Journal of
Mathematical
Inequalities
Volume 8, Number 2 (2014), 219–225 doi:10.7153/jmi-08-14

CONVERSES OF A DISCRETE WIRTINGER TYPE INEQUALITY

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we establish the following converse of Alzer's inequality, which is a discrete analogue of Wirtinger's inequality: Let $z_1, z_2, \cdots, z_n (n \geqslant 2)$ be complex numbers with

$$\sum_{k=1}^{n} z_k = 0,$$

then

$$\sum_{k=1}^{n} |z_k|^2 \geqslant \lambda(n) \min_{1 \leqslant k \leqslant n} \{|z_{k+1} - z_k|^2\}$$

where $z_{n+1} = z_1$ and $\lambda(n) = \frac{n}{4}$, for even n; $\lambda(n) = \frac{n}{4\sin^2 \frac{(n-1)\pi}{2n}}$, for odd n. The constant $\lambda(n)$ is best possible.

1. Introduction

We begin by stating the classical Wirtinger's inequality in the following.

THEOREM A. *Let f be a real-valued function with period 2π and $f' \in L^2[0, 2\pi]$. If*

$$\int_0^{2\pi} f(x)dx = 0,$$

then

$$\int_0^{2\pi} f'(x)^2 dx \geqslant \int_0^{2\pi} f(x)^2 dx \tag{1.1}$$

with equality holding if and only if

$$f(x) = a\cos x + b\sin x,$$

where a, b are real constants.

Because of a wide variety of applications Wirtinger's inequality (1.1) has been generalized in many different directions and improved in many different ways [1, 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15]. Specially, some discrete version of (1.1) are discovered [4, 7, 8, 10, 11, 12, 13, 15].

In 1950, Schoenberg [13, p. 399] established the following discrete analogue of Wirtinger's inequality.

Mathematics subject classification (2010): 26D15, 65D10.

Keywords and phrases: Wirtinger's inequality, Alzer's inequality, discrete analogue of integral inequality.

THEOREM B. *If* $z_1, z_2, \cdots, z_n$ $(n \geqslant 2)$ *are complex numbers with*

$$\sum_{k=1}^{n} z_k = 0,$$

then

$$\sum_{k=1}^{n} |z_{k+1} - z_k|^2 \geqslant 4\sin^2 \frac{\pi}{n} \sum_{k=1}^{n} |z_k|^2 \tag{1.2}$$

where $z_{n+1} = z_1$. *Equality holds in* (1.2) *if and only if* $z_k = a\cos\frac{2\pi k}{n} + b\sin\frac{2\pi k}{n}$, *where* a, b *are real constants.*

In 1992, Alzer [2, p. 86–87] provide a variant of Schoenberg's inequality (1.2), he established the following elegant discrete analogue of Wirtinger inequality.

THEOREM C. *If* $z_1, z_2, \cdots, z_n$ $(n \geqslant 2)$ *are complex numbers with*

$$\sum_{k=1}^{n} z_k = 0,$$

then

$$\sum_{k=1}^{n} |z_{k+1} - z_k|^2 \geqslant \frac{12n}{n^2-1} \max_{1 \leqslant k \leqslant n} |z_k|^2 \tag{1.3}$$

where $z_{n+1} = z_1$. *The constant* $\frac{12n}{n^2-1}$ *is best possible.*

In this paper, we establish the converse of Alzer's inequality (1.3). Our main results are the following three theorems.

THEOREM 1. *If* $z_1, z_2, \cdots, z_n$ $(n \geqslant 2)$ *are complex numbers with*

$$\sum_{k=1}^{n} z_k = 0,$$

then

$$\sum_{k=1}^{n} |z_k|^2 \geqslant \lambda(n) \min_{1 \leqslant k \leqslant n} |z_{k+1} - z_k|^2 \tag{1.4}$$

where $z_{n+1} = z_1$ *and*

$$\lambda(n) = \begin{cases} \dfrac{n}{4}, & \text{for even } n, \\ \dfrac{n}{4\sin^2\frac{(n-1)\pi}{2n}}, & \text{for odd } n. \end{cases}$$

The constant $\lambda(n)$ *is best possible.*

REMARK. Great thanks to the referee, he stated that formulation of theorem 1 could be easily seen from the proof of Theorem 4 in [13, p. 399]. However our proof is new and can be considered as another proof of right hand side of Schoenberg's inequalities.

