Journal of Mathematical Inequalities
Volume 8, Number 1 (2014), 113–122 doi:10.7153/jmi-08-07

# ON $k$–QUASI–PARANORMAL OPERATORS

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(*Communicated by M. Fujii*)

*Abstract.* For a positive integer $k$, an operator $T \in B(\mathscr{H})$ is called $k$-quasi-paranormal if $\|T^{k+1}x\|^2 \leqslant \|T^{k+2}x\|\|T^kx\|$ for all $x \in \mathscr{H}$, which is a common generalization of paranormal and quasi-paranormal. In this paper, firstly we prove some inequalities of this class of operators; secondly we give a necessary and sufficient condition for $T$ to be $k$-quasi-paranormal. Using these results, we prove that: (1) if $\|T^{n+1}\| = \|T\|^{n+1}$ for some positive integer $n \geqslant k$, then a $k$-quasi-paranormal operator $T$ is normaloid; (2) if $E$ is the Riesz idempotent for an isolated point $\lambda_0$ of the spectrum of a $k$-quasi-paranormal operator $T$, then (i) if $\lambda_0 \neq 0$, then $E\mathscr{H} = \ker(T-\lambda_0)$; (ii) if $\lambda_0 = 0$, then $E\mathscr{H} = \ker(T^{k+1})$.

## 1. Introduction

Throughout this paper let $\mathscr{H}$ be a separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $B(\mathscr{H})$ denote the $C^*$-algebra of all bounded linear operators on $\mathscr{H}$. An operator is called paranormal if $\|Tx\|^2 \leqslant \|T^2x\|\|x\|$ for all $x \in \mathscr{H}$. In order to discuss the relations between paranormal and $p$-hyponormal and log-hyponormal operators, Furuta, Ito and Yamazaki [4] introduced class A operators defined by $|T^2| - |T|^2 \geqslant 0$, and they showed that class A is a subclass of paranormal and contains $p$-hyponormal and log-hyponormal operators.

Let $T \in B(\mathscr{H})$ and $\lambda_0$ be an isolated point of $\sigma(T)$. Here $\sigma(T)$ denotes the spectrum of $T$. Then there exists a small enough positive number $r > 0$ such that $\{\lambda \in C : |\lambda-\lambda_0| \leqslant r\} \cap \sigma(T) = \{\lambda_0\}$. Let

$$E = \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=r} (\lambda - T)^{-1} d\lambda.$$

$E$ is called the Riesz idempotent with respect to $\lambda_0$. Stampfli [10] proved that if $T$ is hyponormal(i.e., operators such that $T^*T - TT^* \geqslant 0$), then

$$E \text{ is self-adjoint and } E\mathscr{H} = \ker(T-\lambda_0) = \ker((T-\lambda_0)^*). \tag{1.1}$$

*Mathematics subject classification* (2010): 47B20 47A63.
*Keywords and phrases*: Riesz idempotent, $k$-quasi-paranormal operators, inequality.
This research is supported by the National Natural Science Foundation of China (11071188), (11271112)), the Natural Science Foundation of the Department of Education, Henan Province (2011A110009), (13B110077), the Youth Science Foundation of Henan Normal University and the new teachers Science Foundation of Henan Normal University (No. qd12102).

After that many authors extended this result to many other classes of operators. Chō and Tanahashi [1] proved that (1.1) holds if $T$ is either $p$-hyponormal or log-hyponormal. In the case $\lambda_0 \neq 0$, the result was further shown by Tanahashi and Uchiyama [11] to hold for $p$-quasihyponormal operators, by Tanahashi, Uchiyama and Chō [12] to hold for $(p,k)$-quasihyponormal operators and by Uchiyama and Tanahashi [14, 15] for class A and paranormal operators.

In this paper, we shall study the Riesz idempotent with respect to an isolated point of the spectrum of $k$-quasi-paranormal operators.

