Journal of
Mathematical
Inequalities
Volume 8, Number 2 (2014), 279–285 doi:10.7153/jmi-08-19

MATRIX INEQUALITIES INCLUDING GRAND FURUTA INEQUALITY VIA KARCHER MEAN

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(*Communicated by Masatoshi Fujii*)

Abstract. In our previous paper, we have shown a generalization of Furuta inequality via Karcher mean (Riemannian mean) by using Yamazaki's results which are generalizations of Ando-Hiai inequality and related ones. In this paper, we shall show a generalization of grand Furuta inequality as an extension of our previous result.

1. Introduction

For two positive definite matrices A and B, the weighted geometric mean is defined by $A\sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$ for $\alpha \in [0,1]$. Recently, many authors discuss natural extensions of the (weighted) geometric mean for three or more positive definite matrices [2, 4, 5, 15, 17, 18].

Let $P_m(\mathbb{C})$ be the set of $m\times m$ positive definite matrices on $\mathbb{C}$ and Δ^n be the set of probability vectors (the components satisfy $\sum_i w_i = 1$ and $w_i > 0$ for $i = 1,\dots,n$ if $\omega = (w_1,\dots,w_n) \in \Delta^n$). For $A,B \in P_m(\mathbb{C})$, Riemannian metric between A and B is defined as $\delta_2(A,B) = \|\log A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\|_2$, where $\|X\|_2 = (\operatorname{tr} X^*X)^{\frac{1}{2}}$ (details are in [3]). We remark that $\delta_2(A, A\sharp_\alpha B) = \alpha\delta_2(A,B)$ for $\alpha \in [0,1]$. By using Riemannian metric, weighted Karcher mean of $A_1,\dots,A_n \in P_m(\mathbb{C})$ and $\omega = (w_1,\dots,w_n) \in \Delta^n$ is defined by

$$\Lambda(\omega; A_1,\dots,A_n) = \underset{X\in P_m(\mathbb{C})}{\arg\min} \sum_{i=1}^{n} w_i \delta_2^2(A_i, X),$$

where $\arg\min f(X)$ means the point X_0 which attains minimum value of the function $f(X)$ (see [4, 17, 18]). In particular, we call $\Lambda(\omega; A_1,\dots,A_n)$ Karcher mean if $\omega = (\frac{1}{n},\dots,\frac{1}{n})$. Karcher mean is also called Riemannian mean, the least squares mean, and so on.

Weighted Karcher mean satisfies ten properties proposed in [2], which should be required for a reasonable geometric mean of positive definite matrices, for example, consistency with scalars, monotonicity, congruence invariance, and so on (details are in [14] or other related papers). In the case of two matrices, $\Lambda(\omega; A, B) = A\sharp_\alpha B$ holds for

Mathematics subject classification (2010): Primary 15A45, 47A63. Secondary 15B48, 47A64.
Keywords and phrases: Positive definite matrix, Karcher mean, Furuta inequality and grand Furuta inequality.

Paper JMI-08-19

$\alpha \in [0,1]$ and $\omega = (1-\alpha,\alpha)$. It was obtained in [17, 18] that weighted Karcher mean coincides with the unique positive solution of the following Karcher equation:

$$\sum_{i=1}^{n} w_i \log(X^{\frac{1}{2}} A_i^{-1} X^{\frac{1}{2}}) = 0.$$

Furuta inequality [9] (see also [6, 10, 16, 19]) is established as a generalization of Löwner-Heinz theorem "*$A \geqslant B \geqslant 0$ ensures $A^{\alpha} \geqslant B^{\alpha}$ for any $\alpha \in [0,1]$.*" Ando-Hiai [1] obtained an equivalent inequality to the main result of log majorization, which is called Ando-Hiai inequality. These inequalities can be expressed by using the weighted geometric mean. We remark that these inequalities hold even in the case of bounded linear operators on a complex Hilbert space. In what follows, we denote $A \geqslant 0$ if A is a positive semidefinite matrix (or operator), and we denote $A > 0$ if A is a positive definite matrix (or operator).

