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SOME DYNAMIC HARDY–TYPE INEQUALITIES WITH GENERAL KERNEL

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Abstract. In this paper, we extend some Hardy-type inequalities with certain kernels to arbitrary time scales. Certain classical and some new integral and discrete inequalities are deduced in seek of applications.

1. Introduction

The well-known Hardy inequality as presented in [9] (both in the continuous and discrete settings) has been extensively studied and used as a model for investigation of more general integral inequalities [7, 10–13]. Recently, several papers have treated the unification and extension of Hardy's continuous and discrete integral inequalities by means of the theory of time scales [16–18]. Measure spaces and measurable functions for time scales are discussed in [3, 4, 8]. The aim of this paper is to extend some Hardy-type inequalities with certain kernels to arbitrary time scales. In the next section, we give some preliminaries about the theory of time scales. Our main results are given in Section 3. In Section 4, we discuss inequalities with special kernels. The last section is devoted to particular cases and examples of Hardy-type inequalities on various time scales.

2. Preliminaries

We first briefly introduce some elements of time scale theory. By a time scale $\mathbb{T}$, we mean any nonempty closed subset of $\mathbb{R}$. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, then we say t is right-scattered. If $\rho(t) < t$, then we say t is left-scattered. If $\sigma(t) = t$, then we say t is right-dense. If $\rho(t) = t$, then we say t is left-dense. For

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further properties including the concept of delta differentiation, we refer the reader to [5, 6].

Let $n \in \mathbb{N}$ be fixed. For each $i \in \{1,2,\ldots,n\}$, let $\mathbb{T}_i$ denote a time scale and let σ_i, ρ_i and Δ_i denote the forward jump operator, the backward jump operator, and the delta differentiation operator, respectively. Let us set

$$\Omega^n = \{a = (a_1, a_2, \ldots, a_n) : \ a_i \in \mathbb{T}_i, \ 1 \leqslant i \leqslant n\}.$$

We call Ω^n an n-dimensional time scale. The set Ω^n is a complete metric space with the metric defined by

$$d(a,b) = \left(\sum_{i=1}^{n} |b_i - a_i|^2\right)^{1/2}, \quad a, b \in \Omega^n.$$

In the following, for the convenience of the reader, we briefly describe the Carathéodory construction of a Lebesgue measure in Ω^n. Denote by $\mathscr{F}$ the collection of all n-dimensional time scale intervals in Ω^n of the form

$$V = [a,b) = \times_{i=1}^{n} [a_i, b_i) := [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n)$$

with $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \in \Omega^n$ and $a_i \leqslant b_i$ for all $1 \leqslant i \leqslant n$. If $a_i = b_i$ for some values of i, then $[a,b)$ is understood to be the empty set. Let $m : \mathscr{F} \to [0,\infty)$ be the set function that assigns to each n-dimensional time scale interval $V = [a,b)$ its volume:

$$m(V) = \prod_{i=1}^{n} (b_i - a_i).$$

Then it is not difficult to verify that $\mathscr{F}$ is a semiring of subsets of Ω^n and m is a σ-additive measure on $\mathscr{F}$. Let E be any subset of Ω^n. If there exists at least one finite or countable system of n-dimensional time scale intervals $V_k = [a_k, b_k)$ such that $E \subset \bigcup_{k \in \mathbb{N}} V_k$, then we define the outer measure m^* of E by

$$m^*(E) = \inf \sum_{k \in \mathbb{N}} m(V_k),$$

where the infimum is taken over all coverings of E by a finite or countable system of intervals $V_k \in \mathscr{F}$. If there is no such covering of E, then we put $m^*(E) = \infty$.

A subset A of Ω^n is said to be m^*-measurable (or Δ-measurable) if

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^{\mathrm{c}})$$

holds for all $E \subset \Omega^n$, where A^{c} denotes the complement of A, i.e., $A^{\mathrm{c}} = \Omega^n \setminus A$. The family $\mathscr{M}$ of all m^*-measurable subsets of Ω^n is a σ-algebra, and the restriction of m^* to $\mathscr{M}$, which we denote by μ_Δ, is a σ-additive measure on $\mathscr{M}$. We have $\mathscr{F} \subset \mathscr{M}$ and $\mu_\Delta(V) = m(V)$ for each $V \in \mathscr{F}$. The measure μ_Δ is called the Carathéodory extension of the original measure m defined on the semiring $\mathscr{F}$. The measure μ_Δ obtained in this way is also called the Lebesgue Δ-measure on Ω^n generated by the pair $(\mathscr{F}, m)$. We call $(\Omega^n, \mathscr{M}, \mu_\Delta)$ an n-dimensional time scale measure space.