DEFINITION 1.1. $T \in B(\mathscr{H})$ is called $k$-quasi-paranormal if for a positive integer $k$

$$\|T^{k+1}x\|^2 \leqslant \|T^{k+2}x\|\|T^kx\| \tag{1.2}$$

for all $x \in \mathscr{H}$.

When $k=1$, $T$ is called quasi-paranormal operators. quasi-paranormal and $k$-quasi-paranormal operators have been studied in [6, 9, 18].

It is clear that

$$\begin{aligned} \text{the class of paranormal operators} &\subseteq \text{the class of quasi-paranormal operators} \\ &\subseteq \text{the class of k-quasi-paranormal operators} \\ &\subseteq \text{the class of (k+1)-quasi-paranormal operators.} \end{aligned} \tag{1.3}$$

We show that the inclusion relations in (1.3) are strict, by an example which appeared in [5, 8].

EXAMPLE 1.2. Given a bounded sequence of positive numbers $\{\alpha_i\}_{i=0}^{\infty}$. Let $T$ be the unilateral weighted shift operator on $l^2$ with the canonical orthonormal basis $\{e_n\}_{n=0}^{\infty}$ by $Te_n = \alpha_n e_{n+1}$ for all $n \geqslant 0$, that is,

$$T = \begin{pmatrix} 0 & & & & \\ \alpha_0 & 0 & & & \\ & \alpha_1 & 0 & & \\ & & \alpha_2 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

Straightforward calculations show that $T$ is a $k$-quasi-paranormal operator if and only if $\alpha_k \leqslant \alpha_{k+1} \leqslant \alpha_{k+2} \leqslant \cdots$. So if $\alpha_{k+1} \leqslant \alpha_{k+2} \leqslant \alpha_{k+3} \leqslant \cdots$ and $\alpha_k > \alpha_{k+1}$, then $T$ is a $(k+1)$-quasi-paranormal operator, but not a $k$-quasi-paranormal operator.

In this paper, firstly we prove some inequalities of this class of operators; secondly we give a necessary and sufficient condition for $T$ to be $k$-quasi-paranormal. Using these results, we prove that: (1) if $\|T^{n+1}\| = \|T\|^{n+1}$ for some positive integer $n \geqslant k$, then a $k$-quasi-paranormal operator $T$ is normaloid; (2) if $E$ is the Riesz idempotent for an isolated point $\lambda_0$ of the spectrum of a $k$-quasi-paranormal operator $T$, then (i) if $\lambda_0 \neq 0$, then $E\mathscr{H} = \ker(T-\lambda_0)$; (ii) if $\lambda_0 = 0$, then $E\mathscr{H} = \ker(T^{k+1})$.

## 2. Results

It is well known that $T$ is paranormal if and only if $T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geqslant 0$ for all $\lambda > 0$. Similarly, we have the following result.

THEOREM 2.1. (see [9]) *Let $T \in B(\mathscr{H})$. Then $T$ is $k$-quasi-paranormal if and only if*

$$T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geqslant 0 \tag{2.1}$$

*for all $\lambda > 0$.*

*Proof.* (1.2) is equivalent to the following (2.2):

$$\langle T^{*(k+2)}T^{k+2}x,x\rangle^{\frac{1}{2}}\langle T^{*k}T^k x,x\rangle^{\frac{1}{2}} \geqslant \langle T^{*(k+1)}T^{k+1}x,x\rangle \tag{2.2}$$

for all $x \in \mathscr{H}$. By generalized arithmetic-geometric mean inequality, we have

$$\begin{aligned}\langle T^{*(k+2)}T^{k+2}x,x\rangle^{\frac{1}{2}}\langle T^{*k}T^k x,x\rangle^{\frac{1}{2}} &= \{\lambda^{-1}\langle T^{*(k+2)}T^{k+2}x,x\rangle\}^{\frac{1}{2}}\{\lambda\langle T^{*k}T^k x,x\rangle\}^{\frac{1}{2}}\\ &\leqslant \frac{1}{2}\lambda^{-1}\langle T^{*(k+2)}T^{k+2}x,x\rangle + \frac{1}{2}\lambda\langle T^{*k}T^k x,x\rangle\end{aligned}$$