THEOREM 1.A. (Satellite form of Furuta inequality [9, 16])

$$A \geqslant B \geqslant 0 \text{ with } A > 0 \quad \textit{implies} \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leqslant B \leqslant A \textit{ for } p \geqslant 1 \textit{ and } r \geqslant 0.$$

THEOREM 1.B. (Ando-Hiai inequality [1]) *For $A, B > 0$,*

$$A \sharp_{\alpha} B \leqslant I \textit{ for } \alpha \in [0,1] \quad \textit{implies} \quad A^r \sharp_{\alpha} B^r \leqslant I \textit{ for } r \geqslant 1,$$

or equivalently

$$A \geqslant B > 0 \quad \textit{implies} \quad A^{-r+1} \sharp_{\frac{1}{p}} (A \natural_r B^p) \leqslant A \quad \textit{for } p \geqslant 1 \textit{ and } r \geqslant 1.$$

We remark that we can interpret Theorem 1.A as a consequence of monotonicity of an operator function shown in [7], that is, $A \geqslant B \geqslant 0$ with $A > 0$ ensures that

$$f(p,r) = A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \tag{1.1}$$

is decreasing for $p \geqslant 1$ and $r \geqslant 0$.

Recently, Yamazaki [21] has obtained an excellent generalization of Theorem 1.B and related results in [7, 11] via weighted Karcher mean of n-matrices. By using Yamazaki's results, we have shown a generalization of Theorem 1.A. We remark that $\|\cdot\|_1$ means 1-norm, that is, $\|x\|_1 = \sum_i |x_i|$ for $x = (x_1,\ldots,x_n)$.

THEOREM 1.C. ([14]) *Let $A_1,\ldots,A_n \in P_m(\mathbb{C})$ and $q > 0$. Then $A_i^q \geqslant A_n^q > 0$ for $i = 1,\ldots,n-1$ implies*

$$\Lambda\left(\omega; A_1^{-p_1},\ldots,A_{n-1}^{-p_{n-1}},A_n^{p_n}\right) \leqslant A_n^q \leqslant A_i^q$$

for all $p_i \geqslant 0$ $(i = 1,\ldots,n-1)$ and $p_n > q$, where

$$\widehat{\omega} = \left(\frac{1}{p_1+q},\ldots,\frac{1}{p_{n-1}+q},\frac{n-1}{p_n-q}\right) \textit{ and } \omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1}.$$

Theorem 1.C implies Theorem 1.A by putting $n = 2$, $p_1 = r$, $p_2 = p$ and $q = 1$.

In [12], Furuta has shown an extension of Theorems 1.A and 1.B, which is called grand Furuta inequality (see also [8, 13, 20]). Here, we adopt an expresson in [8], that is, we use the weighted geometric mean $\sharp_\alpha$ and the notation $\natural_s$ defined by $A \natural_s B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^s A^{\frac{1}{2}}$ for $A, B > 0$ and a real number s.

THEOREM 1.D. (Grand Furuta inequality [12]) *If $A \geqslant B \geqslant 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \geqslant 1$,*

$$F(r,s) = A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p)$$

is decreasing for $r \geqslant t$ and $s \geqslant 1$. Moreover,

(i) $A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leqslant (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \leqslant B \leqslant A,$

(ii) $A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leqslant A^{-r+t} \sharp_{\frac{1-t+r}{p-t+r}} B^p \leqslant B \leqslant A$

hold for $r \geqslant t$ and $s \geqslant 1$.

Theorem 1.D leads Theorem 1.A by putting $t = 0$ and $s = 1$, and also Theorem 1.D leads Theorem 1.B by putting $t = 1$ and $s = r$.

In this paper, as an extension of Theorem 1.C, we shall show a generalization of grand Furuta inequality via weighted Karcher mean of n-matrices.

2. Results

THEOREM 2.1. *Let $A_1, \ldots, A_n \in P_m(\mathbb{C})$ and $q > 0$. Then $A_i^q \geqslant A_n^q > 0$ for $i = 1, \ldots, n-1$ implies*

(i)
$$\Lambda\left(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_{n-1}^t \natural_s A_n^{p_n}\right) \leqslant (A_{n-1}^t \natural_s A_n^{p_n})^{\frac{q}{(p_n-t)s+t}} \leqslant A_n^q \leqslant A_i^q,$$

(ii)
$$\Lambda\left(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_{n-1}^t \natural_s A_n^{p_n}\right) \leqslant \Lambda\left(\omega_0; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_n^{p_n}\right) \leqslant A_n^q \leqslant A_i^q$$

for all $t \in [0,q]$, $s \geqslant 1$, $p_i \geqslant 0$ $(i = 1, \ldots, n-1)$ and $p_n > q$, where

$$\widehat{\omega} = \left(\frac{1}{p_1+q}, \ldots, \frac{1}{p_{n-1}+q}, \frac{n-1}{(p_n-t)s+t-q}\right),\ \omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1},$$
$$\widehat{\omega_0} = \left(\frac{1}{p_1+q}, \ldots, \frac{1}{p_{n-1}+q}, \frac{n-1}{p_n-q}\right) \text{ and } \omega_0 = \frac{\widehat{\omega_0}}{\|\widehat{\omega_0}\|_1}.$$