REMARK 2.1. For each $i \in \{1,2,\dots,n\}$, we put $\widetilde{a}_i = \inf \mathbb{T}_i$ and $\widetilde{b}_i = \sup \mathbb{T}_i$. Then we say that $\mathbb{T}_i$ is upper bounded if $\widetilde{b}_i \in \mathbb{R}$ and upper unbounded if $\widetilde{b}_i = \infty$. Analogously, $\mathbb{T}_i$ is lower bounded if $\widetilde{a}_i \in \mathbb{R}$ and lower unbounded if $\widetilde{a}_i = -\infty$. We say that $\mathbb{T}_i$ is bounded if it is upper bounded and lower bounded. Let Ω^n_∞ be the set of all points $b = (b_1, b_2, \dots, b_n) \in \Omega^n$ for which there exists at least one b_i such that $b_i = \widetilde{b}_i$. It is known [4, Theorem 3.1] that if $t = (t_1, t_2, \dots, t_n) \in \Omega^n \setminus \Omega^n_\infty$, then the single-point set $\{t\}$ is Δ-measurable, and its Δ-measure is given by

$$\mu_\Delta(\{t\}) = \prod_{i=1}^{n} (\sigma_i(t_i) - t_i).$$

Obviously, the set $\Omega^n_0 = \Omega^n \setminus \Omega^n_\infty$ can be represented as a finite or countable union of intervals of the family $\mathscr{F}$, and therefore it is Δ-measurable. Furthermore, the set $\Omega^n_\infty = \Omega^n \setminus \Omega^n_0$ is Δ-measurable as the difference of two Δ-measurable sets Ω^n and Ω^n_0, but Ω^n_∞ does not have a finite or countable covering intervals of $\mathscr{F}$. It follows that the set Ω^n_∞ and also any Δ-measurable subset A of Ω^n such that $A \cap \Omega^n_\infty \neq \emptyset$ has Δ-measure infinity. In particular, if $a \in \mathbb{T}$, where $\mathbb{T}$ is an arbitrary time scale, then the set $[a, \infty) = \{t \in \mathbb{T} : a \leqslant t\}$ is Δ-measurable.

We say that an extended real-valued function $f : \Omega^n \to \overline{\mathbb{R}} := [-\infty, \infty]$ is Δ-measurable if for every $\alpha \in \mathbb{R}$, the set

$$f^{-1}([-\infty, \alpha)) = \{t = (t_1, t_2, \dots, t_n) \in \Omega^n : f(t) < \alpha\}$$

is Δ-measurable. It is easy to see that f is Δ-measurable if and only if for each open set $G \subset \mathbb{R}$, the set $f^{-1}(G) = \{t \in \Omega^n : f(t) \in G\}$ is Δ-measurable. Moreover, if $f : \Omega^n \to \mathbb{R}$ is Δ-measurable and $\Phi : I \to \mathbb{R}$ with $I \subset \mathbb{R}$ is a continuous function, then $\Phi \circ f : \Omega^n \to \mathbb{R}$ is Δ-measurable.

Having the σ-additive measure μ_Δ on Ω^n, we possess the corresponding integration theory for functions $f : E \subset \Omega^n \to \mathbb{R}$, according to the general Lebesgue integration theory (see, e.g., [19]). The Lebesgue integral associated with the measure μ_Δ on Ω^n is called the Lebesgue Δ-integral. For a Δ-measurable set $E \subset \Omega^n$ and a Δ-measurable function $f : E \to \mathbb{R}$, the corresponding Δ-integral of f over E will be denoted by

$$\int_E f(t_1, t_2, \dots, t_n)\Delta t_1 \Delta t_2 \cdots \Delta t_n, \quad \int_E f(t)\Delta t \quad \text{or} \quad \int_E f(t)\mu_\Delta(t).$$

So all theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue Δ-integrals on Ω^n. Next, we compare the Lebesgue Δ-integral with the Riemann Δ-integral (see [4, Theorem 3.4]). Let $V = [a, b)$ be an n-dimensional time scale interval in Ω^n and let f be a bounded real-valued function on V. If f is Riemann Δ-integrable over V, then f is Lebesgue Δ-integrable over V and

$$R\int_V f(t)\Delta t = L\int_V f(t)\Delta t,$$

where R and L indicate the Riemann and Lebesgue Δ-integrals, respectively. In particular, if $\mathbb{T}$ is an arbitrary time scale and the interval $[a, b) \subset \mathbb{T}$ contains only isolated points, then

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)} (\sigma(t)-t)f(t).$$

Finally, let $(\Omega,\mathscr{M},\mu_\Delta)$ and $(\Lambda,\mathscr{L},\lambda_\Delta)$ be two finite dimensional time scale measure spaces. We consider the measure space $(\Omega\times\Lambda,\mathscr{M}\times\mathscr{L},\mu_\Delta\times\lambda_\Delta)$, where $\mathscr{M}\times\mathscr{L}$ is σ-algebra product generated by the family $\{E\times F : E\in\mathscr{M}, F\in\mathscr{L}\}$ and

$$(\mu_\Delta\times\lambda_\Delta)(E\times F) = \mu_\Delta(E)\lambda_\Delta(F).$$

Then Fubini's theorem holds. More precisely, if $f:\Omega\times\Lambda\to\mathbb{R}$ is a $\mu_\Delta\times\lambda_\Delta$-integrable function and if we define the function $\varphi(y) = \int_\Omega f(x,y)\Delta x$ for a.e. $y\in\Lambda$ and $\psi(x) = \int_\Lambda f(x,y)\Delta y$ for a.e. $x\in\Omega$, then φ is λ_Δ-integrable on Λ, ψ is μ_Δ-integrable on Ω and

$$\int_\Omega \Delta x \int_\Lambda f(x,y)\Delta y = \int_\Lambda \Delta y \int_\Omega f(x,y)\Delta x.$$

3. Inequalities with general kernels

The following Jensen inequality on time scales is given in [1, Theorem 4.2].