holds for all $x \in \mathscr{H}$ and $\lambda > 0$, so that (2.2) implies the following (2.3):

$$\frac{1}{2}\lambda^{-1}\langle T^{*(k+2)}T^{k+2}x,x\rangle + \frac{1}{2}\lambda\langle T^{*k}T^k x,x\rangle \geqslant \langle T^{*(k+1)}T^{k+1}x,x\rangle \tag{2.3}$$

for all $x \in \mathscr{H}$ and $\lambda > 0$. Conversely, (2.2) follows from (2.3) by putting $\lambda = \{\frac{\langle T^{*(k+2)}T^{k+2}x,x\rangle}{\langle T^{*k}T^k x,x\rangle}\}^{\frac{1}{2}} > 0$ in case $\langle T^{*(k+2)}T^{k+2}x,x\rangle \neq 0$, and letting $\lambda \to +0$ in case $\langle T^{*(k+2)}T^{k+2}x,x\rangle = 0$. Hence (2.2) is equivalent to (2.3). Consequently, the proof of Theorem 2.1 is complete since (2.3) is equivalent to (2.1). □

$T \in B(\mathscr{H})$ is called a $k$-quasi-class A operator for a positive integer $k$ if $T^{*k}(|T^2| - |T|^2)T^k \geqslant 0$, which contains class A and quasi-class A, see [5, 8, 13]. In the following we give the relations between $k$-quasi-paranormal and $k$-quasi-class A operators.

THEOREM 2.2. *Let $T$ be a $k$-quasi-class A operator for a positive integer $k$. Then $T$ is a $k$-quasi-paranormal operator.*

*Proof.* Suppose that $T$ is $k$-quasi-class A operator. Then $T^{*k}(|T^2| - |T|^2)T^k \geqslant 0$. Let $x \in \mathscr{H}$. Then

$$\begin{aligned}\|T^{k+1}x\|^2 &= \langle T^{*k}|T|^2T^k x,x\rangle\\ &\leqslant \langle T^{*k}|T^2|T^k x,x\rangle\\ &\leqslant \||T^2|T^k x\|\|T^k x\|\\ &= \|T^{k+2}x\|\|T^k x\|.\end{aligned}$$

So we have that

$$\|T^{k+1}x\|^2 \leqslant \|T^{k+2}x\|\|T^k x\|,$$

hence $T$ is a $k$-quasi-paranormal operator. □

REMARK. We give an example which is $k$-quasi-paranormal, but not $k$-quasi-class A.

EXAMPLE 2.3. Let $T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in B(l_2 \oplus l_2)$. Then $T$ is $k$-quasi-paranormal, but not $k$-quasi-class A.

By simple calculation we have that

$$T^{*k}|T^2|T^k = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } T^{*k}|T|^2T^k = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} .$$

Hence $T$ not $k$-quasi-class A. However,

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 = \begin{pmatrix} 2-4\lambda+\lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix},$$

we have

$$T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k = \begin{pmatrix} 2(1-\lambda)^2 & 0 \\ 0 & 0 \end{pmatrix} \geqslant 0$$

for all $\lambda > 0$. Therefore $T$ is $k$-quasi-paranormal.