Noting that $\sum_{k=1}^{n}(z_k - c) = 0$, where $c = \frac{z_1+z_2+\cdots+z_n}{n}$, and applying Theorem 1 for n complex numbers $z_1 - c, z_2 - c, \cdots, z_n - c$, we have

THEOREM 2. *If* $z_1, z_2, \cdots, z_n \ (n \geqslant 2)$ *are complex numbers, let* $c = \frac{z_1+z_2+\cdots+z_n}{n}$, *then*

$$\sum_{k=1}^{n} |z_k - c|^2 \geqslant \lambda(n) \min_{1 \leqslant k \leqslant n} |z_{k+1} - z_k|^2 \tag{1.6}$$

where $z_{n+1} = z_1$ *and* $\lambda(n)$ *is given in Theorem 1 and it is best possible.*

For complex numbers $z_1, z_2, \cdots, z_n \ (n \geqslant 2)$, we define that

$$\Delta z_k = z_{k+1} - z_k, \quad \Delta^l z_k = \Delta(\Delta^{l-1} z_k),$$

where $z_j = z_i$ if $j \equiv i \pmod n$. Noting that $\sum_{k=1}^{n} \Delta^l z_k = 0$, and applying Theorem 1 for $\Delta^l z_1, \Delta^l z_2, \cdots, \Delta^l z_n$, we have

THEOREM 3. *If* $z_1, z_2, \cdots, z_n \ (n \geqslant 2)$ *are complex numbers and* l *is a positive integer, then*

$$\sum_{k=1}^{n} |\Delta^l z_k|^2 \geqslant \lambda(n) \min_{1 \leqslant k \leqslant n} |\Delta^{l+1} z_k|^2 \tag{1.7}$$

where $z_j = z_i$ *if* $j \equiv i \pmod n$, $\lambda(n)$ *is given in Theorem 1 and it is best possible.*

2. The proof of the main result

Proof of Theorem 1. Since the inequality (1.4) is homogeneous, we may assume that

$$\min_{1 \leqslant k \leqslant n} |z_{k+1} - z_k|^2 = 1. \tag{2.1}$$

Consider first the case when n is even.

By the parallelogram identity $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$, we have

$$\begin{aligned} \sum_{k=1}^{n} |z_k|^2 &= \frac{1}{2} \sum_{k=1}^{n} (|z_k|^2 + |z_{k+1}|^2) \\ &= \frac{1}{4} \sum_{k=1}^{n} (|z_k + z_{k+1}|^2 + |z_k - z_{k+1}|^2) \\ &\geqslant \frac{1}{4} \sum_{k=1}^{n} |z_k - z_{k+1}|^2 \\ &\geqslant \frac{n}{4} \min_{1 \leqslant k \leqslant n} |z_{k+1} - z_k|^2 = \frac{n}{4}, \end{aligned} \tag{2.2}$$

the equality in the above inequality holds if and only if $z_k + z_{k+1} = 0 \ (k = 1, 2, \cdots, n)$. For example, since n is even, the sequence $(\frac{1}{2}, -\frac{1}{2}, \cdots, \frac{1}{2}, -\frac{1}{2})$ satisfies such condition.

Consider the second case when n is odd.

Let $\theta_k = \arg \frac{z_{k+1}}{z_k} \in [0, 2\pi)$, $k = 1, 2, \cdots, n$.

(1) If $\theta_k \leqslant \frac{\pi}{2}$ or $\theta_k \geqslant \frac{3\pi}{2}$, by (2.1), we obtain

$$\begin{aligned}|z_k|^2+|z_{k+1}|^2 &= |z_k-z_{k+1}|^2+2|z_k||z_{k+1}|\cos\theta_k\\ &\geqslant |z_k-z_{k+1}|^2\\ &\geqslant 1.\end{aligned} \tag{2.3}$$

(2) If $\theta_k \in (\frac{\pi}{2}, \frac{3\pi}{2})$, from the fact that $\cos\theta_k < 0$ and (2.1), we have

$$\begin{aligned}1 \leqslant |z_k-z_{k+1}|^2 &= |z_k|^2+|z_{k+1}|^2-2|z_k||z_{k+1}|\cos\theta_k\\ &\leqslant (|z_k|^2+|z_{k+1}|^2)(1-\cos\theta_k)\\ &= (|z_k|^2+|z_{k+1}|^2)2\sin^2\frac{\theta_k}{2},\end{aligned}$$

this is equivalent to

$$|z_k|^2+|z_{k+1}|^2 \geqslant \frac{1}{2\sin^2\frac{\theta_k}{2}}. \tag{2.4}$$

It implies that

$$|z_k|^2+|z_{k+1}|^2 \geqslant \frac{1}{2}. \tag{2.5}$$

Now we consider the following two cases:

Case 1. If all $\theta_k \in (\frac{\pi}{2}, \frac{3\pi}{2})$, $k = 1,2,\cdots,n$.
By (2.4), we have

$$\sum_{k=1}^{n}|z_k|^2 = \frac{1}{2}\sum_{k=1}^{n}(|z_k|^2+|z_{k+1}|^2) \geqslant \frac{1}{4}\sum_{k=1}^{n}\frac{1}{\sin^2\frac{\theta_k}{2}}. \tag{2.6}$$