THEOREM 3.1. *Assume $\Phi\in C(I,\mathbb{R})$ is convex, where $I\subset\mathbb{R}$ is an interval. Let $(\Lambda,\mathscr{L},\lambda_\Delta)$ be a time scale measure space. Suppose f is λ_Δ-integrable on Λ such that $f(\Lambda)=I$. Moreover, let $h:\Lambda\to\mathbb{R}$ be λ_Δ-integrable such that $\int\limits_\Lambda |h(t)|\Delta t > 0$. Then*

$$\Phi\left(\frac{\int\limits_\Lambda |h(t)|f(t)\Delta t}{\int\limits_\Lambda |h(t)|\Delta t}\right) \leqslant \frac{\int\limits_\Lambda |h(t)|\Phi(f(t))\Delta t}{\int\limits_\Lambda |h(t)|\Delta t}.$$

THEOREM 3.2. *Assume*

$$(\Omega,\mathscr{M},\mu_\Delta) \textit{ and } (\Lambda,\mathscr{L},\lambda_\Delta) \textit{ are two time scale measure spaces,} \tag{3.1}$$

$$k:\Omega\times\Lambda\to\mathbb{R}_+ \textit{ is such that } K(x) := \int_\Lambda k(x,y)\Delta y < \infty,\ x\in\Omega \tag{3.2}$$

and

$$\xi:\Omega\to\mathbb{R}_+ \textit{ is such that } w(y) := \int_\Omega \frac{k(x,y)\xi(x)}{K(x)}\Delta x < \infty,\ y\in\Lambda. \tag{3.3}$$

If $\Phi\in C(I,\mathbb{R})$ is convex, where $I\subset\mathbb{R}$ is an interval, then

$$\int_\Omega \xi(x)\Phi\left(\frac{1}{K(x)}\int_\Lambda k(x,y)f(y)\Delta y\right)\Delta x \leqslant \int_\Lambda w(y)\Phi(f(y))\Delta y$$

holds for all λ_Δ-integrable $f:\Lambda\to\mathbb{R}$ such that $f(\Lambda)\subset I$.

Proof. By using Jensen's inequality given in Theorem 3.1 and the Fubini theorem on time scales, we find that

$$\begin{aligned}
&\int_\Omega \xi(x)\Phi\left(\frac{1}{K(x)}\int_\Lambda k(x,y)f(y)\Delta y\right)\Delta x\\
&= \int_\Omega \xi(x)\Phi\left(\frac{\int_\Lambda |k(x,y)|f(y)\Delta y}{\int_\Lambda |k(x,y)|\Delta y}\right)\Delta x\\
&\leqslant \int_\Omega \xi(x)\frac{\int_\Lambda |k(x,y)|\Phi(f(y))\Delta y}{\int_\Lambda |k(x,y)|\Delta y}\Delta x\\
&= \int_\Omega \frac{\xi(x)}{K(x)}\left(\int_\Lambda k(x,y)\Phi(f(y))\Delta y\right)\Delta x\\
&= \int_\Lambda \Phi(f(y))\left(\int_\Omega \frac{k(x,y)\xi(x)}{K(x)}\Delta x\right)\Delta y\\
&= \int_\Lambda w(y)\Phi(f(y))\Delta y,
\end{aligned}$$

and the proof is complete. □

COROLLARY 3.3. *Assume* (3.1), (3.2) *and* (3.3)*. If* $p>1$*, then*

$$\int_\Omega \xi(x)\left(\frac{1}{K(x)}\int_\Lambda k(x,y)f(y)\Delta y\right)^p \Delta x \leqslant \int_\Lambda w(y)(f(y))^p\Delta y$$

holds for all λ_Δ*-integrable* $f:\Lambda\to\mathbb{R}_+$.

Proof. Use $\Phi(r)=r^p$ and $I=\mathbb{R}_+$ in Theorem 3.2. □

COROLLARY 3.4. *Assume* (3.1), (3.2) *and* (3.3)*. If* $p>1$*, then*

$$\int_\Omega \xi(x)e^{\frac{p}{K(x)}\int_\Lambda k(x,y)\ln(g(y))\Delta y}\Delta x \leqslant \int_\Lambda w(y)(g(y))^p\Delta y$$

holds for all λ_Δ*-integrable* $g:\Lambda\to(0,\infty)$.

Proof. Use $\Phi(r)=e^r$ and $I=\mathbb{R}$ and let $f=\ln(g^p)$ in Theorem 3.2. □

COROLLARY 3.5. *Assume* (3.1), (3.2) *and* (3.3)*. Then*

$$\int_\Omega \xi(x)e^{\frac{1}{K(x)}\int_\Lambda k(x,y)\ln(g(y))\Delta y}\Delta x \leqslant \int_\Lambda w(y)g(y)\Delta y$$

holds for all λ_Δ*-integrable* $g:\Lambda\to(0,\infty)$.

Proof. Use $p=1$ in Corollary 3.4. □

In the following, the entries of a vector $x\in\mathbb{R}^n$ are called x_i, where $1\leqslant i\leqslant n$.