THEOREM 2.4. *Let $T \in B(\mathscr{H})$ be a $k$-quasi-paranormal operator for a positive integer $k$. Then the following assertions hold.*

(1) $\|T^{n+2}\|\|T^n\| \geqslant \|T^{n+1}\|^2$ *for all positive integers $n \geqslant k$.*

(2) *If $T^n = 0$ for some positive integer $n \geqslant k$, then $T^{k+1} = 0$.*

(3) $\|T^{n+1}\| \leqslant \|T^n\| r(T)$ *for all positive integers $n \geqslant k$, where $r(T)$ denotes the spectral radius of $T$.*

*Proof.* The proof is similar to that of [5, Theorem 2.2]. (1) Since that $k$-quasi-paranormal operators are $(k+1)$-quasi-paranormal operators, we only need prove the case $n = k$. It is clear by the definition of $k$-quasi-paranormal operators.

(2) If $n = k, k+1$, it is obvious that $T^{k+1} = 0$. If $T^{k+2} = 0$, then $T^{k+1} = 0$ by (1). The rest of the proof is similar.

(3) We only need to prove the case $n = k$, that is,

$$\|T^{k+1}\| \leqslant \|T^k\| r(T).$$

If $T^n = 0$ for some $n \geqslant k$, then $T^{k+1} = 0$ by (2) and in this case $r(T) = (r(T^{k+1}))^{\frac{1}{k+1}} = 0$. Hence (3) is clear. Therefore we may assume $T^n \neq 0$ for all $n \geqslant k$. Then

$$\frac{\|T^{k+1}\|}{\|T^k\|} \leqslant \frac{\|T^{k+2}\|}{\|T^{k+1}\|} \leqslant \frac{\|T^{k+3}\|}{\|T^{k+2}\|} \leqslant \cdots \leqslant \frac{\|T^{mk}\|}{\|T^{mk-1}\|}$$

by (1), and we have

$$\Big(\frac{\|T^{k+1}\|}{\|T^k\|}\Big)^{mk-k} \leqslant \frac{\|T^{k+1}\|}{\|T^k\|} \times \frac{\|T^{k+2}\|}{\|T^{k+1}\|} \times \cdots \times \frac{\|T^{mk}\|}{\|T^{mk-1}\|} = \frac{\|T^{mk}\|}{\|T^k\|}.$$

Hence

$$\left(\frac{\|T^{k+1}\|}{\|T^k\|}\right)^{k-\frac{k}{m}} \leqslant \frac{\|T^{mk}\|^{\frac{1}{m}}}{\|T^k\|^{\frac{1}{m}}}.$$

By letting $m \longrightarrow \infty$, we have

$$\|T^{k+1}\|^k \leqslant \|T^k\|^k (r(T))^k,$$

that is,

$$\|T^{k+1}\| \leqslant \|T^k\| r(T). \quad \square$$

It is well known that a paranormal operator is normaloid, that is, $\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$ (equivalently $\|T\| = r(T)$). A nonzero nilpotent operator $T$ (satisfying $T^k = 0$) is $k$-quasi-paranormal but not normaloid. By Theorem 2.4, we give a sufficient condition of a $k$-quasi-paranormal operator to be normaloid.

COROLLARY 2.5. *Let $T \in B(\mathscr{H})$ be a $k$-quasi-paranormal operator for a positive integer $k$. If $\|T^{n+1}\| = \|T^n\|\|T\|$ for some positive integer $n \geqslant k$, then $T$ is normaloid. In particularly if $\|T^{n+1}\| = \|T\|^{n+1}$ for some positive integer $n \geqslant k$, then $T$ is normaloid.*

*Proof.* It is clear by (3) of Theorem 2.4. $\square$

In [9], S. Mecheri studied the matrix representation of $k$-quasi-paranormal operator with respect to the direct sum of $\overline{\mathrm{ran}(T^k)}$ and its orthogonal complement. In the following we give an equivalent condition for $T$ to be $k$-quasi-paranormal.