THEOREM 2.2. *Let* $A_1,\ldots,A_n \in P_m(\mathbb{C})$ *and* $q>0$. *If* $A_i^q \geqslant A_n^q > 0$ *for* $i=1,\ldots,n-1$*, then for each* $t\in[0,q]$ *and* $p_n > q$*,*

$$F(p_1,\ldots,p_{n-1},s)=\Lambda\left(\omega;A_1^{-p_1},\ldots,A_{n-1}^{-p_{n-1}},A_{n-1}^{t}\natural_s A_n^{p_n}\right)$$

is decreasing for $p_i \geqslant 0$ $(i=1,\ldots,n-1)$ *and* $s\geqslant 1$*, where*

$$\widehat{\omega}=\left(\tfrac{1}{p_1+q},\ldots,\tfrac{1}{p_{n-1}+q},\tfrac{n-1}{(p_n-t)s+t-q}\right) \text{ and } \omega=\tfrac{\widehat{\omega}}{\|\widehat{\omega}\|_1}.$$

COROLLARY 2.3. *Let* $A_1,\ldots,A_n \in P_m(\mathbb{C})$ *and* $q>0$. *If* $A_i^q \geqslant A_n^q > 0$ *for* $i=1,\ldots,n-1$*, then*

$$f(p_1,\ldots,p_{n-1},p_n)=\Lambda\left(\omega;A_1^{-p_1},\ldots,A_{n-1}^{-p_{n-1}},A_n^{p_n}\right)$$

is decreasing for $p_i \geqslant 0$ $(i=1,\ldots,n-1)$ *and* $p_n>q$*, where*

$$\widehat{\omega}=\left(\tfrac{1}{p_1+q},\ldots,\tfrac{1}{p_{n-1}+q},\tfrac{n-1}{p_n-q}\right) \text{ and } \omega=\tfrac{\widehat{\omega}}{\|\widehat{\omega}\|_1}.$$

Theorem 2.1 implies Theorem 1.C by putting $t=0$ and $s=1$. Theorems 2.1 and 2.2 imply Theorem 1.D by putting $n=2$, $p_1=r-t$, $p_2=p$ and $q=1$. Corollary 2.3 implies (1.1) by putting $n=2$, $p_1=r$, $p_2=p$ and $q=1$. We remark that Corollary 2.3 is already pointed out in our previous paper [14].

We state three matrices case of Theorems 2.1 and 2.2 for readers' sake.

COROLLARY 2.4. *Let* $A_1,A_2,B \in P_m(\mathbb{C})$ *and* $q>0$. *Then* $A_1^q \geqslant B^q > 0$ *and* $A_2^q \geqslant B^q>0$ *implies*

(i) $$\Lambda\left(\omega;A_1^{-r_1+t},A_2^{-r_2+t},A_2^{t}\natural_s B^{p}\right) \leqslant (A_2^t\natural_s B^p)^{\frac{q}{(p-t)s+t}} \leqslant B^q \leqslant A_1^q \text{ (or } A_2^q),$$

(ii) $$\Lambda\left(\omega;A_1^{-r_1+t},A_2^{-r_2+t},A_2^{t}\natural_s B^{p}\right) \leqslant \Lambda\left(\omega_0;A_1^{-r_1+t},A_2^{-r_2+t},B^{p}\right) \leqslant B^q \leqslant A_1^q \text{ (or } A_2^q)$$

for all $t\in[0,q]$, $p>q$, $r_1\geqslant t$, $r_2 \geqslant t$ *and* $s\geqslant 1$*, where*

$$\widehat{\omega}=\left(\tfrac{1}{r_1-t+q},\tfrac{1}{r_2-t+q},\tfrac{2}{(p-t)s+t-q}\right),\ \omega=\tfrac{\widehat{\omega}}{\|\widehat{\omega}\|_1},$$

$$\widehat{\omega_0}=\left(\tfrac{1}{r_1-t+q},\tfrac{1}{r_2-t+q},\tfrac{2}{p-q}\right) \text{ and } \omega_0=\tfrac{\widehat{\omega_0}}{\|\widehat{\omega_0}\|_1}.$$

COROLLARY 2.5. *Let* $A_1,A_2,B \in P_m(\mathbb{C})$ *and* $q>0$. *If* $A_1^q \geqslant B^q>0$ *and* $A_2^q \geqslant B^q > 0$*, then for each* $t\in[0,q]$ *and* $p>q$*,*

$$F(r_1,r_2,s)=\Lambda\left(\omega;A_1^{-r_1+t},A_2^{-r_2+t},A_2^{t}\natural_s B^{p}\right)$$

is decreasing for $r_1\geqslant t$, $r_2\geqslant t$ *and* $s \geqslant 1$*, where*

$$\widehat{\omega}=\left(\tfrac{1}{r_1-t+q},\tfrac{1}{r_2-t+q},\tfrac{2}{(p-t)s+t-q}\right) \text{ and } \omega=\tfrac{\widehat{\omega}}{\|\widehat{\omega}\|_1}.$$

3. Proofs

We use the following two results in order to prove Theorems 2.1 and 2.2.