THEOREM 3.6. *Let* $\mathbb{T}$ *be a time scale and assume*

$$a_i, b_i \in \overline{\mathbb{T}},\ 0 \leqslant a_i < b_i \leqslant \infty,\ 1 \leqslant i \leqslant n,\ \Omega = \Lambda := \times_{i=1}^{n} [a_i, b_i)_{\mathbb{T}}, \tag{3.4}$$

(3.2) *and*

$$u : \Omega \to \mathbb{R}_+ \text{ is such that } v(y) := \int_{\Omega} \frac{y_1 \cdots y_n k(x,y) u(x)}{\sigma(x_1) \cdots \sigma(x_n) K(x)} \Delta x < \infty,\ y \in \Lambda. \tag{3.5}$$

If $\Phi \in \mathrm{C}(I, \mathbb{R})$ *is convex, where* $I \subset \mathbb{R}$ *is an interval, then*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x) \Phi((A_k f)(x)) \frac{\Delta x_1 \cdots \Delta x_n}{\sigma(x_1) \cdots \sigma(x_n)} \leqslant \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} v(y) \Phi(f(y)) \frac{\Delta y_1 \cdots \Delta y_n}{y_1 \cdots y_n} \tag{3.6}$$

holds for all λ_Δ*-integrable* $f : \Lambda \to \mathbb{R}$ *such that* $f(\Lambda) \subset I$, *where*

$$(A_k f)(x) := \frac{1}{K(x)} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} k(x,y) f(y) \Delta y_1 \cdots \Delta y_n.$$

Proof. We replace $\xi(x_1, \ldots, x_n)$ by $\frac{u(x_1, \ldots, x_n)}{\sigma(x_1) \cdots \sigma(x_n)}$ in Theorem 3.2 and notice that therefore

$$w(y_1, \ldots, y_n) = \frac{v(y_1, \ldots, y_n)}{y_1 \cdots y_n}$$

holds. An application of Theorem 3.2 completes the proof. □

COROLLARY 3.7. *Assume* (3.4), (3.2) *and* (3.5). *If* $p > 1$, *then*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x_1, \ldots, x_n) ((A_k f)(x_1, \ldots, x_n))^p \frac{\Delta x_1 \cdots \Delta x_n}{\sigma(x_1) \cdots \sigma(x_n)} \leqslant \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} v(y_1, \ldots, y_n) (f(y_1, \ldots, y_n))^p \frac{\Delta y_1 \cdots \Delta y_n}{y_1 \cdots y_n}$$

holds for all λ_Δ*-integrable* $f : \Lambda \to \mathbb{R}_+$.

Proof. Use $\Phi(r) = r^p$ and $I = \mathbb{R}_+$ in Theorem 3.6. □

COROLLARY 3.8. *Assume* (3.4), (3.2) *and* (3.5). *If* $p > 1$, *then*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x_1, \ldots, x_n) e^{p(A_k \ln(g))(x_1, \ldots, x_n)} \frac{\Delta x_1 \cdots \Delta x_n}{\sigma(x_1) \cdots \sigma(x_n)} \leqslant \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} v(y_1, \ldots, y_n) (g(y_1, \ldots, y_n))^p \frac{\Delta y_1 \cdots \Delta y_n}{y_1 \cdots y_n} \tag{3.7}$$

holds for all λ_Δ*-integrable* $g : \Lambda \to (0, \infty)$.

Proof. Use $\Phi(r) = e^r$ and $I = \mathbb{R}$ and let $f = \ln(g^p)$ in Theorem 3.6. □

EXAMPLE 3.9. If in Corollary 3.8 we take $\mathbb{T} = \mathbb{R}$ and $a_i = 0$ for all $1 \leqslant i \leqslant n$, then (3.7) takes the form

$$\int_0^{b_1} \cdots \int_0^{b_n} u(x_1, \ldots, x_n) e^{p(A_k \ln(g))(x_1, \ldots, x_n)} \frac{\mathrm{d}x_1 \cdots \mathrm{d}x_n}{x_1 \cdots x_n} \leqslant \int_0^{b_1} \cdots \int_0^{b_n} v(y_1, \ldots, y_n)(g(y_1, \ldots, y_n))^p \frac{\mathrm{d}y_1 \cdots \mathrm{d}y_n}{y_1 \cdots y_n}.$$

COROLLARY 3.10. *Assume* (3.4), (3.2) *and* (3.5). *Then*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x_1, \ldots, x_n) e^{(A_k \ln(g))(x_1, \ldots, x_n)} \frac{\Delta x_1 \cdots \Delta x_n}{\sigma(x_1) \cdots \sigma(x_n)} \leqslant \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} v(y_1, \ldots, y_n) g(y_1, \ldots, y_n) \frac{\Delta y_1 \cdots \Delta y_n}{y_1 \cdots y_n}$$

holds for all λ_Δ*-integrable* $g : \Lambda \to (0, \infty)$.