THEOREM 2.6. *Suppose that $T^k$ dose not have dense range. Then $T \in B(\mathscr{H})$ is a $k$-quasi-paranormal operator for a positive integer $k$ if and only if $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathscr{H} = \overline{ran(T^k)} \bigoplus \ker T^{*k}$, where $T_1$ is a paranormal operator on $\overline{ran(T^k)}$ and $T_3^k = 0$, furthermore $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

*Proof.* We first prove the necessary. The necessary has been proved in [9]. We give a proof here for completeness. Suppose that $T \in B(\mathscr{H})$ is a $k$-quasi-paranormal operator for a positive integer $k$. Since that $T^k$ dose not have dense range, we can represent $T$ as the upper triangular matrix

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $\mathscr{H} = \overline{\mathrm{ran}(T^k)} \bigoplus \ker T^{*k}$. We shall prove that $T_1$ is a paranormal operator on $\overline{\mathrm{ran}(T^k)}$ and $T_3^k = 0$. Since $T$ is a $k$-quasi-paranormal operator, it follows from Theorems 2.1 that

$$T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geqslant 0$$

for all $\lambda > 0$. Therefore

$$\langle (T^{*2}T^2 - 2\lambda T^*T + \lambda^2)x, x\rangle = \langle (T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)x, x\rangle \geqslant 0$$

for all $\lambda > 0$ and for all $x \in \overline{\operatorname{ran}(T^k)}$. Hence

$$T_1^{*2}T_1^2 - 2\lambda T_1^* T_1 + \lambda^2 \geqslant 0$$

for all $\lambda > 0$. So we have that $T_1$ is a paranormal operator on $\overline{\operatorname{ran}(T^k)}$. Let $P$ be the orthogonal projection of $\mathscr{H}$ onto $\overline{\operatorname{ran}(T^k)}$. For any $x = (x_1, x_2) \in \mathscr{H}$,

$$\langle T_3^k x_2, x_2\rangle = \langle T^k(I-P)x, (I-P)x\rangle = \langle (I-P)x, T^{*k}(I-P)x\rangle = 0,$$

which implies $T_3^k = 0$.

Since $\sigma(T) \cup \mathfrak{G} = \sigma(T_1) \cup \sigma(T_3)$, where $\mathfrak{G}$ is the union of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1) \cap \sigma(T_3)$ by [7, Corollary 7], and $\sigma(T_3) = 0$ and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points, we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Next prove the sufficiency.

Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathscr{H} = \overline{\operatorname{ran}(T^k)} \bigoplus \ker T^{*k}$, where $T_1$ is a paranormal operator on $\overline{\operatorname{ran}(T^k)}$ and $T_3^k = 0$. Then we have

$$\begin{aligned}
&T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \\
&= \begin{pmatrix} T_1^{*k} & 0 \\ (\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i})^* & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2 & T_1^{*2}T_1T_2 + T_1^{*2}T_2T_3 \\ T_2^*T_1^*T_1^2 + T_3^*T_2^*T_1^2 - 2\lambda T_2^*T_1 & |T_1T_2 + T_2T_3|^2 + |T_3^2|^2 - 2\lambda(T_2^*T_2 + T_3^*T_3) + \lambda^2 \end{pmatrix} \\
&\quad \times \begin{pmatrix} T_1^k & \sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i} \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} T_1^{*k}(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)T_1^k & T_1^{*k}(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i} \\ (\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i})^*(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)T_1^k & D \end{pmatrix},
\end{aligned}$$

where $D = (\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i})^*(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i}$. Let $\lambda > 0$ be arbitrary and $v = x \oplus y$ be a vector in $\mathscr{H} = \overline{\operatorname{ran}(T^k)} \bigoplus \ker T^{*k}$, where $x \in \overline{\operatorname{ran}(T^k)}$

and $y \in \ker T^{*k}$. Then

$$\begin{aligned}
&\langle T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k v, v\rangle \\
&= \langle T_1^{*k}(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)T_1^k x, x\rangle \\
&\quad + \langle T_1^{*k}(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i} y, x\rangle \\
&\quad + \langle (\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i})^*(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)T_1^k x, y\rangle \\
&\quad + \langle (\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i})^*(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i} y, y\rangle \\
&= \langle (T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)(T_1^k x + \sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i} y), T_1^k x + \sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i} y\rangle.
\end{aligned}$$