THEOREM 3.A. ([8]) *Let* $A \geqslant B \geqslant 0$ *with* $A > 0$. *Then*

$$(A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \leqslant B \leqslant A$$

holds for $t \in [0,1]$, $p \geqslant 1$ *and* $s \geqslant 1$.

THEOREM 3.B. ([14]) *Let* $A_1, \ldots, A_n \in P_m(\mathbb{C})$ *and* $\omega = (w_1, \ldots, w_n) \in \Delta^n$. *For each* $i = 1, \ldots, n$ *and* $q \in \mathbb{R}$, *if*

$$\Lambda\left(\omega; A_1^{p_1}, \ldots, A_i^{p_i}, \ldots, A_n^{p_n}\right) \leqslant A_i^q \quad \text{for } p_1, \ldots, p_n \in \mathbb{R} \text{ with } p_i > q,$$

then

$$\begin{aligned}
&\Lambda\left(\omega'; A_1^{p_1}, \ldots, A_{i-1}^{p_{i-1}}, A_i^{p_i'}, A_{i+1}^{p_{i+1}}, \ldots, A_n^{p_n}\right)\\
&\leqslant \Lambda\left(\omega; A_1^{p_1}, \ldots, A_{i-1}^{p_{i-1}}, A_i^{p_i}, A_{i+1}^{p_{i+1}}, \ldots, A_n^{p_n}\right)\\
&\leqslant A_i^q
\end{aligned}$$

for $p_i' \geqslant p_i$, *where* $\widehat{\omega'} = \left(w_1, \ldots, w_{i-1}, \frac{p_i - q}{p_i' - q} w_i, w_{i+1}, \ldots, w_n\right)$ *and* $\omega' = \frac{\widehat{\omega'}}{\|\widehat{\omega'}\|_1}$.

Proof of Theorem 2.1. (i) Put $B = (A_{n-1}^t \natural_s A_n^{p_n})^{\frac{1}{(p_n-t)s+t}}$ and assume that $A_i^q \geqslant A_n^q > 0$ for $q > 0$ and $i = 1, \ldots, n-1$. Then by Theorem 3.A,

$$(A_{n-1}^{q \cdot \frac{t}{q}} \natural_s A_n^{q \cdot \frac{p_n}{q}})^{\frac{1}{(\frac{p_n}{q} - \frac{t}{q})s + \frac{t}{q}}} \leqslant A_n^q,$$

that is,

$$B^q = (A_{n-1}^t \natural_s A_n^{p_n})^{\frac{q}{(p_n-t)s+t}} \leqslant A_n^q \leqslant A_i^q \tag{3.1}$$

holds for $t \in [0,q]$, $p_n > q$ and $s \geqslant 1$.

By Theorem 1.C, (3.1) implies

$$\Lambda\left(\omega'; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, B^{\alpha}\right) \leqslant B^q \leqslant A_n^q \leqslant A_i^q$$

for all $p_i \geqslant 0$ $(i = 1, \ldots, n-1)$ and $\alpha > q$, where $\widehat{\omega'} = \left(\frac{1}{p_1+q}, \ldots, \frac{1}{p_{n-1}+q}, \frac{n-1}{\alpha-q}\right)$ and $\omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1}$. By putting $\alpha = (p_n - t)s + t \, (> q)$,

$$\Lambda\left(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_{n-1}^t \natural_s A_n^{p_n}\right) \leqslant (A_{n-1}^t \natural_s A_n^{p_n})^{\frac{q}{(p_n-t)s+t}} \leqslant A_n^q \leqslant A_i^q$$

for all $t \in [0,q]$, $s \geqslant 1$, $p_i \geqslant 0$ $(i = 1, \ldots, n-1)$ and $p_n > q$, where

$$\widehat{\omega} = \left(\frac{1}{p_1+q}, \ldots, \frac{1}{p_{n-1}+q}, \frac{n-1}{(p_n-t)s+t-q}\right) \text{ and } \omega = \frac{\widehat{\omega}}{\|\widehat{\omega}\|_1}.$$