Proof. Use $p = 1$ in Corollary 3.8. □

4. Inequalities with special kernels

COROLLARY 4.1. *Assume* (3.4), (3.2) *and* (3.5) *with the kernel* k *such that*

$$k(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0 \quad \text{if} \quad a_i \leqslant y_i \leqslant \sigma(x_i) \leqslant b_i,\ 1 \leqslant i \leqslant n. \tag{4.1}$$

If $\Phi \in C(I, \mathbb{R})$ *is convex, where* $I \subset \mathbb{R}$ *is an interval, then* (3.6) *holds for all* λ_Δ*-integrable* $f : \Lambda \to \mathbb{R}$ *such that* $f(\Lambda) \subset I$, *where*

$$K(x) = \int_{\sigma(x_1)}^{b_1} \cdots \int_{\sigma(x_n)}^{b_n} k(x_1, \ldots, x_n, y_1, \ldots, y_n) \Delta y_1 \cdots \Delta y_n,$$

$$v(y) = y_1 \cdots y_n \int_{a_1}^{y_1} \cdots \int_{a_n}^{y_n} \frac{k(x_1, \ldots, x_n, y_1, \ldots, y_n) u(x_1, \ldots, x_n)}{\sigma(x_1) \cdots \sigma(x_n) K(x_1, \ldots, x_n)} \Delta x_1 \cdots \Delta x_n$$

and

$$(A_k f)(x) = \frac{1}{K(x)} \int_{\sigma(x_1)}^{b_1} \cdots \int_{\sigma(x_n)}^{b_n} k(x, y_1, \ldots, y_n) f(y_1, \ldots, y_n) \Delta y_1 \cdots \Delta y_n.$$

Proof. The statement follows from Theorem 3.6 by using (4.1). □

EXAMPLE 4.2. If in Corollary 4.1 we take $\mathbb{T}=\mathbb{R}$ and $b_i=\infty$ for all $1\leqslant i\leqslant n$, then (3.6) takes the form

$$\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}u(x_1,\ldots,x_n)\Phi\left((A_kf)(x_1,\ldots,x_n)\right)\frac{\mathrm{d}x_1\cdots\mathrm{d}x_n}{x_1\cdots x_n}$$
$$\leqslant\int_{a_1}^{\infty}\cdots\int_{a_n}^{\infty}v(y_1,\ldots,y_n)\Phi(f(y_1,\ldots,y_n))\frac{\mathrm{d}y_1\cdots\mathrm{d}y_n}{y_1\cdots y_n},$$

where

$$K(x)=\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}k(x_1,\ldots,x_n,y_1,\ldots,y_n)\mathrm{d}y_1\cdots\mathrm{d}y_n,$$
$$v(y)=y_1\cdots y_n\int_{a_1}^{y_1}\cdots\int_{a_n}^{y_n}\frac{k(x_1,\ldots,x_n,y_1,\ldots,y_n)u(x_1,\ldots,x_n)}{x_1\cdots x_nK(x_1,\ldots,x_n)}\mathrm{d}x_1\cdots\mathrm{d}x_n$$

and

$$(A_kf)(x)=\frac{1}{K(x)}\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}k(x,y_1,\ldots,y_n)f(y_1,\ldots,y_n)\mathrm{d}y_1\cdots\mathrm{d}y_n.$$

This result is the same as [15, inequality (2.2)].

COROLLARY 4.3. *Assume* (3.4), (3.2) *and* (3.5) *with the kernel* k *such that*

$$k(x_1,\ldots,x_n,y_1,\ldots,y_n)=0\quad\text{if}\quad a_i\leqslant\sigma(x_i)\leqslant y_i\leqslant b,\ 1\leqslant i\leqslant n.\tag{4.2}$$

If $\Phi\in\mathrm{C}(I,\mathbb{R})$ *is convex, where* $I\subset\mathbb{R}$ *is an interval, then* (3.6) *holds for all* λ_Δ-*integrable* $f:\Lambda\to\mathbb{R}$ *such that* $f(\Lambda)\subset I$, *where*

$$K(x)=\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_n}^{\sigma(x_n)}k(x_1,\ldots,x_n,y_1,\ldots,y_n)\Delta y_1\cdots\Delta y_n,$$
$$v(y)=y_1\cdots y_n\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}\frac{k(x_1,\ldots,x_n,y_1,\ldots,y_n)u(x_1,\ldots,x_n)}{\sigma(x_1)\cdots\sigma(x_n)K(x_1,\ldots,x_n)}\Delta x_1\cdots\Delta x_n$$

and

$$(A_kf)(x)=\frac{1}{K(x)}\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_n}^{\sigma(x_n)}k(x,y_1,\ldots,y_n)f(y_1,\ldots,y_n)\Delta y_1\cdots\Delta y_n.$$

Proof. The statement follows from Theorem 3.6 by using (4.2). □

EXAMPLE 4.4. If in Corollary 4.3 we take $\mathbb{T}=\mathbb{R}$ and $a_i=0$ for all $1\leqslant i\leqslant n$, then (3.6) takes the form

$$\int_{0}^{b_1}\cdots\int_{0}^{b_n}u(x_1,\ldots,x_n)\Phi\left((A_kf)(x_1,\ldots,x_n)\right)\frac{\mathrm{d}x_1\cdots\mathrm{d}x_n}{x_1\cdots x_n}$$
$$\leqslant\int_{0}^{b_1}\cdots\int_{0}^{b_n}v(y_1,\ldots,y_n)\Phi(f(y_1,\ldots,y_n))\frac{\mathrm{d}y_1\cdots\mathrm{d}y_n}{y_1\cdots y_n},$$

where

$$K(x) = \int_0^{x_1} \cdots \int_0^{x_n} k(x_1,\ldots,x_n,y_1,\ldots,y_n)\mathrm{d}y_1\cdots\mathrm{d}y_n,$$
$$v(y) = y_1\cdots y_n \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \frac{k(x_1,\ldots,x_n,y_1,\ldots,y_n)u(x_1,\ldots,x_n)}{x_1\cdots x_n K(x_1,\ldots,x_n)}\mathrm{d}x_1\cdots\mathrm{d}x_n$$

and

$$(A_k f)(x) = \frac{1}{K(x)} \int_0^{x_1} \cdots \int_0^{x_n} k(x,y_1,\ldots,y_n) f(y_1,\ldots,y_n)\mathrm{d}y_1\cdots\mathrm{d}y_n.$$