Since $T_1$ is a paranormal operator, we have that $T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2 \geqslant 0$ for all $\lambda > 0$. Therefore

$$\langle T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k v, v\rangle \geqslant 0$$

for all $v \in \mathscr{H}$ and for all $\lambda > 0$. Hence

$$T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geqslant 0$$

for $\lambda > 0$. So we have that $T$ is a $k$-quasi-paranormal operator for a positive integer $k$ by Theorem 2.1. □

REMARK. In the proof of Theorem 2.6, let $P$ be the orthogonal projection of $\mathscr{H}$ onto $\overline{\operatorname{ran}(T^k)}$. Then

$$\begin{aligned}
&T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geqslant 0 \\
\iff\ &P(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)P \geqslant 0 \\
\iff\ &T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2 \geqslant 0.
\end{aligned}$$

It is well known that if $T$ is a paranormal operator and $\mathscr{U}$ is a closed invariant subspace of $T$, then $T|\mathscr{U}$ is also paranormal. We shall give a similar result for a $k$-quasi-paranormal operator by Theorem 2.6.

COROLLARY 2.7. *Suppose that $T \in B(\mathscr{H})$ is a $k$-quasi-paranormal operator for a positive integer $k$ and $\mathscr{U}$ is its closed invariant subspace. Then the restriction $T|\mathscr{U}$ of $T$ to $\mathscr{U}$ is also a $k$-quasi-paranormal operator.*

*Proof.* Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathscr{H} = \mathscr{U} \bigoplus (\mathscr{U})^{\perp}$. Since $T$ is a $k$-quasi-paranormal operator for a positive integer $k$, it follows from Theorem 2.1 that

$$T^{*k}(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)T^k \geqslant 0$$

for all $\lambda > 0$. By the proof Theorem 2.6, we have

$$\begin{pmatrix} T_1^{*k}(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)T_1^k & T_1^{*k}(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i} \\ (\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i})^*(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)T_1^k & D \end{pmatrix}$$
$$\geqslant 0,$$

where $D = (\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i})^*(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)\sum_{i=0}^{k-1} T_1^i T_2 T_3^{k-1-i}$. Recall that $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geqslant 0$ if and only if $X, Z \geqslant 0$ and $Y = X^{\frac{1}{2}} W Z^{\frac{1}{2}}$ for some contraction $W$. So we have that

$$T_1^{*k}(T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2)T_1^k \geqslant 0$$

for all $\lambda > 0$. Therefore $T|\mathscr{U} = T_1$ is also a $k$-quasi-paranormal operator. $\square$

In [18], Yuan studied the Riesz idempotent with respect to an isolated point of the spectrum of $k$-quasi-paranormal operators by Bishop's property $(\beta)$. M. Chō and T. Yamazaki proved that class A operators have property $\beta$ in [2] Theorem 3.1; A. Uchiyama and K. Tanahashi proved that paranormal operators have property $\beta$ in [16] Corollary 3.6. Unfortunately there are some mistakes in the proof of these theorems, see detail in [3]. So the Bishop's property $(\beta)$ of the class operators such as class A , paranormal and $k$-quasi-paranormal operators is still an open problem. In the following we have similar result by Theorem 2.4 without Bishop's property $(\beta)$.