(ii) By Theorem 1.C and Löwner-Heinz theorem, $A_i^q \geqslant A_n^q > 0$ for $q > 0$ and $i = 1, \ldots, n-1$ implies

$$\Lambda\left(\omega_0; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_n^{p_n}\right) \leqslant A_n^q = A_n^t \natural_{\frac{q-t}{p_n-t}} A_n^{p_n} \leqslant A_{n-1}^t \natural_{\frac{q-t}{p_n-t}} A_n^{p_n} \tag{3.2}$$

for all $t \in [0,q]$, $p_i \geqslant 0$ $(i = 1, \ldots, n-1)$ and $p_n > q$, where

$$\widehat{\omega_0} = \left(\tfrac{1}{p_1+q}, \ldots, \tfrac{1}{p_{n-1}+q}, \tfrac{n-1}{p_n-q}\right) \text{ and } \omega = \tfrac{\widehat{\omega}}{\|\widehat{\omega}\|_1}.$$

(3.2) holds if and only if

$$\begin{aligned} X &\equiv \Lambda\left(\omega_0; A_{n-1}^{\frac{-t}{2}} A_1^{-p_1} A_{n-1}^{\frac{-t}{2}}, \ldots, A_{n-1}^{-p_{n-1}-t}, A_{n-1}^{\frac{-t}{2}} A_n^{p_n} A_{n-1}^{\frac{-t}{2}}\right) \\ &\leqslant A_{n-1}^{\frac{-t}{2}} A_n^q A_{n-1}^{\frac{-t}{2}} \\ &\leqslant (A_{n-1}^{\frac{-t}{2}} A_n^{p_n} A_{n-1}^{\frac{-t}{2}})^{\frac{q-t}{p_n-t}}. \end{aligned} \tag{3.3}$$

By applying Theorem 3.B to (3.3), we can obtain

$$\Lambda\left(\omega; A_{n-1}^{\frac{-t}{2}} A_1^{-p_1} A_{n-1}^{\frac{-t}{2}}, \ldots, A_{n-1}^{-p_{n-1}-t}, (A_{n-1}^{\frac{-t}{2}} A_n^{p_n} A_{n-1}^{\frac{-t}{2}})^s\right) \leqslant X \tag{3.4}$$

for $s \geqslant 1$, where

$$\widehat{\omega} = \left(\tfrac{1}{p_1+q}, \ldots, \tfrac{1}{p_{n-1}+q}, \tfrac{n-1}{(p_n-t)s+t-q}\right) \text{ and } \omega = \tfrac{\widehat{\omega}}{\|\widehat{\omega}\|_1}.$$

Therefore (3.3) and (3.4) ensures

$$\Lambda\left(\omega; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_{n-1}^t \natural_s A_n^{p_n}\right) \leqslant \Lambda\left(\omega_0; A_1^{-p_1}, \ldots, A_{n-1}^{-p_{n-1}}, A_n^{p_n}\right) \leqslant A_n^q \leqslant A_i^q$$

for all $t \in [0,q]$, $s \geqslant 1$, $p_i \geqslant 0$ $(i = 1, \ldots, n-1)$ and $p_n > q$. $\square$

Proof of Theorem 2.2. (i) or (ii) in Theorem 2.1 ensures

$$\Lambda\left(\omega; A_{n-1}^{\frac{-t}{2}} A_1^{-p_1} A_{n-1}^{\frac{-t}{2}}, \ldots, A_{n-1}^{-p_{n-1}-t}, (A_{n-1}^{\frac{-t}{2}} A_n^{p_n} A_{n-1}^{\frac{-t}{2}})^s\right) \leqslant (A_{n-1}^{\frac{-t}{2}} A_n^{p_n} A_{n-1}^{\frac{-t}{2}})^{\frac{q-t}{p_n-t}} \tag{3.5}$$

by the same way to (3.3), so that we can show monotonicity for $s \geqslant 1$ by applying Theorem 3.B to (3.5).

We can show monotonicity for $p_i \geqslant 0$ $(i = 1, \ldots, n-1)$ by applying Theorem 3.B to (i) or (ii) in Theorem 2.1. $\square$

Proof of Corollary 2.3. Put $t = 0$ and replace $p_n s$ by p_n in Theorem 2.2. $\square$

Proof of Corollaries 2.4, 2.5. Put $n = 3$, $p_1 = r_1 - t$, $p_2 = r_2 - t$, $p_3 = p$ in Theorems 2.1, 2.2, respectively. $\square$

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(Received March 23, 2013)

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