This result is the same as [15, inequality (2.5)]. Special cases are given (for $n=1$) in [10, Theorem 4.1] and (for $k(x,y)=1$) in [14, inequality (2.2)].

REMARK 4.5. Using (4.2) in Example 3.9, we obtain [10, inequality (4.2)] (for $n=1$).

5. Examples and special cases

THEOREM 5.1. *Assume* (3.4) *and*

$$\xi : \Omega \to \mathbb{R}_+ \text{ is such that } \widetilde{w}(y) := \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \frac{\xi(x_1,\ldots,x_n)}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)} \Delta x_1\cdots\Delta x_n < \infty,\ y \in \Lambda. \tag{5.1}$$

If $\Phi \in \mathrm{C}(I,\mathbb{R})$ *is convex, where* $I \subset \mathbb{R}$ *is an interval, then*

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \xi(x_1,\ldots,x_n)\Phi\Big((\widetilde{A}f)(x_1,\ldots,x_n)\Big)\Delta x_1\cdots\Delta x_n$$
$$\leqslant \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \widetilde{w}(y_1,\ldots,y_n)\Phi(f(y_1,\ldots,y_n))\Delta y_1\cdots\Delta y_n \tag{5.2}$$

holds for all λ_Δ*-integrable* $f : \Lambda \to \mathbb{R}$ *such that* $f(\Lambda) \subset I$, *where*

$$(\widetilde{A}f)(x) := \frac{1}{\prod_{i=1}^{n}(\sigma(x_i)-a_i)} \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_n}^{\sigma(x_n)} f(y_1,\ldots,y_n)\Delta y_1\cdots\Delta y_n.$$

Proof. Let K and $A_k f$ be defined as in the statements of Theorem 3.2 and Theorem 3.6, respectively. The statement follows from Theorem 3.2 by using

$$k(x_1,\ldots,x_n,y_1,\ldots,y_n) = \begin{cases} 1 & \text{if} \quad a_i \leqslant y_i < \sigma(x_i) \leqslant b_i,\ 1 \leqslant i \leqslant n \\ 0 & \text{otherwise,} \end{cases} \tag{5.3}$$

since in this case we have

$$K(x_1,\dots,x_n)=\int_{a_1}^{\sigma(x_1)}\cdots\int_{a_n}^{\sigma(x_n)}\Delta y_1\cdots\Delta y_n=\prod_{i=1}^{n}(\sigma(x_i)-a_i)$$

and thus $A_k=\widetilde{A}$ and $w=\widetilde{w}$. $\square$

REMARK 5.2. By using k of the form (5.3), we may also give results corresponding to Corollary 3.3, Corollary 3.4, Corollary 3.5, Theorem 3.6, Corollary 3.7, Corollary 3.8, Corollary 3.10 and Corollary 4.2.

COROLLARY 5.3. *Assume* (3.4) *with* $a_i=0$ *for all* $1\leqslant i\leqslant n$. *If* $\Phi\in \mathrm{C}(I,\mathbb{R})$ *is convex, where* $I\subset\mathbb{R}$ *is an interval, then*

$$\begin{aligned}\int_0^{b_1}\cdots\int_0^{b_n}\Phi\left((\widetilde{A}f)(x_1,\dots,x_n)\right)\frac{\Delta x_1\cdots\Delta x_n}{x_1\cdots x_n}\\ \leqslant\int_0^{b_1}\cdots\int_0^{b_n}\left\{\prod_{i=1}^{n}\left(\frac{1}{y_i}-\frac{1}{b_i}\right)\right\}\Phi(f(y_1,\dots,y_n))\Delta y_1\cdots\Delta y_n \qquad (5.4)\end{aligned}$$

holds for all λ_Δ*-integrable* $f:\Lambda\to\mathbb{R}$ *such that* $f(\Lambda)\subset I$, *where*

$$(\widetilde{A}f)(x):=\frac{1}{\prod_{i=1}^{n}\sigma(x_i)}\int_0^{\sigma(x_1)}\cdots\int_0^{\sigma(x_n)}f(y_1,\dots,y_n)\Delta y_1\cdots\Delta y_n.$$

Proof. The statement follows from Theorem 5.1 by using

$$\xi(x_1,\dots,x_n)=\frac{1}{x_1\cdots x_n},$$

since in this case we have

$$\widetilde{w}(y_1,\dots,y_n)=\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}\frac{1}{\prod_{i=1}^{n}x_i\sigma(x_i)}\Delta x_1\cdots\Delta x_n=\prod_{i=1}^{n}\left(\frac{1}{y_i}-\frac{1}{b_i}\right)$$

as the function $h(x)=1/x$ is known [5, Example 1.25] to have the time scales derivative $h^\Delta(x)=-1/(x\sigma(x))$. $\square$

