensions and sharpenings of Jordan's and Kober's inequalities*, Journal of Inequalities in Pure and Applied Mathematics, vol. 7, no. 2, pp. 1–3, 2006.

[13] L. ZHU, *Inequalities for Hyperbolic Functions and Their Applications*, Journal of Inequalities and Applications, Volume 2010, Article ID 130821.

[14] L. ZHU, *Sharpening of Jordan's inequalities and its applications*, Mathematical Inequalities & Applications, vol. 9, no. 1, pp. 103–106, 2006.

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