THEOREM 2.8. *Suppose $T \in B(\mathscr{H})$ is a $k$-quasi-paranormal operator for a positive integer $k$. Let $\lambda_0$ be an isolated point of $\sigma(T)$ and $E$ the Riesz idempotent for $\lambda_0$. Then the following assertions hold:*

*(i) If $\lambda_0 \neq 0$ , then $E\mathscr{H} = \ker(T - \lambda_0)$.*

*(ii) If $\lambda_0 = 0$ , then $E\mathscr{H} = \ker(T^{k+1})$.*

*Proof.* Suppose that $T$ is a $k$-quasi-paranormal operator. (i) If the range $T^k\mathscr{H}$ is dense, then $T$ is a paranormal operator. Theorem holds by Theorem 3.7 in [17]. Therefore we may assume that $\overline{\operatorname{ran} T^k} \neq \mathscr{H}$. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathscr{H} = \overline{\operatorname{ran}(T^k)} \bigoplus \ker T^{*k}$. By Theorem 2.6 we know that $T_1$ is a paranormal operator and $\sigma(T) = \sigma(T_1) \cup \{0\}$. If $\lambda_0 \neq 0$ is an isolated point of $\sigma(T)$, then $\lambda_0$ is an isolated point of $\sigma(T_1)$. Therefore $\lambda_0$ is a simple pole of the resolvent of $T_1$ and $T_1$ can be written by $T_1 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & T' \end{pmatrix}$ on $\overline{\operatorname{ran}(T^k)} = \ker(T_1 - \lambda_0) \bigoplus \overline{\operatorname{ran}(T_1 - \lambda_0)}$. Hence we have $T - \lambda_0 = \begin{pmatrix} 0 & 0 & T_{21} \\ 0 & T' - \lambda_0 & T_{22} \\ 0 & 0 & T_3 - \lambda_0 \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & C \end{pmatrix}$ on $\mathscr{H} = \ker(T_1 - \lambda_0) \bigoplus \overline{\operatorname{ran}(T_1 - \lambda_0)} \bigoplus \ker(T^{*k})$, where $C = \begin{pmatrix} T' - \lambda_0 & T_{22} \\ 0 & T_3 - \lambda_0 \end{pmatrix}$.

Since $C$ is an invertible operator on $\overline{\mathrm{ran}(T_1-\lambda_0)}\bigoplus\ker(T^{*k})$, it can be easily shown that $p(T-\lambda_0)=q(T-\lambda_0)=1$. Therefore $\lambda_0$ is a simple pole of the resolvent of $T$. Since $E$ is the Riesz idempotent of $T$ with respect to $\lambda_0$, we have $E\mathscr{H}=\ker(T-\lambda_0)$.

(ii) Since $\ker(T^{k+1})\subset E\mathscr{H}$ always holds, it suffices to prove $E\mathscr{H}\subset\ker(T^{k+1})$. It is known that $E\mathscr{H}$ is an invariant subspace of $T$ and $\sigma(T|_{E\mathscr{H}})=\{0\}$. Hence $T|_{E\mathscr{H}}$ is also a $k$-quasi-paranormal operator by Corollary 2.7 and

$$\|(T|_{E\mathscr{H}})^{k+1}\|\leqslant\|(T|_{E\mathscr{H}})^k\|r(T|_{E\mathscr{H}})=0$$

by (3) of Theorem 2.4. Hence

$$(T|_{E\mathscr{H}})^{k+1}=T^{k+1}|_{E\mathscr{H}}=0.$$

This implies $E\mathscr{H}\subset\ker(T^{k+1})$. □

An operator $T$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of $T$.

COROLLARY 2.9. *Let $T\in B(\mathscr{H})$ be a $k$-quasi-paranormal operator for a positive integer $k$. Then $T$ is isoloid.*

*Proof.* Let $\lambda\in\sigma(T)$ be an isolated point. If $\lambda\neq 0$, by (1) of Theorem 2.8, $\ker(T-\lambda)=E\mathscr{H}\neq\{0\}$ for $E\neq 0$. Therefore $\lambda$ is an eigenvalue of $T$. If $\lambda=0$, by (2) of Theorem 2.8, $\ker(T^{k+1})=E\mathscr{H}\neq\{0\}$ for $E\neq 0$. So we have $\ker(T)\neq\{0\}$. Therefore 0 is an eigenvalue of $T$. This completes the proof. □

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(Received January 9, 2013)

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