EXAMPLE 5.4. If $b_i=\infty$ for all $1\leqslant i\leqslant n$ in addition to the assumptions of Corollary 5.3, then (5.4) takes the form

$$\begin{aligned}\int_0^{\infty}\cdots\int_0^{\infty}\Phi\left((\widetilde{A}f)(x_1,\dots,x_n)\right)\frac{\Delta x_1\cdots\Delta x_n}{x_1\cdots x_n}\\ \leqslant\int_0^{\infty}\cdots\int_0^{\infty}\Phi(f(y_1,\dots,y_n))\frac{\Delta y_1\cdots\Delta y_n}{y_1\cdots y_n}.\end{aligned}$$

For $\mathbb{T}=\mathbb{N}$ and $n=1$, this result is given in [2, 11].

Now we give a Hardy–Hilbert-type inequality on time scales.

THEOREM 5.5. *Assume* (3.4) *with* $n=1$, $a_1=0$ *and* $b_1=\infty$. *If we define*

$$K_1(x)=\int_0^\infty \frac{\left(\frac{y}{x}\right)^{-\frac{1}{p}}}{x+y}\Delta y \quad \text{and} \quad K_2(y)=\int_0^\infty \frac{\left(\frac{y}{x}\right)^{1-\frac{1}{p}}}{x+y}\Delta x,$$

then

$$\int_0^\infty (K_1(x))^{1-p}\left(\int_0^\infty \frac{g(y)}{x+y}\Delta y\right)^p \Delta x \leqslant \int_0^\infty K_2(y)(g(y))^p \Delta y \tag{5.5}$$

holds for all λ_Δ*-integrable* $g:\Lambda\to\mathbb{R}_+$.

Proof. We use

$$\xi(x)=\frac{K_1(x)}{x} \quad \text{and} \quad k(x,y)=\begin{cases}\frac{\left(\frac{y}{x}\right)^{-1/p}}{x+y} & \text{if} \quad x\neq 0,\ y\neq 0,\ x+y\neq 0\\ 0 & \text{otherwise}\end{cases}$$

in Corollary 3.3 to obtain

$$\int_0^\infty (K_1(x))^{1-p}\left(\int_0^\infty \frac{\left(\frac{y}{x}\right)^{-1/p} f(y)}{x+y}\Delta y\right)^p \frac{\Delta x}{x} \leqslant \int_0^\infty w(y)(f(y))^p \Delta y, \tag{5.6}$$

where

$$\begin{aligned} w(y) &= \int_0^\infty \frac{k(x,y)\xi(x)}{K_1(x)}\Delta x = \int_0^\infty \frac{k(x,y)\Delta x}{x} \\ &= \frac{1}{y}\int_0^\infty \frac{\left(\frac{y}{x}\right)^{1-1/p}}{x+y}\Delta x = \frac{K_2(y)}{y}. \end{aligned}$$

Using this in (5.6) and letting $f(y)=g(y)y^{-\frac{1}{p}}$, we obtain (5.5). □

EXAMPLE 5.6. If we take $\mathbb{T}=\mathbb{R}$ in Theorem 5.5 and use the known fact that

$$\int_0^\infty \frac{\left(\frac{y}{x}\right)^{-1/p}}{x+y}\mathrm{d}y = \int_0^\infty \frac{\left(\frac{y}{x}\right)^{1-1/p}}{x+y}\mathrm{d}x = \frac{\pi}{\sin(\pi/p)},$$

then (5.5) turns into the classical Hilbert inequality (see e.g., [9])

$$\int_0^\infty \left(\int_0^\infty \frac{g(y)}{x+y}\mathrm{d}y\right)^p \mathrm{d}x \leqslant \left(\frac{\pi}{\sin(\pi/p)}\right)^p \int_0^\infty (g(y))^p \mathrm{d}y.$$

Now we consider some generalizations of Pólya–Knopp type inequalities.

COROLLARY 5.7. *Assume* (3.4) *with* $n=1$, $a_1=a\geqslant 0$, $b_1=\infty$ *and* (5.1)*. If* $\Phi\in C(I,\mathbb{R})$ *is convex, where* $I\subset\mathbb{R}$ *is an interval, then*

$$\int_a^\infty \xi(x)\Phi\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(y)\Delta y\right)\Delta x \leqslant \int_a^\infty\left(\int_y^\infty \frac{\xi(x)\Delta x}{\sigma(x)-a}\right)\Phi(f(y))\Delta y \quad (5.7)$$

holds for all λ_Δ*-integrable* $f:\Lambda\to\mathbb{R}$ *such that* $f(\Lambda)\subset I$.

Proof. The statement follows from Theorem 5.1 by using $n=1$. □

EXAMPLE 5.8. In addition to the assumptions of Corollary 5.7, if $\mathbb{T}$ consists of only isolated points, then (5.7) takes the form

$$\sum_{x\in[a,\infty)_\mathbb{T}} \xi(x)\Phi\left(\frac{1}{\sigma(x)-a}\sum_{y\in[a,x]_\mathbb{T}} f(y)(\sigma(y)-y)\right)(\sigma(x)-x)$$
$$\leqslant \sum_{y\in[a,\infty)_\mathbb{T}}\left(\sum_{x\in[y,\infty)_\mathbb{T}} \xi(x)\frac{\sigma(x)-x}{\sigma(x)-a}\right)\Phi(f(y))(\sigma(y)-y). \quad (5.8)$$

This result is the same as [20, Theorem 1.1], but here we use time scales notation instead of the notation given in [20].