On the other hand, since $z_{n+1} = z_1$, we have

$$\prod_{k=1}^{n}\frac{z_{k+1}}{z_k} = 1.$$

Therefore

$$\sum_{k=1}^{n}\theta_k = \arg\prod_{k=1}^{n}\frac{z_{k+1}}{z_k}+2k\pi = 2k\pi, \tag{2.7}$$

where k is some positive integer and $k < n$.
Noting that $\frac{n\pi}{2} < \sum_{k=1}^{n}\theta_k < \frac{3n\pi}{2}$, it follows that

$$\frac{\pi}{4} < \frac{k\pi}{n} < \frac{3\pi}{4}. \tag{2.8}$$

Since $k < n$ is positive integer and n is odd, by (2.8), we have

$$\left|\sin\frac{k\pi}{n}\right| \leqslant \sin\frac{(n-1)\pi}{2n}. \tag{2.9}$$

Let $f(x) = \frac{1}{\sin^2 x}$, $x \in [\frac{\pi}{4}, \frac{3\pi}{4}]$. It is easy to see that $f(x)$ is convex function. Using Jensen's inequality and combining (2.6), (2.7) and (2.9), we infer that

$$\begin{aligned}
\sum_{k=1}^{n} |z_k|^2 &\geqslant \frac{1}{4}\sum_{k=1}^{n}\frac{1}{\sin^2\frac{\theta_k}{2}} \\
&\geqslant \frac{n}{4}\frac{1}{\sin^2(\frac{\sum_{k=1}^{n}\theta_k}{2n})} \\
&= \frac{n}{4}\frac{1}{\sin^2\frac{k\pi}{n}} \\
&\geqslant \frac{n}{4}\frac{1}{\sin^2\frac{(n-1)\pi}{2n}}.
\end{aligned} \tag{2.10}$$

Case 2. If there exist $\theta_j \notin (\frac{\pi}{2}, \frac{3\pi}{2})$.

Let $I = \{j | \theta_j \notin (\frac{\pi}{2}, \frac{3\pi}{2}),\ j = 1, 2, \cdots, n\}$. By (2.3), for $j \in I$, we have

$$|z_j|^2 + |z_{j+1}|^2 \geqslant 1.$$

But for $j \notin I$, by (2.5), we have

$$|z_j|^2 + |z_{j+1}|^2 \geqslant \frac{1}{2}.$$

Hence

$$\begin{aligned}
\sum_{k=1}^{n} |z_k|^2 &= \frac{1}{2}\Big(\sum_{j\in I}(|z_j|^2 + |z_{j+1}|^2) + \sum_{j\notin I}(|z_j|^2 + |z_{j+1}|^2)\Big) \\
&\geqslant \frac{1}{2}(|I| + \frac{1}{2}(n - |I|)) = \frac{|I| + n}{4} \geqslant \frac{n+1}{4}.
\end{aligned} \tag{2.11}$$

Now we claim that

$$\sin^2\frac{(n-1)\pi}{2n} \geqslant \frac{n}{n+1}. \tag{2.12}$$

In fact, when $n = 3$, by it is obvious that the equality of (2.12) holds; when $n \geqslant 5$, from the fact $\cos x \geqslant 1 - \frac{x^2}{2}$ $(x \in (0, \frac{\pi}{2}))$, we have

$$\sin^2\frac{(n-1)\pi}{2n} = \cos^2\frac{\pi}{2n} \geqslant \left(1 - \frac{(\frac{\pi}{2n})^2}{2}\right) \geqslant 1 - \left(\frac{\pi}{2n}\right)^2 > \frac{n}{n+1}.$$

Hence (2.12) is proved.

By (2.11) and (2.12), we implies that

$$\sum_{k=1}^{n}|z_k|^2 \geqslant \frac{n+1}{4} \geqslant \frac{n}{4\sin^2\frac{(n-1)\pi}{2n}}.$$

Thus, we have proved that when n is odd, the inequality

$$\sum_{k=1}^{n}|z_k|^2 \geqslant \frac{n}{4\sin^2\frac{(n-1)\pi}{2n}} \tag{2.13}$$

holds. On the other hand, taking

$$z_k = \frac{e^{\frac{i(n-1)k\pi}{n}}}{2\sin\frac{(n-1)\pi}{2n}}, \quad k=1,2,\cdots,n,$$

it is easy to see that for all $k=1,2,\cdots,n$,

$$|z_k - z_{k+1}| = 1, \quad \sum_{k=1}^{n} z_k = 0$$

and the equality of (2.13) holds. Hence the constant $\lambda(n)$ is best possible.

We complete the proof of Theorem 1. $\square$

Acknowledgements.

The authors are grateful to the referees for their valuable comments and advises for improvements.

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(Received February 6, 2013)

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