REMARK 5.9. As in Example 5.8, one can write the discrete version of (5.2).

In the following three examples, we consider Example 5.8 with $\Phi(r)=r^p$, where $p>1$.

EXAMPLE 5.10. For $\mathbb{T}=h\mathbb{N}=\{hn:n\in\mathbb{N}\}$ with $h>0$, $a=h$, and

$$\xi(x)=\frac{1}{\sigma(x)},$$

(5.8) takes the form

$$\sum_{n=1}^\infty \frac{1}{n+1}\left(\frac{1}{n}\sum_{k=1}^n f(kh)\right)^p \leqslant \sum_{n=1}^\infty \frac{(f(nh))^p}{n}. \quad (5.9)$$

EXAMPLE 5.11. For $\mathbb{T}=\mathbb{N}^2=\{n^2:n\in\mathbb{N}\}$ with $a=1$ and

$$\xi(x)=\frac{2(\sigma(x)-1)}{(\sigma(x)-x)^2(2\sqrt{x}+3)},$$

(5.8) takes the form

$$\sum_{n=1}^\infty \frac{2\,(n(n+2))^{1-p}}{(2n+1)(2n+3)}\left(\sum_{k=1}^n (2k+1)f(k^2)\right)^p \leqslant \sum_{n=1}^\infty (f(n^2))^p.$$

If instead

$$\xi(x) = \frac{\sigma(x)-1}{x\sigma(x)},$$

then (5.8) takes the form

$$\sum_{n=1}^{\infty} \frac{(2n+1)(n+2)^{1-p}}{n^{p+1}(n+1)^2} \left(\sum_{k=1}^{n} (2k+1) f(k^2) \right)^p \leqslant \sum_{n=1}^{\infty} \frac{2n+1}{n^2} (f(n^2))^p.$$

EXAMPLE 5.12. For $\mathbb{T} = q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}\}$ with $q > 1$, $a = q$ and

$$\xi(x) = \frac{\sigma(x)-a}{\sigma(x)(\sigma(x)-x)},$$

(5.8) takes the form

$$\sum_{n=1}^{\infty} q^{-n}(q-1)^p (q^n-1)^{1-p} \left(\sum_{k=1}^{n} q^{k-1} f(q^k) \right)^p \leqslant \sum_{n=1}^{\infty} (f(q^n))^p. \tag{5.10}$$

In the following three examples, we consider Example 5.8 with $\Phi(r) = e^r$ and $f(y) = \ln(g(y))$ for $g(y) > 0$.

EXAMPLE 5.13. For $\mathbb{T}$, a and ξ as in Example 5.10, (5.8) takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left(\prod_{k=1}^{n} g(kh) \right)^{\frac{1}{n}} \leqslant \sum_{n=1}^{\infty} \frac{g(nh)}{n}. \tag{5.11}$$

If we let $\varphi(y) = g(y)/y$ in (5.11), then we get

$$\sum_{n=1}^{\infty} \frac{1}{n+1} \left(n! \prod_{k=1}^{n} \varphi(kh) \right)^{\frac{1}{n}} \leqslant \sum_{n=1}^{\infty} \varphi(nh). \tag{5.12}$$

Since $e^{-1} < (n!)^{\frac{1}{n}}/(n+1)$, from (5.12) we obtain

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \varphi(kh) \right)^{\frac{1}{n}} \leqslant e \sum_{n=1}^{\infty} \varphi(nh),$$

which is the well-known Carleman inequality [10, p. 141].

EXAMPLE 5.14. For $\mathbb{T}$, a and the two choices of ξ as in Example 5.11, (5.8) takes the forms

$$\sum_{n=1}^{\infty} \frac{2n(n+2)}{(2n+1)(2n+3)} \left(\prod_{k=1}^{n} (g(k^2))^{2k+1} \right)^{\frac{1}{n(n+2)}} \leqslant \sum_{n=1}^{\infty} g(n^2)$$

and

$$\sum_{n=1}^{\infty} \frac{(2n+1)(n+2)}{n(n+1)^2} \left(\prod_{k=1}^{n} (g(k^2))^{2k+1} \right)^{\frac{1}{n(n+2)}} \leqslant \sum_{n=1}^{\infty} \frac{2n+1}{n^2} g(n^2).$$

EXAMPLE 5.15. For $\mathbb{T}$, a and ξ as in Example 5.12, (5.8) takes the form

$$\sum_{n=1}^{\infty} q^{-n}(q^n-1)\left(\prod_{k=1}^{n}(g(q^k))^{q^{k-1}}\right)^{\frac{q-1}{q^n-1}} \leqslant \sum_{n=1}^{\infty} g(q^n). \tag{5.13}$$

REMARK 5.16. For $h=1$, inequalities (5.9) and (5.12) are given in [13, (12.6), (12.7), p. 153]. Also, (5.10) and (5.13) are the same as [13, (12.1), (12.2), p. 153].

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